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SCALES OF WEIGHT CHARACTERIZATIONS FOR SOME MULTIDIMENSIONAL DISCRETE HARDY AND CARLEMAN TYPE INEQUALITIES

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Abstract. Some new scales of weight characterizations for the discrete multidimensional Hardy inequality are proved for the case $1 < p \le q < \infty$. As limit cases some scales of weight characterizations for the multidimensional Carleman inequality are obtained for the case $0 < p \le q < \infty$.

რეზიუმე. ნაშრომში მიღებულია წონითი დახასიათების ახალი სკალები დისკრეტული მრავალგანზომილებიანი ჰარდის უტოლობისათვის იმ შემთხვევაში, როცა $1 < p \le q < \infty$. როგორც ზღვარითი შემთხვევა ანალოგიური სკალა დადგენილია მრავალგანზომილებიანი კარლემანის უტოლობისათვის, როცა $0 < p \le q < \infty$.

1. Introduction

Consider the inequality

$$\sum_{n=1}^{\infty}\Big(\frac{1}{n}\sum_{k=1}^{n}a_k\Big)^p \le \Big(\frac{p}{p-1}\Big)^p\sum_{n=1}^{\infty}a_n^p \qquad (a_n \ge 0,\ \ p > 1)\,. \tag{1.1}$$

In fact, for $p = 2$ G. H. Hardy dealt with the inequality (1.1) already in his paper [8] from 1919 (see also his 1920 paper [9]) but his ideas and motivation (to find an elementary proof of Hilbert's inequality) traces back even to his paper [7] from 1915. Concerning the history and developments of (1.1) we refer to [16], [17] and [18].

Another inequality which is related to (1.1) is Carleman's inequality from 1922 (see [6]):

$$\sum_{n=1}^{\infty}\Big(\prod_{k=1}^{n}a_k\Big)^{\frac{1}{k}} \le e\sum_{n=1}^{\infty}a_n. \tag{1.2}$$

In his 1925 note [10] G. H. Hardy mentioned that G. Pólya pointed out to him the fact that (1.2) is just a natural end point inequality of the inequality

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(1.1). For more information about the history and developments of (1.2) see [13], [14], [26] and the recent thesis [12] by M. Johansson.

Moreover, the weighted version of (1.1)

$$\Big(\sum_{n=1}^{\infty}\Big(\sum_{k=1}^{n}a_k\Big)^q u_n\Big)^{\frac{1}{q}} \le C\Big(\sum_{n=1}^{\infty}a_n^p v_n\Big)^{\frac{1}{p}} \tag{1.3}$$

has been extensively studied e.g. by G. Bennett [1] and [2], M. S. Braverman and V. D. Stepanov [3], M. L. Goldman [4] and for some recent results see also [19]. The continuous version of (1.3) has been studied and generalized by many authors, see e.g. [5], [15], [22], [25], [26], and also the books [14] and [20] on this subject and the references therein. In 1985, E. Sawyer [24] derived necessary and sufficient conditions for the two-dimensional Hardy inequality, where he showed that three independent conditions must be satisfied.

Concerning multidimensional versions of (1.3) there are only a few results in literature. Some sufficient conditions in the two-dimensional case were proved by Y. Rakotondratsimba [23].

We state the following slight improvement of a recent result presented in [19]:

Theorem 1.1. *Let $1 < p \le q < \infty$, $s_1, s_2 \in (1,p)$ and let $\{a_{n_1 n_2}\}$, $n_1, n_2 = 1, 2, \dots$ be an arbitrary nonnegative sequence. Moreover, assume that $\{u_{n_1,n_2}\}$, $n_1, n_2 = 1, 2, \dots$, $\{v_{n_1}\}, n_1 = 1, 2, \dots$, and $\{\omega_{n_2}\}, n_2 = 1, 2, \dots$, are fixed nonnegative sequences. Then the inequality*

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}a_{k_1,k_2}\Big)^q u_{n_1,n_2}\Big)^{1/q} \le$$
$$\le C\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}a_{n_1,n_2}^p v_{n_1}\omega_{n_2}\Big)^{1/p} \tag{1.4}$$

holds for some finite constant C if and only if

$$A(s_1,s_2) := \sup_{N_1,N_2>0}\Big(\sum_{k_1=1}^{N_1}v_{k_1}^{1-p'}\Big)^{\frac{s_1-1}{p}}\Big(\sum_{k_2=1}^{N_2}\omega_{k_2}^{1-p'}\Big)^{\frac{s_2-1}{p}} \times$$
$$\times\Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}u_{n_1,n_2}\Big(\sum_{k_1=1}^{n_1}v_{k_1}^{1-p'}\Big)^{\frac{q(p-s_1)}{p}}\Big(\sum_{k_2=1}^{n_2}\omega_{k_2}^{1-p'}\Big)^{\frac{q(p-s_2)}{p}}\Big)^{1/q} < \infty. \tag{1.5}$$

Moreover, if C is the best constant in (1.4), then

$$\sup_{1<s_1,s_2<p}\Big(\frac{s_1-1}{s_1}\Big)^{\frac{1}{p}}\Big(\frac{s_2-1}{s_2}\Big)^{\frac{1}{p}}A(s_1,s_2) \le C \le$$
$$\le \inf_{1<s_1,s_2<p}A(s_1,s_2)\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}. \tag{1.6}$$

Remark 1.1. For the case when $w_{n_1} = v_{n_1}$, $n = 1, 2, \ldots$, Theorem 1.1 was stated in [19, Theorem 2] and a sketch of the proof was also given there. In Section 2 of this paper we include a proof of Theorem 1.1 with all details.

As a limit result of Theorem 1.1 the following result of Carleman type is obtained (c.f. [19, Proposition 3]):

Theorem 1.2. *Let* $0 < p \leq q < \infty$ *and* $s_1, s_2 > 1$. *Moreover, let* $\{a_{n_1,n_2}\}$, $n_1, n_2 = 1, 2, \ldots$, *be an arbitrary nonnegative sequence and let* $\{u_{n_1,n_2}\}$ *and* $\{v_{n_1,n_2}\}$ *be fixed nonnegative sequences, where* $v_{n_1,n_2} > 0$, $n_1, n_2 = 1, 2, \ldots$. *Then the Carleman type inequality*

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\prod_{k_1=1}^{n_1}\prod_{k_2=1}^{n_2}a_{k_1,k_2}\Big)^{\frac{q}{n_1 n_2}}u_{n_1 n_2}\Big)^{\frac{1}{q}} \leq C\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}a^p_{n_1,n_2}v_{n_1}, n_2\Big)^{\frac{1}{p}} \quad (1.7)$$

holds for some finite constant $C > 0$ *if and only if*

$$B(s_1, s_2) :=$$

$$= \sup_{N_1, N_2 > 0} N_1^{\frac{s_1-1}{p}} N_2^{\frac{s_2-1}{p}} \Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty} n_1^{-\frac{s_1 q}{p}} n_2^{-\frac{s_2 q}{p}} w_{n_1}, n_2\Big)^{\frac{1}{q}} < \infty, \quad (1.8)$$

where

$$w_{n_1,n_2} = u_{n_1,n_2}\Big(\prod_{k_1=1}^{n_1}\prod_{k_2=1}^{n_2} v_{k_1,k_2}\Big)^{\frac{-q}{n_1 n_2 p}}. \quad (1.9)$$

Moreover, for the best constant C *in* (1.7), *we have the following estimates:*

$$\sup_{s_1,s_2>1}\Big(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\Big)^{\frac{1}{p}}\Big(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\Big)^{\frac{1}{p}} B(s_1, s_2) \leq$$

$$\leq C \leq \inf_{s_1,s_2>1} e^{\frac{s_1+s_2-2}{p}} B(s_1, s_2) \quad (1.10)$$

Remark 1.2. Theorem 1.2 is a generalization of the result of H. P. Heinig, R. Kerman and M. Krbec [11], where they gave a necessary and sufficient condition for the case $p = q = 1$. In one dimension, Theorem 1.2 is a discrete correspondence of a result by B. Opic and P. Gurka [21].

Remark 1.3. Note that Theorem 1.1 is formulated only with a product weight sequence $\{v_{n_1}\omega_{n_2}\}$ on the right hand side in (1.4). This is crucial because as in the continuous case (see the Sawyer result [24]) it seems impossible to avoid to have three conditions when characterizing (1.4) with $v_{n_1}\omega_{n_2}$ replaced by a general weight $v_{n_1 n_2}$. However, as we see in Theorem 1.2 in the limit case one condition is sufficient to characterize (1.7) for a general weight $v_{n_1 n_2}$. A similar observation for the continuous case can be found in the PhD thesis of A. Wedestig [26].

The aim of this paper is to follow-up and improve the paper [19]. In Section 2 we give complete proofs of Theorems 1.1 and 1.2. In Section 3 we state and prove the general multidimensional versions of these results.

2. Proofs of Theorems 1.1 and 1.2

Before we prove these theorems we refer to the following elementary Lemma:

Lemma 2.1. *Let* $A_k = \sum\limits_{n=1}^{k} a_n$, $A_0 = 0$ *and for* $n = 1, 2, ...$ *let* $a_n > 0$.

a) *If* $0<d<1$, *then, for* $k=1,2,\ldots$, $dA_k^{d-1}a_k \le A_k^d - A_{k-1}^d \le dA_{k-1}^{d-1}a_k$.

b) *If* $d<0$ *or* $d>1$ *then, for* $k=1,2,\ldots$, $dA_{k-1}^{d-1}a_k \le A_k^d - A_{k-1}^d \le dA_k^{d-1}a_k$.

