# GENERALIZED GOURSAT PROBLEM FOR A SPATIAL FOURTH ORDER HYPERBOLIC EQUATION WITH DOMINATED LOW TERMS 

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#### Abstract

In this paper the correctness of the generalized Goursat problem for a spatial fourth order hyperbolic equation with dominated low terms is considered. The effect of influence of low terms on the correctness of the problem is shown.





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In the space $R^{3}$ of independent variables $x_{1}, x_{2}$ and $x_{3}$ let

$$
\begin{gathered}
D:=\left\{x:=\left(x_{1}, x_{2}, x_{3}\right) \in R^{3} ; \quad R:=\right]-\infty ;+\infty\left[\begin{array}{ll}
; & \left.x_{i}>0 ; \quad i=1,2,3\right\}, \\
\Omega:=\left\{(\xi, \eta) \in R^{2}: \quad 0<\xi, \eta\right\}, & S_{i}:=D \cap\left\{x \in R^{3}: \quad x_{i}=0\right\}, \quad i=1,2,3 ; \\
\Gamma_{i}:=S_{j} \cap S_{k}, & i \neq j, k ; \quad i, j, k=1,2,3 .
\end{array}\right.
\end{gathered}
$$

For the class of functions $\varphi$, continuous in $\bar{D}$ with its own partial derivatives $D_{x_{1}}^{i} \varphi, D_{x_{2}}^{j} \varphi, D_{x_{3}}^{k} \varphi, 0 \leq i \leq m, 0 \leq j \leq n, 0 \leq k \leq l$ we use the symbol $C^{m, n, l}(\bar{D}), m, n, l=0,1, \ldots$, while the symbol $C_{0}^{m, n, l}(\bar{D}), m, n$, $l=0,1, \ldots$ denotes the subspace of those functions of the class $C^{m, n, l}(\bar{D})$ which vanish on $\Gamma_{i}, i=1,2,3$, with their partial derivatives. Suppose that the functions $\mathrm{a}^{i, j, k}$ belong to the class $C^{i, j, k}(\bar{D}), i=0,1,2 ; j, k=0,1$; $i+j+k \neq 4$ and $f \in C_{0}(\bar{D})$.

For the equation

$$
\begin{equation*}
\frac{\partial^{4}}{\partial x_{1}^{2} \partial x_{2} \partial x_{3}} \mathrm{u}(\mathrm{x})+\sum_{\substack{i+j+k \leq 3 \\ i=0,1,2 ; j, k=0,1}} \mathrm{a}^{i, j, k}(\mathrm{x}) \frac{\partial^{i+j+k}}{\partial x_{1}^{i} \partial x_{2}^{j} \partial x_{3}^{k}} \mathrm{u}(\mathrm{x})=\mathrm{f}(\mathrm{x}) \tag{1}
\end{equation*}
$$

we consider the following generalized Goursat problem: find in the domain $D$ a regular solution of equation (1) of the class $C_{0}^{2,1,1}(\bar{D})$ which satisfies

[^0]the following boundary conditions:
\[

$$
\begin{align*}
& \left.\left(\sum_{\substack{i+j+k \leq 3 \\
i=0,1,2 ; j, k=0,1}} b_{m}^{i, j, k} \frac{\partial^{i+j+k}}{\partial x_{1}^{i} \partial x_{2}^{j} \partial x_{3}^{k}} u\right)\right|_{x_{m}=0}=f_{m}, \quad m=2,3  \tag{2}\\
& \left.\left(\sum_{\substack{i+j+k \leq 3 \\
i=0,1,2 ; j, k=0,1}} b_{1 n}^{i, j, k} \frac{\partial^{i+j+k}}{\partial x_{1}^{i} \partial x_{2}^{j} \partial x_{3}^{k}} u\right)\right|_{x_{1}=0}=f_{1 n}(x), \quad n=1,2, \tag{3}
\end{align*}
$$
\]

where $b_{m}^{i, j, k} \in C^{i, j, k}\left(S_{m}\right), b_{1 n}^{i, j, k} \in C^{i, j, k}\left(S_{1}\right), f_{m} \in C_{0}\left(S_{m}\right), f_{1 n} \in C_{0}\left(S_{1}\right)$, $m=2,3 ; n=1,2$ are the given functions.

The problem (1)-(3) represents the variant of the two- and three- dimensional generalized Goursat problems for linear hyperbolic equations considered in [1-3]. Some spatial boundary problems for the third order hyperbolic equations are considered in $[4,5]$.

## Theorem 1. If conditions

$$
\operatorname{det}\left(\begin{array}{ll}
b_{11}^{0,1,1} & b_{11}^{1,1,1} \\
b_{12}^{0,1,1} & b_{12}^{1,1,1}
\end{array}\right) \neq 0, \quad b_{2}^{2,0,1} \neq 0, \quad b_{3}^{2,1,0} \neq 0
$$

hold, then the problem (1)-(3) is uniquely solvable in the class $C_{0}^{2,1,1}(\bar{D})$.
Proof. Let us show that in the above conditions it is possible to define uniquely the Goursat traces on characteristic planes $x_{i}=0, i=1,2,3$. Towards this end, we introduce the functions

$$
\alpha^{1}:=\left.u\right|_{S_{1}}, \quad \alpha^{2}:=\left.u_{x_{1}}\right|_{S_{1}}, \quad \alpha^{3}:=\left.u_{x_{1} x_{1}}\right|_{S_{1}}
$$