Proof. The proof follows by just using the mean value theorem in an appropriate way, for details see [19]. □

Proof of Theorem 1.1. For simplicity we only present the proof for the case when $\omega_k = v_k$, $k = 1, 2$, but the proof in the general case is completely the same.

Put $b_{n_1}, n_2^p = a_{n_1}, n_2^p v_{n_1} v_{n_2}$ in (1.4). Then (1.4) is equivalent to

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} b_{k_1,k_2} v_{k_1}^{-\frac{1}{p}} v_{k_2}^{-\frac{1}{p}}\Big)^q u_{n_1,n_2}\Big)^{1/q} \le$$
$$\le C\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} b_{n_1,n_2}^p\Big)^{1/p}. \tag{2.1}$$

Assume that (1.5) holds and let $V_{n_i} = \sum\limits_{k=1}^{n_i} v_{k_i}^{1-p'}$, $i = 1, 2$. Applying Hölder's inequality, Lemma 1(a) with $a_k = v_k^{1-p'}$ and $d = d_i = \frac{p-s_i}{p-1}$, $i = 1, 2$ (note that $0 < d < 1$) and Minkowski's inequality to the left hand side of (2.1) we find that

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} b_{k_1,k_2} v_{k_1}^{-\frac{1}{p}} v_{k_2}^{-\frac{1}{p}}\Big)^q u_{n_1,n_2}\Big)^{\frac{1}{q}} =$$
$$=\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} b_{k_1,k_2} V_{k_1}^{\frac{s_1-1}{p}} V_{k_2}^{\frac{s_2-1}{p}} V_{k_1}^{-\frac{s_1-1}{p}} V_{k_2}^{-\frac{s_2-1}{p}} v_{k_1}^{-\frac{1}{p}} v_{k_2}^{-\frac{1}{p}}\Big)^q u_{n_1,n_2}\Big)^{\frac{1}{q}} \le$$
$$\le \Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} b_{k_1,k_2}^p V_{k_1}^{s_1-1} V_{k_2}^{s_2-1}\Big)^{\frac{q}{p}}\Big(\sum_{k_1=1}^{n_1} V_{k_1}^{-\frac{(s_1-1)p'}{p}} v_{k_1}^{-\frac{p'}{p}}\Big)^{\frac{q}{p'}} \times$$
$$\times\Big(\sum_{k_2=1}^{n_2} V_{k_2}^{-\frac{(s_2-1)p'}{p}} v_{k_2}^{-\frac{p'}{p}}\Big)^{\frac{q}{p'}} u_{n_1,n_2}\Big)^{\frac{1}{q}} \le$$

$$\leq \Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} b_{k_1,k_2}^p V_{k_1}^{s_1-1}V_{k_2}^{s_2-1}\Big)^{\frac{q}{p}}\times$$

$$\times\Big(\sum_{k_1=1}^{n_1} V_{k_1}^{(\frac{p-s_1}{p-1})} - V_{k_1-1}^{(\frac{p-s_1}{p-1})}\Big)^{\frac{q}{p'}}\Big(\sum_{k_2=1}^{n_2} V_{k_2}^{(\frac{p-s_2}{p-1})} - V_{k_2-1}^{(\frac{p-s_2}{p-1})}\Big)^{\frac{q}{p'}} u_{n_1,n_2}\Big)^{\frac{1}{q}} =$$

$$= \Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}\times$$

$$\times\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} b_{k_1,k_2}^p V_{k_1}^{s_1-1}V_{k_2}^{s_2-1}\Big)^{\frac{q}{p}} V_{n_1}^{(\frac{p-s_1}{p})q} V_{n_2}^{(\frac{p-s_2}{p})q} u_{n_1,n_2}\Big)^{\frac{1}{q}} \leq$$

$$\leq \Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}\times$$

$$\times\Big(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} b_{k_1,k_2}^p V_{k_1}^{s_1-1}V_{k_2}^{s_2-1}\Big(\sum_{n_1=k_1}^{\infty}\sum_{n_2=k_2}^{\infty} V_{n_1}^{(\frac{p-s_1}{p})q} V_{n_2}^{(\frac{p-s_2}{p})q} u_{n_1,n_2}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}} \leq$$

$$\leq \Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}\times$$

$$\times \sup_{k_1,k_2>0} V_{k_1}^{\frac{s_1-1}{p}} V_{k_2}^{\frac{s_2-1}{p}}\Big(\sum_{n_1=k_1}^{\infty}\sum_{n_2=k_2}^{\infty} V_{n_1}^{(\frac{p-s_1}{p})q} V_{n_2}^{(\frac{p-s_2}{p})q} u_{n_1,n_2}\Big)^{\frac{1}{q}}\times$$

$$\times\Big(\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} b_{k_1,k_2}^p\Big)^{\frac{1}{p}} =$$

$$= \Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}} A_3(s_1,s_2)\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} b_{n_1,n_2}^p\Big)^{\frac{1}{p}}. \qquad (2.2)$$

Hence, by taking infimum over $s_1, s_2 \in (1,p)$, (2.1) and, thus, (1.4) holds with a constant C satisfying the right hand side inequality in (1.6).

Next, assume that (1.4) and, thus, (2.1) holds and apply the following test sequence to (2.1):

$$b_{k_1k_2}^p := \begin{cases} V_{N_1}^{-s_1} v_{k_1}^{1-p'} V_{N_2}^{-s_2} v_{k_2}^{1-p'} & if \begin{cases} k_1 = 1,\ldots,N_1, \\ k_2 = 1,\ldots,N_2 \end{cases} \\ V_{N_1}^{-s_1} v_{k_1}^{1-p'} V_{k_2}^{-s_2} v_{k_2}^{1-p'} & if \begin{cases} k_1 = 1,\ldots,N_1, \\ k_2 = N_2+1,\ldots, \end{cases} \\ V_{k_1}^{-s_1} v_{k_1}^{1-p'} V_{N_2}^{-s_2} v_{k_2}^{1-p'} & if \begin{cases} k_1 = N_1+1,\ldots, \\ k_2 = 1,\ldots,N_2 \end{cases} \\ V_{k_1}^{-s_1} v_{k_1}^{1-p'} V_{k_2}^{-s_2} v_{k_2}^{1-p'} & if \begin{cases} k_1 = N_1+1,\ldots, \\ k_2 = N_2+1,\ldots\ . \end{cases} \end{cases}$$

(Here N_1 and N_2 are fixed natural numbers).

For the right hand side of (2.1), by applying Lemma 1(b) we obtain that

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} b^p_{n_1,n_2}\Big)^{1/p} =$$

$$= \Big(\sum_{n_1=1}^{N_1} V_{N_1}^{-s_1} v_{n_1}^{1-p'} \sum_{n_2=1}^{N_2} V_{N_2}^{-s_2} v_{n_2}^{1-p'} + \sum_{n_1=1}^{N_1} V_{N_1}^{-s_1} v_{n_1}^{1-p'} \sum_{n_2=N_2+1}^{\infty} V_{n_2}^{-s_2} v_{n_2}^{1-p'} +$$

$$+ \sum_{n_2=1}^{N_2} V_{N_2}^{-s_2} v_{n_2}^{1-p'} \sum_{n_1=N_1+1}^{\infty} V_{n_1}^{-s_1} v_{n_1}^{1-p'} + \sum_{n_1=N_1+1}^{\infty} V_{n_1}^{-s_1} v_{n_1}^{1-p'} \sum_{n_2=N_2+1}^{\infty} V_{n_2}^{-s_2} v_{n_2}^{1-p'}\Big)^{1/p} \leq$$

$$\leq \Big(V_{N_1}^{1-s_1} V_{N_2}^{1-s_2} + \Big(\frac{1}{s_2-1}\Big) V_{N_1}^{1-s_1} V_{N_2}^{1-s_2} +$$

$$+ \Big(\frac{1}{s_1-1}\Big) V_{N_1}^{1-s_1} V_{N_2}^{1-s_2} + \Big(\frac{1}{s_1-1}\Big)\Big(\frac{1}{s_2-1}\Big) V_{N_1}^{1-s_1} V_{N_2}^{1-s_2}\Big)^{\frac{1}{p}} =$$

$$= \Big(1 + \frac{1}{s_1-1} + \frac{1}{s_2-1} + \Big(\frac{1}{s_1-1}\Big)\Big(\frac{1}{s_2-1}\Big)\Big)^{\frac{1}{p}} V_{N_1}^{\frac{1-s_1}{p}} V_{N_2}^{\frac{1-s_2}{p}}.$$

Hence,

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} b^p_{n_1,n_2}\Big)^{1/p} \leq \Big(\frac{s_1}{s_1-1}\Big)^{\frac{1}{p}} \Big(\frac{s_2}{s_2-1}\Big)^{\frac{1}{p}} V_{N_1}^{\frac{1-s_1}{p}} V_{N_2}^{\frac{1-s_2}{p}}. \tag{2.3}$$

For the left hand side of (2.1) we have

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} b_{k_1,k_2} v_{k_1}^{-\frac{1}{p}} v_{k_2}^{-\frac{1}{p}}\Big)^q u_{n_1,n_2}\Big)^{1/q} \geq$$

$$\geq \Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} b_{k_1,k_2} v_{k_1}^{-\frac{1}{p}} v_{k_2}^{-\frac{1}{p}}\Big)^q u_{n_1,n_2}\Big)^{1/q} =$$

$$= \Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}\Big(\sum_{k_1=1}^{N_1} V_{N_1}^{-\frac{s_1}{p}} v_{k_1}^{1-p'} \sum_{k_2=1}^{N_2} V_{N_2}^{-\frac{s_2}{p}} v_{k_2}^{1-p'} +$$

$$+ \sum_{k_1=1}^{N_1} V_{N_1}^{-\frac{s_1}{p}} v_{k_1}^{1-p'} \sum_{k_2=N_2+1}^{n_2} V_{k_2}^{-\frac{s_2}{p}} v_{k_2}^{1-p'} +$$

$$+ \sum_{k_1=N_1+1}^{n_1} V_{k_1}^{-\frac{s_1}{p}} v_{k_1}^{1-p'} \sum_{m_2=1}^{N_2} V_{N_2}^{-\frac{s_2}{p}} v_{k_2}^{1-p'} +$$

$$+ \sum_{k_1=N_1+1}^{n_1} V_{k_1}^{-\frac{s_1}{p}} v_{k_1}^{1-p'} \sum_{k_2=N_2+1}^{n_2} V_{k_2}^{-\frac{s_2}{p}} v_{k_2}^{1-p'}\Big)^q u_{n_1 n_2}\Big)^{\frac{1}{q}} \geq$$

$$\geq \Big(\sum_{n_1=k_1}^{\infty}\sum_{n_2=k_2}^{\infty}\Big(V_{N_1}^{\frac{p-s_1}{p}}V_{N_2}^{\frac{p-s_2}{p}}+V_{N_1}^{\frac{p-s_1}{p}}V_{n_2}^{-\frac{s_2}{p}}\sum_{k_2=N_2+1}^{n_2}v_{k_2}^{1-p'}+V_{N_2}^{\frac{p-s_2}{p}}V_{n_1}^{-\frac{s_1}{p}}\sum_{k_1=N_1+1}^{n_1}v_{k_1}^{1-p'}+$$
$$+V_{n_1}^{-\frac{s_1}{p}}V_{n_2}^{-\frac{s_2}{p}}\sum_{k_1=N_1+1}^{n_1}v_{k_1}^{1-p'}\sum_{k_2=N_2+1}^{n_2}v_{k_2}^{1-p'}\Big)^q u_{n_1,n_2}\Big)^{\frac{1}{q}}=$$
$$=\Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}\Big(V_{N_1}^{\frac{p-s_1}{p}}V_{N_2}^{\frac{p-s_2}{p}}+V_{N_1}^{\frac{p-s_1}{p}}V_{n_2}^{-\frac{s_2}{p}}(V_{n_2}-V_{N_2})+$$
$$+V_{N_2}^{\frac{p-s_2}{p}}V_{n_1}^{-\frac{s_1}{p}}(V_{n_1}-V_{N_1})+V_{n_1}^{-\frac{s_1}{p}}(V_{n_1}-V_{N_1})V_{n_2}^{-\frac{s_2}{p}}(V_{n_2}-V_{N_2})\Big)^q u_{n_1n_2}\Big)^{\frac{1}{q}}=$$
$$=\Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}\Big(V_{N_1}^{\frac{p-s_1}{p}}V_{N_2}^{\frac{p-s_2}{p}}+V_{N_1}^{\frac{p-s_1}{p}}V_{n_2}^{\frac{p-s_2}{p}}-V_{N_1}^{\frac{p-s_1}{p}}V_{n_2}^{-\frac{s_2}{p}}V_{N_2}+$$
$$+V_{N_2}^{\frac{p-s_2}{p}}V_{n_1}^{\frac{p-s_1}{p}}-V_{N_2}^{\frac{p-s_2}{p}}V_{n_1}^{-\frac{s_1}{p}}V_{N_1}+$$
$$+(V_{n_1}^{\frac{p-s_1}{p}}-V_{n_1}^{-\frac{s_1}{p}}V_{N_1})(V_{n_2}^{\frac{p-s_2}{p}}-V_{n_2}^{-\frac{s_2}{p}}V_{N_2})\Big)^q u_{n_1n_2}\Big)^{\frac{1}{q}}\geq$$
$$\geq\Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}\Big(V_{N_1}^{\frac{p-s_1}{p}}V_{N_2}^{\frac{p-s_2}{p}}+V_{N_1}^{\frac{p-s_1}{p}}V_{n_2}^{\frac{p-s_2}{p}}-V_{N_1}^{\frac{p-s_1}{p}}V_{N_2}^{-\frac{s_2}{p}}V_{N_2}+$$
$$+V_{N_2}^{\frac{p-s_2}{p}}V_{n_1}^{\frac{p-s_1}{p}}-V_{N_2}^{\frac{p-s_2}{p}}V_{N_1}^{-\frac{s_1}{p}}V_{N_1}+$$
$$+(V_{n_1}^{\frac{p-s_1}{p}}-V_{N_1}^{-\frac{s_1}{p}}V_{N_1})(V_{n_2}^{\frac{p-s_2}{p}}-V_{N_2}^{-\frac{s_2}{p}}V_{N_2})\Big)^q u_{n_1n_2}\Big)^{\frac{1}{q}}=$$
$$=\Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}V_{n_1}^{\frac{q(p-s_1)}{p}}V_{n_2}^{\frac{q(p-s_2)}{p}}u_{n_1n_2}\Big)^{\frac{1}{q}}. \tag{2.4}$$

Hence, according (2.1), (2.3) and (2.4),

$$\Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}V_{n_1}^{\frac{q(p-s_1)}{p}}V_{n_2}^{\frac{q(p-s_2)}{p}}u_{n_1,n_2}\Big)^{\frac{1}{q}}\leq C\Big(\frac{s_1}{s_1-1}\Big)^{\frac{1}{p}}\Big(\frac{s_2}{s_2-1}\Big)^{\frac{1}{p}}V_{N_1}^{\frac{1-s_1}{p}}V_{N_2}^{\frac{1-s_2}{p}},$$

so that

$$\Big(\frac{s_1-1}{s_1}\Big)^{\frac{1}{p}}\Big(\frac{s_2-1}{s_2}\Big)^{\frac{1}{p}}V_{N_1}^{\frac{s_1-1}{p}}V_{N_2}^{\frac{s_2-1}{p}}\times$$
$$\times\Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}V_{n_1}^{\frac{q(p-s_1)}{p}}V_{n_2}^{\frac{q(p-s_2)}{p}}u_{n_1,n_2}\Big)^{\frac{1}{q}}\leq C.$$

Hence, by taking supremum over $N_1, N_2 > 0$ and supremum over $s_1, s_2 \in (1, p)$ we conclude that (1.5) and the left hand side of the estimate (1.6) hold.

Summing up, we have proved that (1.4) is equivalent to (1.5) and that (1.6) holds. The proof is complete. $\square$

Proof of Theorem 1.2. Assume that (1.8) holds. Replace $a^p_{n_1,n_2}v_{n_1,n_2}$ with $b^p_{n_1,n_2}$ in (1.7). Then (1.7) is equivalent to

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\prod_{k_1=1}^{n_1}\prod_{k_2=1}^{n_2} b_{k_1,k_2}\Big)^{\frac{q}{n_1n_2}} w_{n_1,n_2}\Big)^{\frac{1}{q}} \le C\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} b^p_{n_1,n_2}\Big)^{\frac{1}{p}}, \tag{2.5}$$

where w_{n_1,n_2} is defined by (1.9).

In Theorem 1.1 put $u_{n_1,n_2} = w_{n_1,n_2}n_1^{-q}n_2^{-q}$ and $v_{n_1} = \omega_{n_2} = 1$, $n_1, n_2 = 1, 2 \ldots$, and we find that

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\frac{1}{n_1n_2}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}a_{k_1,k_2}\Big)^q w_{n_1,n_2}\Big)^{1/q} \le C\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}a^p_{n_1,n_2}\Big)^{1/p} \tag{2.6}$$

holds if and only if

$$A(s_1,s_2) :=$$
$$= \sup_{N_1,N_2>0} N_1^{\frac{s_1-1}{p}} N_2^{\frac{s_2-1}{p}}\Big(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty} n_1^{-\frac{s_1q}{p}} n_2^{-\frac{s_2q}{p}} w_{n_1,n_2}\Big)^{\frac{1}{q}} < \infty \tag{2.7}$$

and the best possible constant C in (2.6) can be estimated as follows:

$$\sup_{1<s_1,s_2<p}\Big(\frac{s_1-1}{s_1}\Big)^{\frac{1}{p}}\Big(\frac{s_2-1}{s_2}\Big)^{\frac{1}{p}}A(s_1,s_2) \le C \le$$
$$\le \inf_{1<s_1,s_2<p}\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\Big(\frac{p-1}{p-s_2}\Big)^{\frac{1}{p'}}A(s_1,s_2). \tag{2.8}$$

Now we replace a_{k_1,k_2} by $b^\alpha_{k_1,k_2}$ in (2.6) with $0 < \alpha < p$ and after that we replace p by $\frac{p}{\alpha}$ and q by $\frac{q}{\alpha}$ in (2.6)–(2.8) and since $1 < \frac{p}{\alpha} \le \frac{q}{\alpha} < \infty$, in view of Theorem 1.1, we find that if $1 < s_1, s_2 < \frac{p}{\alpha}$, then the inequality