Then with respect to the unknown function

$$
\chi:=\left(\begin{array}{l}
\alpha^{1} \\
\alpha^{2} \\
\alpha^{3}
\end{array}\right)
$$

from equation (1) and conditions (3) we have

$$
\begin{equation*}
A \chi_{x_{2} x_{3}}+B \chi_{x_{2}}+C \chi_{x_{3}}+D \chi=G \tag{4}
\end{equation*}
$$

where

$$
\begin{gathered}
A:=\left(\begin{array}{ccc}
\left.a^{0,1,1}\right|_{S_{1}} & \left.a^{1,1,1}\right|_{S_{1}} & 1 \\
b_{11}^{0,1,1} & b_{11}^{1,1,1} & 0 \\
b_{12}^{0,1,1} & b_{12}^{1,1,1} & 0
\end{array}\right), \quad B:=\left(\begin{array}{ccc}
\left.a^{0,1,0}\right|_{S_{1}} & \left.a^{1,1,0}\right|_{S_{1}} & \left.a^{2,1,0}\right|_{S_{1}} \\
b_{11}^{0,1,0} & b_{11}^{1,1,0} & b_{11}^{2,1,0} \\
b_{12}^{0,1,0} & b_{12}^{1,1,0} & b_{12}^{2,1,0}
\end{array}\right) \\
C:=\left(\begin{array}{ccc}
\left.a^{0,0,1}\right|_{S_{1}} & \left.a^{1,0,1}\right|_{S_{1}} & \left.a^{2,0,1}\right|_{S_{1}} \\
b_{11}^{0,0,1} & b_{11}^{1,0,1} & b_{11}^{2,0,1} \\
b_{12}^{0,0,1} & b_{12}^{1,0,1} & b_{12}^{2,0,1}
\end{array}\right)
\end{gathered}
$$

$$
\begin{gathered}
D:=\left(\begin{array}{ccc}
\left.a^{0,0,0}\right|_{S_{1}} & \left.a^{1,0,0}\right|_{S_{1}} & \left.a^{2,0,0}\right|_{S_{1}} \\
b_{11}^{0,0,0} & b_{11}^{1,0,0} & b_{11}^{2,0,0} \\
b_{12}^{0,0,0} & b_{12}^{1,0,0} & b_{12}^{2,0,0}
\end{array}\right) \\
G:=\left(\begin{array}{c}
\left.f\right|_{S_{1}} \\
f_{11} \\
f_{12}
\end{array}\right)
\end{gathered}
$$

It is easy to see that the function $\chi$ satisfies the homogenous Goursat conditions

$$
\begin{equation*}
\left.\chi\right|_{\Gamma_{2} \cup \Gamma_{3}}=0 . \tag{5}
\end{equation*}
$$

According to the conditions of the theorem, $\operatorname{det} A \neq 0$, and therefore the problem (4), (5) is uniquely solvable, and thus we can define uniquely the Goursat traces $\left.u\right|_{x_{1}=0},\left.u_{x_{1}}\right|_{x_{1}=0}$.

Further, we introduce the functions

$$
\beta^{1}:=\left.u\right|_{S_{3}}, \quad \beta^{2}:=\left.u_{x_{3}}\right|_{S_{3}} .
$$

Now, with respect to the unknown function $\varphi:=\binom{\beta^{1}}{\beta^{2}}$, from equation (1) and from the second condition of (2) we obtain

$$
\begin{equation*}
A^{\prime} \varphi_{x_{1} x_{1} x_{2}}+B^{\prime} \varphi_{x_{1} x_{1}}+C^{\prime} \varphi_{x_{1} x_{2}}+D^{\prime} \varphi_{x_{1}}+E^{\prime} \varphi_{x_{2}}+F^{\prime} \varphi=G^{\prime} \tag{6}
\end{equation*}
$$

where

$$
\begin{gathered}
A^{\prime}:=\left(\begin{array}{cc}
\left.a^{2,1,0}\right|_{S_{3}} & 1 \\
b_{3}^{2,1,0} & 0
\end{array}\right), \quad B^{\prime}:=\left(\begin{array}{cc}
\left.a^{2,0,0}\right|_{S_{3}} & \left.a^{2,0,1}\right|_{S_{3}} \\
b_{3}^{2,0,0} & b_{3}^{2,0,1}
\end{array}\right), \\
C^{\prime}:=\left(\begin{array}{cc}
\left.a^{1,1,0}\right|_{S_{3}} & \left.a^{1,1,1}\right|_{S_{3}} \\
b_{3}^{1,1,0} & b_{3}^{1,1,1}
\end{array}\right), \quad D^{\prime}:=\left(\begin{array}{cc}
\left.a^{1,0,0}\right|_{S_{3}} & \left.a^{1,0,1}\right|_{S_{3}} \\
b_{3}^{1,0,0} & b_{3}^{1,0,1}
\end{array}\right), \\
E^{\prime}:=\left(\begin{array}{cc}
\left.a^{0,1,0}\right|_{S_{3}} & \left.a^{0,1,1}\right|_{S_{3}} \\
b_{3}^{0,1,0} & b_{3}^{0,1,1}
\end{array}\right), \quad F^{\prime}:=\left(\begin{array}{cc}
\left.a^{0,0,0}\right|_{S_{3}} & \left.a^{0,0,1}\right|_{S_{3}} \\
b_{3}^{0,0,0} & b_{3}^{0,0,1}
\end{array}\right), \\
G^{\prime}:=\binom{\left.f\right|_{S_{3}}}{f_{3}}
\end{gathered}
$$

It is easy to verify that the following Goursat conditions

$$
\begin{equation*}
\left.\varphi\right|_{x_{1}=0}=0,\left.\quad \varphi\right|_{x_{2}=0}=0,\left.\quad \varphi_{x_{1}}\right|_{x_{1}=0}=0 \tag{7}
\end{equation*}
$$

hold. According to the conditions of the theorem, $\operatorname{det} A^{\prime} \neq 0$, and therefore the problem (6), (7) is uniquely solvable and hence the Goursat trace $\left.u\right|_{x_{3}=0}$ is uniquely defined [3].