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\frac{1}{n_1n_2}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}b^\alpha_{k_1,k_2}\Big)^{\frac{q}{\alpha}} w_{n_1,n_2}\Big)^{1/q} \le$$
$$\le C_\alpha\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}b^p_{n_1,n_2}\Big)^{1/p} \tag{2.9}$$

holds for $b_{k_1,k_2} > 0$ if and only if $A(s_1,s_2) < \infty$. Now $A^{\frac{1}{\alpha}}(s_1,s_2) = B(s_1,s_2)$ for all $\alpha > 0$ and w_{n_1,n_2} is defined by (1.9), so according to the right hand side of estimate (2.8), we have that the best possible constant C_α in (2.9) satisfies

$$C_\alpha \le \inf_{1<s_1,s_2<\frac{p}{\alpha}}\Big(\frac{p-\alpha}{p-\alpha s_1}\Big)^{\frac{p-\alpha}{p\alpha}}\Big(\frac{p-\alpha}{p-\alpha s_2}\Big)^{\frac{p-\alpha}{p\alpha}}B(s_1,s_2).$$

Moreover, we note that

$$\Big(\frac{1}{n_1n_2}\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} b_{k_1k_2}^{\alpha}\Big)^{\frac{q}{\alpha}}\downarrow\Big(\prod_{k_1=1}^{n_1}\prod_{k_2=1}^{n_2} b_{k_1k_2}\Big)^{\frac{q}{n_1n_2}}, \text{ as } \alpha\longrightarrow 0^+$$

(the scale of power means converges to the geometric mean) and

$$\Big(\frac{p-\alpha}{p-\alpha s_1}\Big)^{\frac{p-\alpha}{\alpha p}}\Big(\frac{p-\alpha}{p-\alpha s_2}\Big)^{\frac{p-\alpha}{\alpha p}}\longrightarrow e^{\frac{s_1+s_2-2}{p}}\ as\ \alpha\longrightarrow 0^+.$$

Hence, it follows that (2.5) and, thus, (1.7) hold with a constant satisfying the upper estimate in (1.10).

Next we assume that (1.7) holds. To prove the lower bound of the best constant C in (1.10) we apply the following test sequence to (2.5):

$$b_{k_1,k_2}^p := $$
$$=\begin{cases} N_1^{-1}N_2^{-1} & \text{for } k_1=1,\dots,N_1;\ \ k_2=1,\dots,N_2\\ N_1^{-1}e^{-s_2}N_2^{s_2-1}k_2^{-s_2} & \text{for } k_1=1,\dots,N_1;\ \ k_2=N_2+1,\dots\\ N_2^{-1}e^{-s_1}N_1^{s_1-1}k_1^{-s_1} & \text{for } k_1=N_1+1,\dots;\ \ k_2=1,\dots,N_2\\ e^{-(s_1+s_2)}N_1^{s_1-1}k_1^{-s_1}N_2^{s_2-1}k_2^{-s_2} & \text{for } k_1=N_1+1,\dots;\ \ k_2=N_2+1,\dots \end{cases}$$

(Here N_1 and N_2 are fixed natural numbers).

Applying the test sequence to the right hand side of (2.5), we have

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} b_{n_1,n_2}^p\Big)^{1/p} =$$
$$=\Bigg(\sum_{n_1=1}^{N_1}\sum_{n_2=1}^{N_2} N_1^{-1}N_2^{-1}+\sum_{n_1=1}^{N_1}\sum_{n_2=N_2+1}^{\infty} N_1^{-1}e^{-s_2}N_2^{s_2-1}n_2^{-s_2}+$$
$$+\sum_{n_2=1}^{N_2}\sum_{n_1=N_1+1}^{\infty} N_2^{-1}e^{-s_1}N_1^{s_1-1}n_1^{-s_1}+$$
$$+\sum_{n_1=N_1+1}^{\infty}\sum_{n_2=N_2+1}^{\infty} e^{-(s_1+s_2)}N_1^{s_1-1}n_1^{-s_1}N_2^{s_2-1}n_2^{-s_2}\Bigg)^{\frac{1}{p}} =$$
$$=\Bigg(1+e^{-s_2}N_2^{s_2-1}\sum_{n_2=N_2+1}^{\infty} n_2^{-s_2}+e^{-s_1}N_1^{s_1-1}\sum_{n_1=N_1+1}^{\infty} n_1^{-s_1}+$$
$$+e^{-(s_1+s_2)}N_1^{s_1-1}N_2^{s_2-1}\sum_{n_1=N_1+1}^{\infty} n_1^{-s_1}\sum_{n_2=N_2+1}^{\infty} n_2^{-s_2}\Bigg)^{\frac{1}{p}}. \qquad (2.10)$$

For $s>1$,

$$n^{-s} \le \int_{n-1}^{n} x^{-s}dx = \Big[\frac{x^{1-s}}{1-s}\Big]_{n-1}^{n} = \frac{(n-1)^{1-s}-n^{1-s}}{s-1}.$$

and

$$\sum_{n=N+1}^{\infty} n^{-s} \le \int_{N}^{\infty} x^{-s}dx = \Big[\frac{x^{1-s}}{1-s}\Big]_{N}^{\infty} = \frac{N^{1-s}}{s-1}, \tag{2.11}$$

where $n=n_i$, $N=N_i$, $s=s_i$, $i=1,2$.

By using (2.11) in (2.10) we obtain

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} b_{n_1,n_2}^p\Big)^{1/p} \le \left(1+\frac{e^{-s_2}}{s_2-1}+\frac{e^{-s_1}}{s_1-1}+\frac{e^{-(s_1+s_2)}}{(s_1-1)(s_2-1)}\right)^{\frac{1}{p}} =$$

$$= \Big(\frac{1+e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)}\Big)^{\frac{1}{p}}\Big(\frac{1+e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)}\Big)^{\frac{1}{p}}. \tag{2.12}$$

Moreover, the left hand side of (2.5), can be estimated as follows:

$$\left(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Big(\prod_{k_1=1}^{n_1}\prod_{k_2=1}^{n_2} b_{k_1,k_2}^p\Big)^{\frac{q}{pn_1n_2}} w_{n_1,n_2}\right)^{\frac{1}{q}} \ge$$

$$\ge \left(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}\left(\prod_{k_1=1}^{n_1}\prod_{k_2=1}^{n_2} b_{k_1,k_2}^p\right)^{\frac{q}{pn_1n_2}} w_{n_1,n_2}\right)^{\frac{1}{q}} =$$

$$= \left(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}\left(\exp\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\ln b_{k_1,k_2}^p\right)^{\frac{q}{pn_1n_2}} w_{n_1,n_2}\right)^{\frac{1}{q}}. \tag{2.13}$$

Applying the test sequence to the inner summation of (2.13), we get that

$$\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\ln b_{k_1k_2}^p =$$

$$= \sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\ln\big(N_1^{-1}N_2^{-1}\big) + \sum_{k_1=1}^{N_1}\sum_{k_2=N_2+1}^{n_2}\ln\big(N_1^{-1}e^{-s_2}N_2^{s_2-1}k_2^{-s_2}\big)+$$

$$+\sum_{k_1=N_1+1}^{n_1}\sum_{k_2=1}^{N_2}\ln\big(N_2^{-1}e^{-s_1}N_1^{s_1-1}k_1^{-s_1}\big)+$$

$$+\sum_{k_1=N_1+1}^{n_1}\sum_{k_2=N_2+1}^{n_2}\ln\big(e^{-(s_1+s_2)}N_1^{s_1-1}N_2^{s_2-1}k_1^{-s_1}k_2^{-s_2}\big). \tag{2.14}$$

Using the mean value theorem, we find that

$$\sum_{k=N+1}^{n} \ln k = \sum_{k=N+2}^{n+1} k \ln(k-1) - \sum_{k=N+1}^{n} k \ln k =$$

$$= \sum_{k=N+1}^{n} k\big[\ln(k-1) - \ln k\big] - (N+1)\ln N + (n+1)\ln n =$$

$$= \sum_{k=N+1}^{n} -k \int\limits_{k-1}^{k} \frac{1}{x}\, dx + (n+1)\ln n - (N+1)\ln N \leq$$

$$\leq \sum_{k=N+1}^{n} (-1) + (n+1)\ln n - (N+1)\ln N$$

so that

$$- \sum_{k=N+1}^{n} \ln k \geq (n-N) - (n+1)\ln n + (N+1)\ln N. \tag{2.15}$$

We now consider (2.14) termwise. Below we represent the first, second, third and forth terms of the right hand side of (2.14) by $\sum_1$, $\sum_2$, $\sum_3$ and $\sum_4$, respectively. We find that

$$\sum_{1} = \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{N_2} \ln\left(N_1^{-1} N_2^{-1}\right) = -N_1 N_2 \ln N_1 - N_1 N_2 \ln N_2,$$

and, in view of (2.15),

$$\sum_{2} = \sum_{k_1=1}^{N_1} \sum_{k_2=N_2+1}^{n_2} \ln\left(N_1^{-1} e^{-s_2} N_2^{s_2-1} k_2^{-s_2}\right) =$$

$$= -N_1(n_2 - N_2)\ln N_1 + s_2 N_1 (N_2 - n_2) +$$

$$+(s_2-1) N_1 (n_2 - N_2) \ln N_2 - s_2 N_1 \sum_{k_2=N_2+1}^{n_2} \ln k_2 \geq$$

$$\geq -N_1(n_2-N_2)\ln N_1 - s_2 N_1 (n_2 - N_2) + (s_2-1)N_1(n_2-N_2)\ln N_2 +$$

$$+ N_1 s_2 (n_2 - N_2) - N_1 s_2 (n_2+1)\ln n_2 + N_1 s_2 (N_2+1)\ln N_2 =$$

$$= -N_1(n_2-N_2)\ln N_1 + (s_2-1)N_1(n_2-N_2)\ln N_2 -$$

$$- N_1 s_2 (n_2+1)\ln n_2 + N_1 s_2 (N_2+1) \ln N_2.$$

Similarly, we obtain that

$$\sum_{3} \geq -N_2(n_1-N_1)\ln N_2 + (s_1-1)N_2(n_1-N_1)\ln N_1 -$$

$$- N_2 s_1 \left(n_1+1\right)\ln n_1 + N_2 s_1 \left(N_1+1\right)\ln N_1$$

and

$$\sum_4 \geq (s_1-1)(n_1-N_1)(n_2-N_2)\ln N_1 +$$
$$+(s_2-1)(n_1-N_1)(n_2-N_2)\ln N_2 - s_1(n_2-N_2)(n_1+1)\ln n_1 +$$
$$+s_1(n_2-N_2)(N_1+1)\ln N_1 - s_2(n_1-N_1)(n_2+1)\ln n_2 +$$
$$+s_2(n_1-N_1)(N_2+1)\ln N_2.$$

Summing up and simplifying, we have, for any $n_1 \geq N_1$, $n_2 \geq N_2$,

$$\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2} \ln b_{k_1k_2}^p = \sum_1 + \sum_2 + \sum_3 + \sum_4 \geq$$
$$\geq (n_1n_2+n_2)\ln n_1^{-s_1} + (n_1n_2+n_1)\ln n_2^{-s_2} +$$
$$+(n_2s_1+(s_1-1)n_1n_2)\ln N_1 + (n_1s_2+(s_2-1)n_1n_2)\ln N_2 \geq$$
$$\geq n_1n_2\ln n_1^{-s_1} + n_1n_2\ln n_2^{-s_2} + n_1n_2\ln N_1^{s_1-1} + n_1n_2\ln N_2^{s_2-1} =$$
$$= n_1n_2\ln\left(n_1^{-s_1}n_2^{-s_2}N_1^{s_1-1}N_2^{s_2-1}\right). \tag{2.16}$$

Hence

$$\left(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}\left(\exp\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}\ln b_{k_1,k_2}^p\right)^{\frac{q}{pn_1n_2}} w_{n_1,n_2}\right)^{\frac{1}{q}} \geq$$
$$\geq \left(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty}\left(\exp\ln\left(n_1^{-s_1}n_2^{-s_2}N_1^{s_1-1}N_2^{s_2-1}\right)^{n_1n_2}\right)^{\frac{q}{pn_1n_2}} w_{n_1,n_2}\right)^{\frac{1}{q}} =$$
$$= \left(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty} n_1^{-\frac{qs_1}{p}} n_2^{-\frac{qs_2}{p}} N_1^{\frac{q(s_1-1)}{p}} N_2^{\frac{q(s_2-1)}{p}} w_{n_1,n_2}\right)^{\frac{1}{q}} =$$
$$= N_1^{\frac{s_1-1}{p}} N_2^{\frac{s_2-1}{p}} \left(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty} n_1^{-\frac{qs_1}{p}} n_2^{-\frac{qs_2}{p}} w_{n_1,n_2}\right)^{\frac{1}{q}}. \tag{2.17}$$

Combining (2.17), (2.13), (2.12) and (1.7) we get that

$$N_1^{\frac{s_1-1}{p}} N_2^{\frac{s_2-1}{p}} \left(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty} n_1^{-\frac{qs_1}{p}} n_2^{-\frac{qs_2}{p}} w_{n_1,n_2}\right)^{\frac{1}{q}}$$
$$\leq C\left(\frac{1+e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)}\right)^{\frac{1}{p}}\left(\frac{1+e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)}\right)^{\frac{1}{p}},$$

that is,

$$\left(\frac{e^{s_1}(s_1-1)}{1+e^{s_1}(s_1-1)}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)}{1+e^{s_2}(s_2-1)}\right)^{\frac{1}{p}} \times$$
$$\times N_1^{\frac{s_1-1}{p}} N_2^{\frac{s_2-1}{p}} \left(\sum_{n_1=N_1}^{\infty}\sum_{n_2=N_2}^{\infty} n_1^{-\frac{qs_1}{p}} n_2^{-\frac{qs_2}{p}} w_{n_1,n_2}\right)^{\frac{1}{q}} \leq C. \tag{2.18}$$

Thus, by taking supremum over $N_1, N_2 > 0$ and supremum over $s_1, s_2 \in (1,p)$ we find that (1.8) and also the left hand side inequality (1.10) hold. Hence, we have proved that (1.7) is equivalent to (1.8) and that (1.10) holds. The proof is complete. □

3. The multidimensional main results

We state and prove the n-dimensional versions of Theorems 1.1 and 1.2.

Theorem 3.1. *Let $M \in \mathbb{Z}_+$, $1 < p \le q < \infty$, $s_1, \dots, s_M \in (1,p)$ and let $\{a_{n_1,\dots,n_M}\}$, $n_1, \dots, n_M = 1, 2, \dots,$ be an arbitrary nonnegative sequence. Moreover, let $\{u_{n_1,\dots,n_M}\}$, $n_1, \dots, n_M = 1, 2, \dots,$ and $\{v_{i,n_k}\}_{n_k=1}^{\infty}$, $i = 1, 2, \dots, M$ be fixed weight sequences. Then the inequality*

$$\left(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M} a_{k_1,\dots,k_M}\Big)^q u_{n_1,\dots,n_M}\right)^{\frac{1}{q}} \le$$
$$\le C\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty} a^p_{n_1,\dots,n_M} v_{1,n_1}\cdots v_{M,n_M}\Big)^{\frac{1}{p}} \tag{3.1}$$

holds for some finite constant $C > 0$ if and only if

$$A_3\left(s_1, \dots, s_n\right) :=$$
$$= \sup_{N_1,\dots,N_M>0}\Big(\sum_{k_1=1}^{N_1} v_{1,k_1}^{1-p'}\Big)^{\frac{s_1-1}{p}}\cdots\Big(\sum_{k_M=1}^{N_M} v_{M,k_M}^{1-p'}\Big)^{\frac{s_M-1}{p}}\times$$
$$\times\Bigg(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty} u_{n_1,\dots,n_M}\Big(\sum_{k_1=1}^{n_1} v_{1,k_1}^{1-p'}\Big)^{\frac{q(p-s_1)}{p}}\times$$
$$\times\Big(\sum_{k_2=1}^{n_2} v_{2,k_2}^{1-p'}\Big)^{\frac{q(p-s_2)}{p}}\cdots\Big(\sum_{k_M=1}^{n_M} v_{M,k_M}^{1-p'}\Big)^{\frac{q(p-s_M)}{p}}\Bigg)^{\frac{1}{q}} < \infty. \tag{3.2}$$

Moreover, if C is the best constant in (3.1)*, then*

$$\sup_{1<s_1,\dots,s_M<p}\Big(\frac{s_1-1}{s_1}\Big)^{\frac{1}{p}}\cdots\Big(\frac{s_M-1}{s_M}\Big)^{\frac{1}{p}} A_3(s_1,\dots,s_M) \le C \le$$
$$\le \inf_{1<s_1,\dots,s_M<p} A_3(s_1,\dots,s_M)\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\cdots\Big(\frac{p-1}{p-s_M}\Big)^{\frac{1}{p'}}. \tag{3.3}$$

The limit result of Theorem 3.1 reads as follows:

Theorem 3.2. *Let $M \in \mathbb{Z}_+$, $0 < p \le q < \infty$, $s_1, \dots, s_M > 1$, and let $\{a_{n_1,\dots,n_M}\}$, $n_1, \dots, n_M = 1, 2, \dots,$ be an arbitrary nonnegative sequence. Moreover, let $\{u_{n_1,\dots,n_M}\}$, $n_1, \dots, n_M = 1, 2, \dots,$ and $\{v_{n_1,\dots,n_M}\}$,*

$n_1,\dots,n_M=1,2,\dots,$ be fixed nonnegative sequences, where $v_{n_1,\dots,n_M}>0$, $n_1,\dots,n_M=1,2,\dots$. Then the Carleman type inequality

$$\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\prod_{k_1=1}^{n_1}\cdots\prod_{k_M=1}^{n_M}a_{k_1,\dots,k_M}\Big)^{\frac{q}{n_1\cdots n_M}}u_{n_1,\dots,n_M}\Big)^{\frac{1}{q}}\le$$
$$\le C\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}a^p_{n_1,\dots,n_M}v_{n_1,\dots,n_M}\Big)^{\frac{1}{p}} \tag{3.4}$$

holds for some finite constant $C>0$ if and only if

$$B(s_1,\dots,s_M):=\sup_{N_1,\dots,N_M>0}N_1^{\frac{s_1-1}{p}}\cdots N_M^{\frac{s_M-1}{p}}\times$$
$$\times\Big(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty}n_1^{-\frac{qs_1}{p}}\cdots n_M^{-\frac{qs_M}{p}}w_{n_1,\dots,n_M}\Big)^{\frac{1}{q}}<\infty, \tag{3.5}$$

where

$$w_{n_1,\dots,n_M}:=u_{n_1,\dots,n_M}\Big(\prod_{k_1=1}^{n_1}\cdots\prod_{k_M=1}^{n_M}v_{k_1,\dots,k_M}\Big)^{-\frac{q}{pn_1\cdots n_M}}. \tag{3.6}$$