Further, let us introduce the functions

$$
\gamma^{1}:=\left.u\right|_{S_{2}}, \quad \gamma^{2}:=\left.u_{x_{2}}\right|_{S_{2}}
$$

With respect to unknown function $\psi:=\binom{\gamma^{1}}{\gamma^{2}}$, from equation (1) and from the first condition of (2) we get

$$
\begin{equation*}
A^{\prime \prime} \psi_{x_{1} x_{1} x_{3}}+B^{\prime \prime} \psi_{x_{1} x_{1}}+C^{\prime \prime} \psi_{x_{1} x_{3}}+D^{\prime \prime} \psi_{x_{1}}+E^{\prime \prime} \psi_{x_{3}}+F^{\prime \prime} \psi=G^{\prime \prime} \tag{8}
\end{equation*}
$$

where

$$
\begin{gathered}
A^{\prime \prime}:=\left(\begin{array}{cc}
\left.a^{2,1,0}\right|_{S_{2}} & 1 \\
b_{2}^{2,1,0} & 0
\end{array}\right), \quad B^{\prime \prime}:=\left(\begin{array}{cc}
\left.a^{2,0,0}\right|_{S_{2}} & \left.a^{2,0,1}\right|_{S_{2}} \\
b_{2}^{2,0,0} & b_{2}^{2,0,1}
\end{array}\right) \\
C^{\prime \prime}:=\left(\begin{array}{cc}
\left.a^{1,1,0}\right|_{S_{2}} & \left.a^{1,1,1}\right|_{S_{2}} \\
b_{2}^{1,1,0} & b_{2}^{1,1,1}
\end{array}\right), \quad D^{\prime \prime}:=\left(\begin{array}{cc}
\left.a^{1,0,0}\right|_{S_{2}} & \left.a^{1,0,1}\right|_{S_{2}} \\
b_{2}^{1,0,0} & b_{2}^{1,0,1}
\end{array}\right), \\
E^{\prime \prime}:=\left(\begin{array}{cc}
\left.a^{0,1,0}\right|_{S_{2}} & \left.a^{0,1,1}\right|_{S_{2}} \\
b_{2}^{0,1,0} & b_{2}^{0,1,1}
\end{array}\right), \quad F^{\prime \prime}:=\left(\begin{array}{cc}
\left.a^{0,0,0}\right|_{S_{2}} & \left.a^{0,0,1}\right|_{S_{2}} \\
b_{2}^{0,0,0} & b_{2}^{0,0,1}
\end{array}\right), \\
G^{\prime \prime}:=\binom{\left.f\right|_{S_{2}}}{f_{2}} .
\end{gathered}
$$

It is easy to verify that there hold the following Goursat conditions

$$
\begin{equation*}
\left.\psi\right|_{x_{1}=0}=0,\left.\quad \psi\right|_{x_{3}=0}=0,\left.\quad \psi_{x_{1}}\right|_{x_{1}=0}=0 \tag{9}
\end{equation*}
$$

hold. Since $\operatorname{det} A^{\prime \prime} \neq 0$, the problem (8), (9) is uniquely solvable and therefore the Goursat trace $\left.u\right|_{x_{2}=0}$ is uniquely defined.

As is known, in this case the Goursat problem [7]

$$
\left\{\begin{array}{l}
\frac{\partial^{4}}{\partial x_{1}^{2} \partial x_{2} \partial x_{3}} u(x)+\sum_{\substack{i+j+k \leq 3 \\
i=0,1,2 ; j, k=0,1,}} a^{i, j, k}(x) \frac{\partial^{i+j+k}}{\partial x_{1}^{i} \partial x_{2}^{j} \partial x_{3}^{k}} u(x)=f(x) \\
\left.u\right|_{x_{1}=0}=\sigma^{1},\left.\quad u\right|_{x_{2}=0}=\sigma^{2},\left.\quad u\right|_{x_{3}=0}=\sigma^{3},\left.\quad u_{x_{1}}\right|_{x_{1}=0}=\sigma_{1}^{1}
\end{array}\right.
$$

is uniquely solvable.
Now let us show that the above-constructed solution of the Goursat problem is of class $C_{0}^{2,1,1}$. To this end, we consider, for example, the edge $\Gamma_{1}$ and show that

$$
\begin{equation*}
\left.\frac{\partial^{i+j+k}}{\partial x_{1}^{i} \partial x_{2}^{j} \partial x_{3}^{k}} u\right|_{\Gamma_{1}}=0, \quad i+j+k \leq 4, \quad i=0,1,2 ; \quad j, k=0,1 . \tag{10}
\end{equation*}
$$

As it can be seen from the process of construction of the solution, there take place the following conditions:

$$
\left.u\right|_{\Gamma_{1}}=0,\left.\quad u_{x_{1}}\right|_{\Gamma_{1}}=0,\left.\quad u_{x_{2}}\right|_{\Gamma_{1}}=0,\left.\quad u_{x_{3}}\right|_{\Gamma_{1}}=0
$$

It is easy to see that these conditions ensure the fulfillment of (10), except

$$
\left.u_{x_{1} x_{2} x_{3}}\right|_{\Gamma_{1}}=0,\left.\quad u_{x_{2} x_{3}}\right|_{\Gamma_{1}}=0 .
$$

Let us show the last conditions. On the edge $\Gamma_{1}$ equation (8) reads as

$$
\left.\left(A^{\prime \prime} \psi_{x_{1} x_{1} x_{3}}+C^{\prime \prime} \psi_{x_{1} x_{3}}+E^{\prime \prime} \psi_{x_{3}}\right)\right|_{\Gamma_{1}}=0
$$

Introducing a new function $\Psi:=\left.\psi_{x_{3}}\right|_{\Gamma_{1}}$, we obtain the ordinary second order differential equation

$$
\begin{equation*}
A^{\prime \prime} \Psi_{x_{1} x_{1}}+C^{\prime \prime} \Psi_{x_{1}}+E^{\prime \prime} \Psi=0 \tag{11}
\end{equation*}
$$

From conditions (3), when $x_{1}=x_{2}=x_{3}=0$, we have

$$
\left\{\begin{array}{l}
\left.b_{11}^{1,1,1} u_{x_{1} x_{2} x_{3}}\right|_{x=0}+\left.b_{11}^{0,1,1} u_{x_{2} x_{3}}\right|_{x=0}=0 \\
\left.b_{12}^{1,1,1} u_{x_{1} x_{2} x_{3}}\right|_{x=0}+\left.b_{12}^{0,1,1} u_{x_{2} x_{3}}\right|_{x=0}=0
\end{array}\right.
$$

Since the determinant of the system is nonzero, we have the unique solution

$$
\left.u_{x_{1} x_{2} x_{3}}\right|_{x=0}=\left.u_{x_{2} x_{3}}\right|_{x=0}=0
$$

and therefore for equation (11) we have the Cauchy problem

$$
\Psi(0)=\Psi_{x_{1}}(0)=0
$$

Thus $\left.\Psi\right|_{\Gamma_{1}}=0$, and hence $\left.\psi_{x_{3}}\right|_{\Gamma_{1}}=0$. Finally, $\left.u_{x_{2} x_{3}}\right|_{\Gamma_{1}}=0$, $\left.u_{x_{1} x_{2} x_{3}}\right|_{\Gamma_{1}}=0$.

By analogy we can verify that conditions (10) are valid for the rest edges $\Gamma_{2}$ and $\Gamma_{3}$.

Thus the theorem is proven.
Let us show by a simple equation

$$
\begin{equation*}
u_{x_{1} x_{1} x_{2} x_{3}}=F, \quad F \in C_{0}(\bar{D}) \tag{12}
\end{equation*}
$$

that the low terms in the boundary conditions (2), (3) affect the correctness of the problem (12), (2), (3).

It is easy to see that the regular solution of equation (12) of class $C_{0}^{2,1,1}(\bar{D})$ can be represented in the following form [7]:

$$
\begin{gather*}
u\left(x_{1}, x_{2}, x_{3}\right)=\int_{0}^{x_{1}} \int_{0}^{x_{2}}\left(x_{1}-\xi_{1}\right) \varphi\left(\xi_{1}, \xi_{2}\right) d \xi_{1} d \xi_{2}+ \\
+\int_{0}^{x_{1}} \int_{0}^{x_{3}}\left(x_{1}-\xi_{1}\right) \psi\left(\xi_{1}, \xi_{3}\right) d \xi_{1} d \xi_{3}+\int_{0}^{x_{2}} \int_{0}^{x_{3}} \chi\left(\xi_{2}, \xi_{3}\right) d \xi_{2} d \xi_{3}+ \\
+x_{1} \int_{0}^{x_{2}} \int_{0}^{x_{3}} \widetilde{\chi}\left(\xi_{2}, \xi_{3}\right) d \xi_{2} d \xi_{3}+\int_{0}^{x_{1}} \int_{0}^{x_{2}} \int_{0}^{x_{3}}\left(x_{1}-\xi_{1}\right) F\left(\xi_{1}, \xi_{2}, \xi_{3}\right) d \xi_{1} d \xi_{2} d \xi_{3} \tag{13}
\end{gather*}
$$

where

$$
\begin{array}{r}
\varphi\left(x_{1}, x_{2}\right):=u_{x_{1} x_{2}}\left(x_{1}, x_{2}, 0\right), \quad \psi\left(x_{1}, x_{3}\right):=u_{x_{1} x_{3}}\left(x_{1}, 0, x_{3}\right), \\
\chi\left(x_{2}, x_{3}\right):=u_{x_{2} x_{3}}\left(0, x_{2}, x_{3}\right), \quad \widetilde{\chi}\left(x_{2}, x_{3}\right):=u_{x_{1} x_{2} x_{3}}\left(0, x_{2}, x_{3}\right),
\end{array}
$$

and formula (13) establishes one-to-one correspondence between the solutions of equation (12) of the class $C_{0}^{2,1,1}(\bar{D})$ and the functions $\varphi, \psi, \chi, \widetilde{\chi}$ belonging to the class $C_{0}(\bar{\Omega})$.