Moreover, for the best constant C in (3.4) we have the following estimates:

$$\sup_{s_1,\dots,s_M>1}\left(\frac{e^{s_1}(s_1-1)}{1+e^{s_1}(s_1-1)}\right)^{\frac{1}{p}}\cdots\left(\frac{e^{s_M}(s_M-1)}{1+e^{s_M}(s_M-1)}\right)^{\frac{1}{p}}\times$$
$$\times B(s_1,\dots,s_M)\le C\le\inf_{s_1,\dots,s_M>1}e^{\frac{s_1+\cdots+s_M-M}{p}}B(s_1,\dots,s_M). \tag{3.7}$$

Proof of Theorem 3.1 The result is known for $M=1$ (see [19, Theorem 1]) and $M=2$ (see Theorem 1.1) so we assume that $M=3,4,\dots$.

Put $b^p_{n_1,\dots,n_M}=a^p_{n_1,\dots,n_M}v_{1,n_1}\cdots v_{M,n_M}$ in (3.1). Then (3.1) is equivalent to

$$\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M}b_{k_1,\dots,k_M}v_{1,k_1}^{-\frac{1}{p}}\cdots v_{M,k_M}^{-\frac{1}{p}}\Big)^q u_{n_1,\dots,n_M}\Big)^{\frac{1}{q}}\le$$
$$\le C\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}b^p_{n_1,\dots,n_M}\Big)^{\frac{1}{p}}. \tag{3.8}$$

Assume that (3.2) holds and let $V_{m,n_j}=\sum_{k_i=1}^{n_j}v_{m,k_i}^{1-p'}$ for $i,j,m=1,2,\dots,M$, $V_{m,0}=0$, $m=1,2,\dots,M$. Applying Hölder's inequality, Lemma 1(a) with $a_k=a_{k_i}=v_{m,k_i}^{1-p'}$, and $d=d_i=\frac{p-s_i}{p-1}$, $i,m=1,2,\dots,M$ (note that $0<d<1$) and Minkowski's inequality to the left hand side of (3.8) we have that

$$\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M}b_{k_1,\dots,k_M}v_{1,k_1}^{-\frac{1}{p}}\cdots v_{M,k_M}^{-\frac{1}{p}}\Big)^q u_{n_1,\dots,n_M}\Big)^{\frac{1}{q}}=$$

$$=\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M}b_{k_1,\dots,k_M}V_{1,k_1}^{\frac{s_1-1}{p}}\cdots V_{M,k_M}^{\frac{s_M-1}{p}}\times$$

$$\times V_{1,k_1}^{-\frac{s_1-1}{p}}\cdots V_{M,k_M}^{-\frac{s_M-1}{p}}v_{1,k_1}^{-\frac{1}{p}}\cdots v_{M,k_M}^{-\frac{1}{p}}\Big)^q u_{n_1,\dots,n_M}\Big)^{\frac{1}{q}}\le$$

$$\le\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M}b^p_{k_1,\dots,k_M}V_{1,k_1}^{s_1-1}\cdots V_{M,k_M}^{s_M-1}\Big)^{\frac{q}{p}}\times$$

$$\times\Big(\sum_{k_1=1}^{n_1}V_{1,k_1}^{-\frac{(s_1-1)p'}{p}}v_{1,k_1}^{1-p'}\Big)^{\frac{q}{p'}}\cdots\Big(\sum_{k_M=1}^{n_M}V_{M,k_M}^{-\frac{(s_M-1)p'}{p}}v_{M,k_M}^{1-p'}\Big)^{\frac{q}{p'}}u_{n_1,\dots,n_M}\Big)^{\frac{1}{q}}\le$$

$$\le\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\cdots\Big(\frac{p-1}{p-s_M}\Big)^{\frac{1}{p'}}\times$$

$$\times\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M}b^p_{k_1,\dots,k_M}V_{1,k_1}^{s_1-1}\cdots V_{M,k_M}^{s_M-1}\Big)^{\frac{q}{p}}\times$$

$$\times\Big(\sum_{k_1=1}^{n_1}\Big(V_{1,k_1}^{(\frac{p-s_1}{p-1})}-V_{1,k_1-1}^{(\frac{p-s_1}{p-1})}\Big)\Big)^{\frac{q}{p'}}\Big(\sum_{k_2=1}^{n_2}\Big(V_{2,k_2}^{(\frac{p-s_2}{p-1})}-V_{2,k_2-1}^{(\frac{p-s_2}{p-1})}\Big)\Big)^{\frac{q}{p'}}\times\cdots$$

$$\cdots\times\Big(\sum_{k_M=1}^{n_M}\Big(V_{M,k_M}^{(\frac{p-s_M}{p-1})}-V_{M,k_M-1}^{(\frac{p-s_M}{p-1})}\Big)\Big)^{\frac{q}{p'}}u_{n_1,\dots,n_M}\Big)^{\frac{1}{q}}=$$

$$=\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\cdots\Big(\frac{p-1}{p-s_M}\Big)^{\frac{1}{p'}}\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\times$$

$$\times\Big(\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M}b^p_{k_1,\dots,k_M}V_{1,k_1}^{s_1-1}\cdots V_{M,k_M}^{s_M-1}\Big)^{\frac{q}{p}}\times$$

$$\times V_{1,n_1}^{(\frac{p-s_1}{p-1})\frac{q}{p'}}\cdots V_{M,n_M}^{(\frac{p-s_M}{p-1})\frac{q}{p'}}u_{n_1,\dots,n_M}\Big)^{\frac{1}{q}}\le$$

$$\le\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\cdots\Big(\frac{p-1}{p-s_M}\Big)^{\frac{1}{p'}}\Big(\sum_{k_1=1}^{\infty}\cdots\sum_{k_M=1}^{\infty}b^p_{k_1,\dots,k_M}V_{1,k_1}^{s_1-1}\cdots V_{M,k_M}^{s_M-1}\times$$

$$\times\Big(\sum_{n_1=k_1}^{\infty}\cdots\sum_{n_M=k_M}^{\infty}V_{1,n_1}^{\frac{q(p-s_1)}{p}}\cdots V_{M,n_M}^{\frac{q(p-s_M)}{p}}u_{n_1\cdots n_M}\Big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}\le$$

$$\le\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\cdots\Big(\frac{p-1}{p-s_M}\Big)^{\frac{1}{p'}}\times$$

$$\times \sup_{k_1,\ldots,k_M>0} V_{1,k_1}^{\frac{s_1-1}{p}} \cdots V_{M,k_M}^{\frac{s_M-1}{p}} \Big(\sum_{n_1=k_1}^{\infty} \cdots \sum_{n_M=k_M}^{\infty} V_{1,n_1}^{\frac{q(p-s_1)}{p}} \cdots V_{M,n_M}^{\frac{q(p-s_M)}{p}} u_{n_1\cdots n_M} \Big)^{\frac{1}{q}} \times$$

$$\times \Big(\sum_{k_1=1}^{\infty} \cdots \sum_{k_M=1}^{\infty} b_{k_1,\ldots,k_M}^p \Big)^{\frac{1}{p}} =$$

$$= \Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}} \cdots \Big(\frac{p-1}{p-s_M}\Big)^{\frac{1}{p'}} A_3(s_1,\ldots,s_M) \Big(\sum_{n_1=1}^{\infty} \cdots \sum_{n_M=1}^{\infty} b_{n_1\cdots n_M}^p \Big)^{\frac{1}{p}}.$$

Hence, by taking infimum over $s_1,\ldots,s_M \in (1,p)$, (3.8) and, thus, (3.1) holds with a constant C satisfying the right hand side inequality in (3.4).

Now assume (3.1) and, thus, (3.8) holds. Similar to the two-dimensional case we consider the following test sequence:

$$b_{k_1,\ldots,k_M}^p := \\ = \left\{ \prod_{i=1}^{M} \begin{pmatrix} V_{m,N_i}^{-s_i} v_{m,k_i}^{1-p'} & for \quad k_i = 1,2,\ldots N_i \\ V_{m,k_i}^{-s_i} v_{m,k_i}^{1-p'} & for \quad k_i = N_i+1,\ldots \end{pmatrix}, \quad m = 1,\ldots,M \right) \quad . \tag{3.9}$$

We claim that

$$\sum_{k_1=1}^{n_1} \cdots \sum_{k_M=1}^{n_M} b_{k_1,\ldots,k_M} v_{1,k_1}^{-\frac{1}{p}} \cdots v_{M,k_M}^{-\frac{1}{p}} \geq V_{1,n_1}^{\frac{p-s_1}{p}} \cdots V_{M,n_M}^{\frac{p-s_M}{p}}. \tag{3.10}$$

Using the calculations in (2.4) we see that (3.10) holds for $M = 2$.