Remark 1. Here we take into account the fact that $R(x ; \xi)=x_{1}-\xi_{1}$ is the Riemann function for equation (12).

Substituting (13) in the boundary conditions (2), (3), we obtain the system of the third kind Volterra integral equations

$$
\begin{gather*}
b_{11}^{0,1,1}\left(x_{2}, x_{3}\right) \chi\left(x_{2}, x_{3}\right)+b_{11}^{1,1,1}\left(x_{2}, x_{3}\right) \widetilde{\chi}\left(x_{2}, x_{3}\right)+ \\
+b_{11}^{0,0,1}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \chi\left(\xi_{2}, x_{3}\right) d \xi_{2}+b_{11}^{1,0,1}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \widetilde{\chi}\left(\xi_{2}, x_{3}\right) d \xi_{2}+ \\
+b_{11}^{0,1,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{3}} \chi\left(x_{2}, \xi_{3}\right) d \xi_{3}+b_{11}^{1,1,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{3}} \widetilde{\chi}\left(x_{2}, \xi_{3}\right) d \xi_{3}+ \\
+b_{11}^{0,0,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \int_{0}^{x_{3}} \chi\left(\xi_{2}, \xi_{3}\right) d \xi_{2} d \xi_{3}+ \\
+b_{11}^{1,0,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \int_{0}^{x_{3}} \widetilde{\chi}\left(\xi_{2}, \xi_{3}\right) d \xi_{2} d \xi_{3}=\widetilde{f}_{11}\left(x_{2}, x_{3}\right), \\
b_{12}^{0,1,1}\left(x_{2}, x_{3}\right) \chi\left(x_{2}, x_{3}\right)+b_{12}^{1,1,1}\left(x_{2}, x_{3}\right) \widetilde{\chi}\left(x_{2}, x_{3}\right)+ \\
+b_{12}^{0,0,1}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \chi\left(\xi_{2}, x_{3}\right) d \xi_{2}+b_{12}^{1,0,1}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \widetilde{\chi}\left(\xi_{2}, x_{3}\right) d \xi_{2}+ \\
+b_{12}^{0,1,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{3}} \chi\left(x_{2}, \xi_{3}\right) d \xi_{3}+b_{12}^{1,1,0}\left(x_{2}, x_{3}\right) \int_{0} \widetilde{\chi}\left(x_{2}, \xi_{3}\right) d \xi_{3}+ \\
+b_{12}^{0,0,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \int_{0}^{x_{3}} \chi\left(\xi_{2}, \xi_{3}\right) d \xi_{2} d \xi_{3}+ \\
+b_{12}^{1,0,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \int_{0}^{x_{3}} \widetilde{\chi}\left(\xi_{2}, \xi_{3}\right) d \xi_{2} d \xi_{3}=\tilde{f}_{12}\left(x_{2}, x_{3}\right), \tag{14}
\end{gather*}
$$

$$
\begin{gathered}
b_{2}^{2,0,1}\left(x_{1}, x_{3}\right) \psi\left(x_{1}, x_{3}\right)+b_{2}^{1,0,1}\left(x_{1}, x_{3}\right) \int_{0}^{x_{1}} \psi\left(\xi_{1}, x_{3}\right) d \xi_{1}+ \\
+b_{2}^{0,0,1}\left(x_{1}, x_{3}\right) \int_{0}^{x_{1}}\left(x_{1}-\xi_{1}\right) \psi\left(\xi_{1}, x_{3}\right) d \xi_{1}+b_{2}^{2,0,0}\left(x_{1}, x_{3}\right) \int_{0}^{x_{3}} \psi\left(x_{1}, \xi_{3}\right) d \xi_{3}+ \\
+b_{2}^{1,0,0}\left(x_{1}, x_{3}\right) \int_{0}^{x_{1}} \int_{0}^{x_{3}} \psi\left(\xi_{1}, \xi_{3}\right) d \xi_{1} d \xi_{3}+ \\
+b_{2}^{0,0,0}\left(x_{1}, x_{3}\right) \int_{0}^{x_{1}} \int_{0}^{x_{3}}\left(x_{1}-\xi_{1}\right) \psi\left(\xi_{1}, \xi_{3}\right) d \xi_{1} d \xi_{3}=\tilde{f}_{2}\left(x_{1}, x_{3}\right) \\
+b_{3}^{0,1,0}\left(x_{1}, x_{2}\right) \varphi\left(x_{1}, x_{2}\right)+b_{3}^{1,1,0}\left(x_{1}, x_{2}\right) \int_{0}^{x_{1}} \varphi\left(\xi_{1}, x_{2}\right) d \xi_{1}+ \\
x_{1} \\
\int_{0}^{\left(x_{1}, x_{2}\right)}\left(x_{1}-\xi_{1}\right) \varphi\left(\xi_{1}, x_{2}\right) d \xi_{1}+b_{3}^{2,0,0}\left(x_{1}, x_{2}\right) \int_{0}^{x_{2}} \varphi\left(x_{1}, \xi_{2}\right) d \xi_{2}+ \\
+b_{3}^{1,0,0}\left(x_{1}, x_{2}\right) \int_{0}^{x_{1}} \int_{0}^{x_{2}} \varphi\left(\xi_{1}, \xi_{2}\right) d \xi_{1} d \xi_{2}+ \\
+b_{3}^{0,0,0}\left(x_{1}, x_{2}\right) \int_{0}^{x_{1}} \int_{0}^{x_{2}}\left(x_{1}-\xi_{1}\right) \varphi\left(\xi_{1}, \xi_{2}\right) d \xi_{1} d \xi_{2}=\widetilde{f}_{3}\left(x_{1}, x_{2}\right)
\end{gathered}
$$

where

$$
\begin{gathered}
\tilde{f}_{11}\left(x_{2}, x_{3}\right)=f_{11}\left(x_{2}, x_{3}\right)-b_{11}^{2,0,1}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} F\left(0, \xi_{2}, x_{3}\right) d \xi_{2}- \\
b_{11}^{2,1,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{3}} F\left(0, x_{2}, \xi_{3}\right) d \xi_{3}-b_{11}^{2,0,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \int_{0}^{x_{3}} F\left(0, \xi_{2}, \xi_{3}\right) d \xi_{2} d \xi_{3} \\
\tilde{f}_{12}\left(x_{2}, x_{3}\right)=f_{12}\left(x_{2}, x_{3}\right)-b_{12}^{2,0,1}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} F\left(0, \xi_{2}, x_{3}\right) d \xi_{2}- \\
-b_{12}^{2,1,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{3}} F\left(0, x_{2}, \xi_{3}\right) d \xi_{3}-b_{12}^{2,0,0}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \int_{0}^{x_{3}} F\left(0, \xi_{2}, \xi_{3}\right) d \xi_{2} d \xi_{3}
\end{gathered}
$$

$$
\begin{gathered}
\tilde{f}_{2}\left(x_{1}, x_{3}\right)=f_{2}\left(x_{1}, x_{3}\right)-b_{2}^{1,1,1}\left(x_{1}, x_{3}\right) \int_{0}^{x_{1}} F\left(\xi_{1}, 0, x_{3}\right) d \xi_{1}- \\
-b_{2}^{2,1,0}\left(x_{1}, x_{3}\right) \int_{0}^{x_{3}} F\left(x_{1}, 0, \xi_{3}\right) d \xi_{3}-b_{2}^{1,1,0}\left(x_{1}, x_{3}\right) \int_{0}^{x_{1}} \int_{0}^{x_{3}} F\left(\xi_{1}, 0, \xi_{3}\right) d \xi_{1} d \xi_{3}- \\
-b_{2}^{0,1,1}\left(x_{1}, x_{3}\right) \int_{0}^{x_{1}}\left(x_{1}-\xi_{1}\right) F\left(\xi_{1}, 0, x_{3}\right) d \xi_{1}- \\
-b_{2}^{0,1,0}\left(x_{1}, x_{3}\right) \int_{0}^{x_{1}} \int_{0}^{x_{3}}\left(x_{1}-\xi_{1}\right) F\left(\xi_{1}, 0, \xi_{3}\right) d \xi_{1} d \xi_{3} \\
\tilde{f}_{3}\left(x_{1}, x_{2}\right)=f_{3}\left(x_{1}, x_{2}\right)-b_{3}^{1,1,1}\left(x_{1}, x_{2}\right) \int_{0}^{x_{1}} F\left(\xi_{1}, x_{2}, 0\right) d \xi_{1}- \\
-b_{3}^{2,0,1}\left(x_{1}, x_{2}\right) \int_{0}^{x_{2}} F\left(x_{1}, \xi_{2}, 0\right) d \xi_{2}-b_{3}^{1,0,1}\left(x_{1}, x_{2}\right) \int_{0}^{x_{2}} \int_{0} F\left(\xi_{1}, \xi_{2}, 0\right) d \xi_{1} d \xi_{2}- \\
-b_{3}^{0,1,1}\left(x_{1}, x_{2}\right) \int_{0}^{x_{1}}\left(x_{1}-\xi_{1}\right) F\left(\xi_{1}, x_{2}, 0\right) d \xi_{1}- \\
-b_{3}^{0,0,1}\left(x_{1}, x_{2}\right) \int_{0}^{x_{1}} \int_{0}^{x_{2}}\left(x_{1}-\xi_{1}\right) F\left(\xi_{1}, \xi_{2}, 0\right) d \xi_{1} d \xi_{2}
\end{gathered}
$$

Introducing the notation

$$
\begin{gathered}
A:=\left(\begin{array}{ll}
b_{11}^{0,1,1} & b_{11}^{1,1,1} \\
b_{12}^{0,1,1} & b_{12}^{1,1,1}
\end{array}\right), A_{2}:=\left(\begin{array}{ll}
b_{11}^{0,1,1} & b_{11}^{1,1,1} \\
b_{12}^{0,1,1} & b_{12}^{1,1,1}
\end{array}\right), A_{3}:=\left(\begin{array}{ll}
b_{11}^{0,1,0} & b_{11}^{1,1,0} \\
b_{12}^{0,1,0} & b_{12}^{1,1,0}
\end{array}\right), \\
A_{23}:=\left(\begin{array}{ll}
b_{11}^{0,0,0} & b_{11}^{1,0,0} \\
b_{12}^{0,0,0} & b_{12}^{1,0,0}
\end{array}\right), \mathrm{X}:=\binom{\chi}{\widetilde{\chi}}, F:=\binom{\widetilde{f}_{11}}{\widetilde{f}_{12}}
\end{gathered}
$$

the first two integral equations can be rewritten in the form

$$
\begin{gathered}
A\left(x_{2}, x_{3}\right) X\left(x_{2}, x_{3}\right)+A_{2}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} X\left(\xi_{2}, x_{3}\right) d \xi_{2}+ \\
+A_{3}\left(x_{2}, x_{3}\right) \int_{0}^{x_{3}} X\left(x_{2}, \xi_{3}\right) d \xi_{3}+
\end{gathered}
$$

$$
+A_{23}\left(x_{2}, x_{3}\right) \int_{0}^{x_{2}} \int_{0}^{x_{3}} X\left(\xi_{2}, \xi_{3}\right) d \xi_{2} d \xi_{3}=F\left(x_{2}, x_{3}\right), \quad\left(x_{2}, x_{3}\right) \in \bar{\Omega}
$$