We assume that (3.10) holds with M replaced by $M-1$, $M \geq 3$. By using this assumption, the relation $-\frac{1}{p} + \frac{1-p'}{p} = 1 - p'$ and the fact that $V_{M,n}$ is nondecreasing, we get

$$\sum_{k_1=1}^{n_1} \cdots \sum_{k_M=1}^{n_M} b_{k_1,\ldots,k_M} v_{1,k_1}^{-\frac{1}{p}} \cdots v_{M,k_M}^{-\frac{1}{p}} \geq$$

$$\geq V_{1,n_1}^{\frac{p-s_1}{p}} \cdots V_{M-1,n_{M-1}}^{\frac{p-s_{M-1}}{p}} \Big(\sum_{k_M=1}^{N_M} V_{M,N_M}^{-\frac{s_M}{p}} v_{M,k_M}^{1-p'} + \sum_{k_M=N_{M+1}}^{n_M} V_{M,k_M}^{-\frac{s_M}{p}} v_{M,k_M}^{1-p'} \Big) \geq$$

$$\geq V_{1,n_1}^{\frac{p-s_1}{p}} \cdots V_{M-1,n_{M-1}}^{\frac{p-s_{M-1}}{p}} \big(V_{M,N_M}^{\frac{p-s_M}{p}} + V_{M,n_M}^{-\frac{s_M}{p}} (V_{M,n_M} - V_{M,N_M}) \big) =$$

$$= V_{1,n_1}^{\frac{p-s_1}{p}} \cdots V_{M-1,n_{M-1}}^{\frac{p-s_{M-1}}{p}} \big(V_{M,n_M}^{\frac{p-s_M}{p}} + V_{M,N_M}^{\frac{p-s_M}{p}} - V_{M,n_M}^{-\frac{s_M}{p}} V_{M,N_M} \big) \geq$$

$$\geq V_{1,n_1}^{\frac{p-s_1}{p}} \cdots V_{M-1,n_{M-1}}^{\frac{p-s_{M-1}}{p}} \big(V_{M,n_M}^{\frac{p-s_M}{p}} + V_{M,N_M}^{\frac{p-s_M}{p}} - V_{M,N_M}^{-\frac{s_M}{p}} V_{M,N_M} \big) =$$

$$= V_{1,n_1}^{\frac{p-s_1}{p}} \cdots V_{M-1,n_{M-1}}^{\frac{p-s_{M-1}}{p}} V_{M,n_M}^{\frac{p-s_M}{p}}.$$

Hence, by the induction axiom (3.10) holds for each M, $M \geq 2$. We conclude that

$$\Big(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty}\Big(\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M} b_{k_1,\ldots,k_M} v_{1,k_1}^{-\frac{1}{p}}\cdots v_{M,k_M}^{-\frac{1}{p}}\Big)^q u_{n_1,\ldots,n_M}\Big)^{\frac{1}{q}} =$$

$$= \Big(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty}\Big(V_{1,n_1}^{\frac{q(p-s_1)}{p}}\cdots V_{M,n_M}^{\frac{q(p-s_n)}{p}}\Big)u_{n_1,\ldots,n_M}\Big)^{\frac{1}{q}}. \tag{3.11}$$

Moreover, applying the test sequence (3.9) to the right hand side of (3.8) and using Lemma 1(b) with $d = d_i = 1 - s_i < 0$, $i = 1, 2, \ldots, M$, we obtain that

$$\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\cdots\sum_{n_M=1}^{\infty} b^p_{n_1,\ldots,n_M}\Big)^{\frac{1}{p}} =$$

$$= \Big(\Big(\sum_{n_1=1}^{N_1} V_{1,N_1}^{-s_1} v_{1,n_1}^{1-p'} + \sum_{n_1=N_1+1}^{\infty} V_{1,n_1}^{-s_1} v_{1,n_1}^{1-p'}\Big) \times \cdots$$

$$\cdots \times \Big(\sum_{n_M=1}^{N_M} V_{M,n_M}^{-s_n} v_{M,n_M}^{1-p'} + \sum_{n_M=N_M+1}^{\infty} V_{M,n_M}^{-s_n} v_{M,n_M}^{1-p'}\Big)\Big)^{\frac{1}{p}} \leq$$

$$\leq \Big(\Big(V_{1,N_1}^{1-s_1} + \frac{1}{s_1-1}V_{1,N_1}^{1-s_1}\Big)\cdots\Big(V_{M,N_M}^{1-s_M} + \frac{1}{s_M-1}V_{M,N_M}^{1-s_M}\Big)\Big)^{\frac{1}{p}} =$$

$$= \Big(\frac{s_1}{s_1-1}\Big)^{\frac{1}{p}}\cdots\Big(\frac{s_M}{s_M-1}\Big)^{\frac{1}{p}} V_{1,N_1}^{\frac{1-s_1}{p}}\cdots V_{M,N_M}^{\frac{1-s_M}{p}}. \tag{3.12}$$

Hence, according (3.11), (3.12) and (3.8),

$$\Big(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty}(V_{1,n_1}^{\frac{q(p-s_1)}{p}}\cdots V_{M,n_M}^{\frac{q(p-s_M)}{p}})u_{n_1,\ldots,n_M}\Big)^{\frac{1}{q}} \leq$$

$$\leq C\Big(\frac{s_1}{s_1-1}\Big)^{\frac{1}{p}}\cdots\Big(\frac{s_M}{s_M-1}\Big)^{\frac{1}{p}} V_{1,N_1}^{\frac{1-s_1}{p}}\cdots V_{M,N_M}^{\frac{1-s_M}{p}}$$

so that

$$\Big(\frac{s_1-1}{s_1}\Big)^{\frac{1}{p}}\cdots\Big(\frac{s_M-1}{s_M}\Big)^{\frac{1}{p}} V_{1,N_1}^{\frac{s_1-1}{p}}\cdots V_{M,N_M}^{\frac{s_M-1}{p}} \times$$

$$\times\Big(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty}(V_{1,n_1}^{\frac{q(p-s_1)}{p}}\cdots V_{M,n_M}^{\frac{q(p-s_M)}{p}})u_{n_1,\ldots,n_M}\Big)^{\frac{1}{q}} \leq C.$$

Thus, by taking supremum over $N_1, \ldots, N_M > 0$ and supremum over $s_1, \ldots, s_M \in (1, p)$ we find that (3.2) and the left hand side of the estimate (3.4) hold. Hence, we have proved that (3.1) is equivalent to (3.2) and that (3.4) holds. The proof is complete. □

Proof of Theorem 3.2 Assume that (3.5) holds. The result is known for $M = 1$ (see [19, Pr oposition 1]) and $M = 2$ (see Theorem 1.2) so we assume that $M = 3, 4, \ldots$. Put $b^p_{n_1,\ldots,n_M} = a^p_{n_1,\ldots,n_M} v_{n_1,\ldots,n_M}$ in (3.4). Then (3.4) is equivalent to

$$\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\prod_{k_1=1}^{n_1}\cdots\prod_{k_M=1}^{n_M} b_{k_1,\ldots,k_M}\Big)^{\frac{q}{n_1\cdots n_M}} w_{n_1,\ldots,n_M}\Big)^{\frac{1}{q}} \le$$
$$\le C\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty} b^p_{n_1,\ldots,n_M}\Big)^{\frac{1}{p}}, \tag{3.13}$$

where $w_{n_1,\ldots,n_M}$ is defined in (3.6).

Now we use Theorem 3.1 with $u_{n_1,\ldots,n_M} = w_{n_1,\ldots,n_M} n_1^{-q}\cdots n_M^{-q}$ and $v_{1,n_1} = v_{2,n_2} = \cdots = v_{M,n_M} = 1$, and obtain

$$\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\frac{1}{n_1\cdots n_M}\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M} a_{k_1,\ldots,k_M}\Big)^{q} w_{n_1,\ldots,n_M}\Big)^{\frac{1}{q}} \le$$
$$\le C\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty} a^p_{n_1,\ldots,n_M}\Big)^{\frac{1}{p}} \tag{3.14}$$

holds for some finite constant C if and only if

$$A(s_1,\ldots,s_M) := \sup_{N_1,\ldots,N_M>0} N_1^{\frac{s_1-1}{p}}\cdots N_M^{\frac{s_M-1}{p}} \times$$
$$\Big(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty}(n_1^{-\frac{qs_1}{p}}\cdots n_M^{-\frac{qs_M}{p}})w_{n_1,\ldots,n_M}\Big)^{\frac{1}{q}} < \infty \tag{3.15}$$

and the best possible constant C in (3.14) can be estimated as follows:

$$\sup_{1<s_1,\ldots,s_M<p}\Big(\frac{s_1-1}{s_1}\Big)^{\frac{1}{p}}\cdots\Big(\frac{s_M-1}{s_M}\Big)^{\frac{1}{p}} A(s_1,\ldots,s_M) \le C \le$$
$$\le \inf_{1<s_1,\ldots,s_M<p} A_3(s_1,\ldots,s_M)\Big(\frac{p-1}{p-s_1}\Big)^{\frac{1}{p'}}\cdots\Big(\frac{p-1}{p-s_M}\Big)^{\frac{1}{p'}}. \tag{3.16}$$