As is known, if $\operatorname{det} A\left(x_{2}, x_{3}\right) \neq 0,\left(x_{2}, x_{3}\right) \in \bar{\Omega}$, then the last Volterra integral equation is uniquely solvable. As for the third and the fourth Volterra integral equations, in (14) if $b_{2}^{2,0,1} \neq 0$ and $b_{3}^{2,1,0} \neq 0$ everywhere, then they are uniquely solvable.

Further, let us show that if the condition $b_{3}^{2,1,0} \neq 0$ in Theorem 1 is violated, the problem (1)-(3) under appropriate conditions can nevertheless be still correct. Thus we suppose that $b_{3}^{2,1,0}=0$ everywhere, and the rest coefficients $b_{3}^{i, j, k}, i=0,1,2 ; j, k=0,1 ; i+j+k \neq 4$ in the left-hand side of the fourth equation of (14) are constant. The fourth integral equation will be read as

$$
\begin{align*}
& b_{3}^{1,1,0} \int_{0}^{x_{1}} \varphi\left(\xi_{1}, x_{2}\right) d \xi_{1}+b_{3}^{0,1,0} \int_{0}^{x_{1}}\left(x_{1}-\xi_{1}\right) \varphi\left(\xi_{1}, x_{2}\right) d \xi_{1}+ \\
& +b_{3}^{2,0,0} \int_{0}^{x_{2}} \varphi\left(x_{1}, \xi_{2}\right) d \xi_{2}+b_{3}^{1,0,0} \int_{0}^{x_{1}} \int_{0}^{x_{2}} \varphi\left(\xi_{1}, \xi_{2}\right) d \xi_{1} d \xi_{2}+ \\
& +b_{3}^{0,0,0}\left(x_{1}, x_{2}\right) \int_{0}^{x_{1}} \int_{0}^{x_{2}}\left(x_{1}-\xi_{1}\right) \varphi\left(\xi_{1}, \xi_{2}\right) d \xi_{1} d \xi_{2}=\tilde{f}_{3}\left(x_{1}, x_{2}\right) . \tag{15}
\end{align*}
$$

Introducing the function

$$
\Phi\left(x_{1}, x_{2}\right):=\int_{0}^{x_{1}} \int_{0}^{x_{2}}\left(x_{1}-\xi_{1}\right) \varphi\left(\xi_{1}, \xi_{2}\right) d \xi_{1} d \xi_{2}, \quad\left(x_{1}, x_{2}\right) \in \bar{\Omega}
$$

equation (15) can be rewritten as follows:

$$
\begin{equation*}
b_{3}^{2,0,0} \Phi_{x_{1} x_{1}}+b_{3}^{1,1,0} \Phi_{x_{1} x_{2}}+b_{3}^{1,0,0} \Phi_{x_{1}}+b_{3}^{0,1,0} \Phi_{x_{2}}+b_{3}^{0,0,0} \Phi=\widetilde{f}_{3} . \tag{16}
\end{equation*}
$$

Noticing that

$$
\begin{equation*}
\left.\Phi\right|_{x_{1}=0}=\left.\Phi\right|_{x_{2}=0}=0 \tag{17}
\end{equation*}
$$

we can conclude that the solvability of equation (15) is reduced to that of equation (16) with the boundary conditions (17). Note that the derivative $\Phi_{x_{1} x_{1} x_{2}}$ should be continuous, and this is provided by imposing additional conditions of smoothness with respect to the variable $x_{1}$ on the functions $f_{3}$ and $F$. Besides, due to the definition of these functions we have $\left.\Phi_{x_{1}}\right|_{x_{1}=0}=$ 0.

It is easy to see that if $b_{3}^{2,0,0}=0$ and $b_{3}^{1,1,0} \neq 0$, then the problem (16), (17) represents the Goursat problem, and therefore it is uniquely solvable.

Suppose now that $b_{3}^{2,0,0} \cdot b_{3}^{1,1,0}<0$, and not restricting the generality of reasoning we assume that $b_{3}^{2,0,0}<0, b_{3}^{1,1,0}>0$. With respect to new variables $\xi=x_{2}-\frac{b_{3}^{1,1,0}}{b_{3}^{2,0,0}} x_{1}, \eta=x_{2}$ equation (16) is read as

$$
\begin{equation*}
\Phi_{\xi \eta}^{*}+\alpha \Phi_{\xi}^{*}+\beta \Phi_{\eta}^{*}+\gamma \Phi^{*}=\widetilde{f}^{*} \tag{18}
\end{equation*}
$$

where $\alpha, \beta, \gamma$ are certain constants and $\widetilde{f}^{*}$ is the known function. It is easy to verify that the conditions

$$
\begin{equation*}
\left.\Phi^{*}\right|_{\eta=0}=\left.\Phi^{*}\right|_{\xi=\eta}=0 \tag{19}
\end{equation*}
$$

are valid. For equation (18) we consider the Goursat problem with the following conditions:

$$
\begin{equation*}
\left.\Phi^{*}\right|_{\eta=0}=0,\left.\quad \Phi^{*}\right|_{\xi=0}=\mu, \quad \mu(0)=0 \tag{20}
\end{equation*}
$$

where the function $\mu$ is such that the solution of the Goursat problem (18), (20) satisfies conditions (19). As is known [6], the solution of the Goursat problem (18), (20) reads as

$$
\begin{aligned}
\Phi^{*}(\xi, \eta)=R(0, \eta ; \xi, \eta) \mu(\eta)+ & \int_{0}^{\eta}\left[\alpha R(0, \tau ; \xi, \eta)-\frac{\partial}{\partial \tau} R(0, \tau ; \xi, \eta)\right] \mu(\tau) d \tau+ \\
& +\int_{0}^{\xi} d t \int_{0}^{\eta} R(t, \tau ; \xi, \eta) \widetilde{f}^{*}(t, \tau) d \tau
\end{aligned}
$$

and for $\xi=\eta$, due to (19),

$$
\begin{align*}
& R(0, \eta ; \eta, \eta) \mu(\eta)+\int_{0}^{\eta}\left[\alpha R(0, \tau ; \eta, \eta)-\frac{\partial}{\partial \tau} R(0, \tau ; \eta, \eta)\right] \mu(\tau) d \tau+ \\
& +\int_{0}^{\eta} d t \int_{0}^{\eta} R(t, \tau ; \eta \eta) \widetilde{f}^{*}(t, \tau) d \tau=0 \tag{21}
\end{align*}
$$

Noticing also that for the Riemann function we have

$$
R(0, \eta ; \eta, \eta)=\exp \left\{-\int_{0}^{\eta} \beta(\tau, \eta) d \tau\right\} \neq 0
$$

we see that the solution of the integral Volterra equation (21) satisfies conditions (19) and therefore the problem (18), (19) is uniquely solvable.

Thus, for the case when $b_{3}^{2,1,0}=0, b_{3}^{2,0,0}=0$ and $b_{3}^{1,1,0} \neq 0$ the problem $(16),(17)$ is solvable and therefore equation (15) is solvable too.

Further, consider the case when $b_{3}^{2,1,0}=b_{3}^{1,1,0}=b_{3}^{2,0,0}=0$. Then equation (16) reads as

$$
\begin{equation*}
b_{3}^{1,0,0} \Phi_{x_{1}}+b_{3}^{0,1,0} \Phi_{x_{2}}+b_{3}^{0,0,0} \Phi=\widetilde{f}_{3}, \tag{22}
\end{equation*}
$$

with

$$
\begin{equation*}
\Phi\left(x_{1}, 0\right)=0, \quad \Phi\left(0, x_{2}\right)=0, \quad x_{1}, x_{2} \in R_{+} . \tag{23}
\end{equation*}
$$

This case is considered in [3]. It is found that conditions $b_{3}^{0,1,0} \cdot b_{3}^{1,0,0} \geq 0$, $\left|b_{3}^{0,1,0}\right|+\left|b_{3}^{1,0,0}\right| \neq 0$ or $b_{3}^{0,1,0} \cdot b_{3}^{1,0,0}<0$ together with the equality

$$
\begin{equation*}
\int_{0}^{x_{2}} \exp \left\{\frac{b_{3}^{0,0,0}}{b_{3}^{0,1,0}}\left(\xi_{2}-x_{2}\right)\right\} \widetilde{f}_{3}\left(x_{1}+\frac{b_{3}^{1,0,0}}{b_{3}^{0,1,0}}\left(\xi_{2}-x_{2}\right), \xi_{2}\right) d \xi_{2}=0, \quad x_{2} \in R+ \tag{24}
\end{equation*}
$$

ensure the solvability of the problem (22), (23) and, hence, of equation (15).
Thus the following theorem is valid
Theorem 2. If $\operatorname{det}\left(\begin{array}{cc}b_{11}^{0,1,1} & b_{11}^{1,1,1} \\ b_{12}^{0,1,1} & b_{12}^{1,1,1}\end{array}\right) \neq 0, \quad b_{2}^{2,0,1} \neq 0$ and any of the following conditions
a) $b_{3}^{2,1,0}=b_{3}^{2,0,0}=0, b_{3}^{1,1,0} \neq 0$,
b) $b_{3}^{2,1,0}=0, b_{3}^{2,0,0} \cdot b_{3}^{1,1,0}<0$,
c) $b_{3}^{2,1,0}=b_{3}^{1,1,0}=b_{3}^{2,0,0}=0, b_{3}^{0,1,0} \cdot b_{3}^{1,0,0} \geq 0,\left|b_{3}^{0,1,0}\right|+\left|b_{3}^{1,0,0}\right| \neq 0$,
d) $b_{3}^{2,1,0}=b_{3}^{1,1,0}=b_{3}^{2,0,0}=0, b_{3}^{0,1,0} \cdot b_{3}^{1,0,0}<0$ together with equality (24), take place, then the problem (12), (2), (3) is uniquely solvable in the class $C_{0}^{2,1,1}(\bar{D})$ for any $f_{2} \in C_{0}\left(S_{2}\right), f_{3} \in C_{0}^{1,0}\left(S_{3}\right), f_{1 n} \in C_{0}\left(S_{1}\right), n=1,2$, $F \in C_{0}^{1,0,0}(\bar{D})$.

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