Now we replace $a_{k_1,\ldots,k_M}$ by $b^{\alpha}_{k_1,\ldots,k_M}$ in (3.14) with $0<\alpha<p$ and after that we replace p by $\frac{p}{\alpha}$ and q by $\frac{q}{\alpha}$ in (3.14)–(3.16) and we find that for $1 < s_1,\ldots,s_M < \frac{p}{\alpha}$ we have that

$$\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\Big(\frac{1}{n_1\cdots n_M}\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M} b^{\alpha}_{k_1,\ldots,k_M}\Big)^{\frac{q}{\alpha}} w_{n_1,\ldots,n_M}\Big)^{\frac{1}{q}} \le$$
$$\le C_{\alpha}\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty} b^p_{n_1,\ldots,n_M}\Big)^{\frac{1}{p}} \tag{3.17}$$

holds for $b_{k_1,\ldots,k_M} > 0$ if and only if

$$A_\alpha(s_1,\ldots,s_M) := \sup_{N_1,\ldots,N_M>0} N_1^{\alpha(\frac{s_1-1}{p})},\ldots,N_M^{\alpha(\frac{s_M-1}{p})}\times$$
$$\times\Big(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty}(n_1^{\frac{-qs_1}{p}}\cdots n_M^{\frac{-qs_M}{p}})w_{n_1,\ldots,n_M}\Big)^{\frac{\alpha}{q}} < \infty.$$

Since now $A_\alpha^{\frac{1}{\alpha}} = B$ for all $\alpha > 0$ and $w_{n_1,\ldots,n_M}$ is defined in (3.6), the upper estimate of the best possible constant C_α in (3.17) can be estimated as follows:

$$C_\alpha \le \inf_{1<s_1,\ldots,s_M<p} B\left(s_1,\ldots,s_M\right)\left(\frac{p-\alpha}{p-\alpha s_1}\right)^{\frac{p-\alpha}{p\alpha}}\cdots\left(\frac{p-\alpha}{p-\alpha s_M}\right)^{\frac{p-\alpha}{p\alpha}}.$$

Moreover, we note that

$$\Big(\frac{1}{n_1\cdots n_M}\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M}b_{k_1,\ldots,k_M}^{\alpha}\Big)^{\frac{q}{\alpha}}\downarrow\Big(\prod_{k_1=1}^{n_1}\cdots\prod_{k_M=1}^{n_M}b_{k_1\cdots k_M}\Big)^{\frac{q}{n_1\cdots n_M}}$$

(the scale of power means converges to the geometric mean) and

$$\Big(\frac{p-\alpha}{p-\alpha s_1}\Big)^{\frac{p-\alpha}{p\alpha}}\cdots\Big(\frac{p-\alpha}{p-\alpha s_M}\Big)^{\frac{p-\alpha}{p\alpha}} \longrightarrow e^{\frac{s_1+\cdots+s_M-M}{p}} \quad \text{as } \alpha\longrightarrow 0^+.$$

Thus, we get

$$C \le \inf_{s_1,\ldots,s_M>1} B(s_1,\ldots,s_M)e^{\frac{s_1+\cdots+s_M-M}{p}}. \tag{3.18}$$

Hence, it follows that (3.4) holds with a constant satisfying the upper estimate in (3.7).

Next we assume that (3.4) and, thus, (3.13) holds and apply the following test sequence to (3.13):

$$b_{k_1,\ldots,k_M}^p := \left\{\prod_{i=1}^{M}\begin{pmatrix} N_i^{-1} & for & k_i = 1,\ldots,N_i \\ e^{-s_i}N_i^{s_i-1}k_i^{-s_i} & for & k_i = N_i+1,\ldots\end{pmatrix}\right. . \tag{3.19}$$

We apply the test sequence (3.19) to the right hand side of (3.13) and use estimate (2.11) in the calculation. Then we have, since $b_{k_1,\ldots,k_M}^p$ is of product type, that

$$\Big(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}b_{n_1,\ldots,n_M}^{p}\Big)^{\frac{1}{p}} = \Big(\Big(\sum_{n_1=1}^{N_1}N_1^{-1}+\sum_{n_1=N_1+1}^{\infty}e^{-s_1}N_1^{s_1-1}n_1^{-s_1}\Big)\times\cdots$$
$$\cdots\times\Big(\sum_{n_M=1}^{N_M}N_M^{-1}+\sum_{n_M=N_M+1}^{\infty}e^{-s_M}N_M^{s_M-1}n_M^{-s_M}\Big)\Big)^{\frac{1}{p}} =$$

$$= \left(\left(1+e^{-s_1}N_1^{s_1-1}\sum_{n_1=N_1+1}^{\infty} n_1^{-s_1}\right)\times\cdots\left(1+e^{-s_M}N_M^{s_M-1}\sum_{n_M=N_M+1}^{\infty} n_M^{-s_M}\right)\right)^{\frac{1}{p}} \le$$

$$\le \left(1+\frac{e^{-s_1}}{s_1-1}\right)^{\frac{1}{p}}\cdots\left(1+\frac{e^{-s_M}}{s_M-1}\right)^{\frac{1}{p}} =$$

$$= \left(\frac{1+e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)}\right)^{\frac{1}{p}}\cdots\left(\frac{1+e^{s_M}(s_M-1)}{e^{s_M}(s_M-1)}\right)^{\frac{1}{p}}. \tag{3.20}$$

Moreover, the left hand side of (3.13) can be estimated as follows:

$$\left(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\left(\prod_{k_1=1}^{n_1}\cdots\prod_{k_M=1}^{n_M} b^p_{k_1,\ldots,k_M}\right)^{\frac{q}{pn_1\cdots n_M}} w_{n_1,\ldots,n_M}\right)^{\frac{1}{q}} \ge$$

$$\ge \left(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty}\left(\prod_{k_1=1}^{n_1}\cdots\prod_{k_M=1}^{n_M} b^p_{k_1,\ldots,k_M}\right)^{\frac{q}{pn_1\cdots n_M}} w_{n_1,\ldots,n_M}\right)^{\frac{1}{q}} =$$

$$= \left(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty}\left(\exp\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M}\ln b^p_{k_1,\ldots,k_M}\right)^{\frac{q}{pn_1\cdots n_M}} w_{n_1,\ldots,n_M}\right)^{\frac{1}{q}}. \tag{3.21}$$

By following the argumentation in the proof of Theorem 1.2 we see that it is sufficient to prove that

$$\sum_{k_1=1}^{n_1}\cdots\sum_{k_M=1}^{n_M}\ln b^p_{k_1,\ldots,k_M} \ge$$

$$\ge n_1\cdots n_M \ln\left(n_1^{-s_1}\cdots n_M^{-s_M}N_1^{s_1-1}\cdots N_M^{s_M-1}\right). \tag{3.22}$$

We have already proved that this formula holds for $M = 2$ (see (2.16)). Moreover, by mathematical induction as in the proof of Theorem 1.2, we easily find that it is true in general.

Using (3.21) and (3.22), we find that the left hand side of (3.13) can be estimated as follows:

$$\left(\sum_{n_1=1}^{\infty}\cdots\sum_{n_M=1}^{\infty}\left(\prod_{k_1=1}^{n_1}\cdots\prod_{k_M=1}^{n_M} b^p_{k_1,\ldots,k_M}\right)^{\frac{q}{pn_1\cdots n_M}} w_{n_1,\ldots,n_M}\right)^{\frac{1}{q}} \ge$$

$$\ge N_1^{\frac{s_1-1}{p}}\cdots N_M^{\frac{s_M-1}{p}}\left(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty} n_1^{-\frac{qs_1}{p}}\cdots n_M^{-\frac{qs_M}{p}} w_{n_1,\ldots,n_M}\right)^{\frac{1}{q}}. \tag{3.23}$$

Now, by combining (3.23), (3.20) and (3.4), we obtain that

$$N_1^{\frac{s_1-1}{p}}\cdots N_M^{\frac{s_M-1}{p}}\left(\sum_{n_1=N_1}^{\infty}\cdots\sum_{n_M=N_M}^{\infty} n_1^{-\frac{qs_1}{p}}\cdots n_M^{-\frac{qs_M}{p}} w_{n_1,\ldots,n_M}\right)^{\frac{1}{q}} \le$$

$$\le C\left(\frac{1+e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)}\right)^{\frac{1}{p}}\cdots\left(\frac{1+e^{s_M}(s_M-1)}{e^{s_M}(s_M-1)}\right)^{\frac{1}{p}},$$

that is

$$\left(\frac{e^{s_1}(s_1-1)}{1+e^{s_1}(s_1-1)}\right)^{\frac{1}{p}} \cdots \left(\frac{e^{s_M}(s_M-1)}{1+e^{s_M}(s_M-1)}\right)^{\frac{1}{p}} \times$$
$$\times N_1^{\frac{s_1-1}{p}} \cdots N_M^{\frac{s_M-1}{p}} \left(\sum_{n_1=N_1}^{\infty} \cdots \sum_{n_M=N_M}^{\infty} n_1^{-\frac{qs_1}{p}} \cdots n_M^{-\frac{qs_M}{p}} w_{n_1,\ldots,n_M} \right)^{\frac{1}{q}} \leq C.$$

Hence, by taking supremum over $N_1, \ldots, N_M > 0$ and supremum over $s_1, \ldots, s_M \in (1,p)$ we find that (3.5) and the left hand side of (3.7) hold. The proof is complete. □

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