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DYNAMICAL STABILITY OF ORTHOTROPIC SHELLS OF ROTATION, CLOSE BY THEIR SHAPE TO THE CYLINDRICAL ONES, UNDER THE ACTION OF MERIDIONAL STRESSES

S. KUKUDZHANOV

Abstract. In the present paper we investigate the influence of meridional forces on the form of wave formation, eigen oscillations, on magnitude of lower frequencies and on the boundaries of dynamical instability of orthotropic shells of revolution, close to cylindrical ones.

რეზიუმე. ნაშრომში განხილულია მერიდიანული ძალების გავლენა ცილინდრულ ფორმასთან მიახლოებული ბრუნვითი ორთოტროპული გარსის საზღვარზე მოდებული ტალღური წარმოქმნის ფორმაზე, უმცირესი რხევების სიდიდეზე და დინამიურ მდგრადობაზე.

We investigate eigen oscillations and dynamical stability of orthotropic shells of revolution, close by their shape to cylindrical ones, under the action of meridional stresses uniformly distributed over the shell faces. We consider the shells of middle length whose middle surface element is described by the parabolic function. On the basis of the theory of shallow shells the resolving equation for oscillations of the corresponding prestressed shell is obtained. This equation in an isotropic case differs from the well-known one [1] by an additive term which may be of the same order as another terms. We consider the shells of positive and negative Gaussian curvature. Shell edges are assumed to be simply supported. In dimensionless form we present formulas and universal curves of dependence of the least frequency, forms of wave formation and boundaries of dynamical instability on the parameters of orthotropy, preliminary stress, Gaussian curvature and on the amplitude of shell deviation from the cylinder. It is shown that in the presence of preliminary stresses the orthotropy parameters and shell deviation from cylindrical form (order of thickness) may essentially change the lower frequencies, the forms of wave formation and boundaries of dynamical instability of the corresponding prestressed orthotropic cylindrical shell. It should be noted that

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for convex shells under preliminary pressure the influence of an elastic axial parameter is stronger than of an elastic circumferential parameter, while for concave shells the situation is opposite. However, for shells under preliminary tension the leading role of one or another parameter of orthotropy may be changed depending on the magnitude of preliminary stress and Gaussian curvature.

1. Let us consider the shell whose midlength surface is formed by rotating a square parabola around the z-axis of the Cartesian coordinate system x, y, z, with the origin at the middle of the segment of the axis of revolution (Fig.1). It is assumed that radius R of the middle surface cross-section of the shell is defined by the equality

$$R = r + \delta_0\big[1 - \xi^2(r/\ell)^2\big] \tag{1.1}$$

where r is radius of the face cross-section; δ_0 is maximal deviation (for $\delta_0 > 0$ the shell is convex, while for $\delta_0 < 0$ it is concave); $L = 2\ell$ is the shell length; $\xi = z/r$. We consider the midlength shells ([2], [3]) and assume that

$$\big(\delta_0/r\big)^2, \quad \big(\delta_0/\ell\big)^2 \ll 1. \tag{1.2}$$

In the capacity of basic equations of oscillations we take the equations of the theory of shallow shells ([4]). For the midlength shells under consideration, the forms of oscillations corresponding to the lower frequencies, are accompanied with feebly marked longitudinal wave formation compared to circumferential one. Therefore the following relation is valid:

$$\frac{\partial^2 f}{\partial \xi^2} \ll \frac{\partial^2 f}{\partial \varphi^2} \quad (f = w, \psi) \tag{1.3}$$

where w and ψ are, respectively, the functions of radial displacement and stress. As a result, the system of equations for the shells under consideration is reduced to the equation

$$\begin{gathered}
\varepsilon \frac{\partial^8 w}{\partial \varphi^8} + \frac{E_1}{E_2}\Big(\frac{\partial^4 w}{\partial \xi^4} + 4\delta \frac{\partial^4 w}{\partial \xi^2 \partial \varphi^2} + 4\delta^2 \frac{\partial^4 w}{\partial \varphi^4}\Big) - \\
- t_1^0 \frac{\partial^6 w}{\partial \xi^2 \partial \varphi^4} - t_2^0 \frac{\partial^6 w}{\partial \varphi^6} + \frac{\rho r^2}{E_2} \frac{\partial}{\partial t^2}\Big(\frac{\partial^4 w}{\partial \varphi^4}\Big) = 0 \qquad (1.4) \\
\varepsilon = \frac{h^2}{12 r^2 (1 - \nu^2)}, \quad \delta = \delta_0 r/\ell^2, \quad t_i^0 = T_i^0/E_2 h \quad (i = 1, 2)
\end{gathered}$$

where E_1, E_2, ν_1, ν_2 are the elastic moduli and Poisson coefficients in the axial and circumferential direction ($E_1\nu_2 = E_2\nu_1$); h is the shell thickness; φ is the angular coordinate; T_1^0, T_2^0 are meridional and circumferential stresses of the initial state; t is time and ρ is density.

The additive term in the above equation, compared with the equation given in [1] for an isotropic shell, is the fourth term which is, owing to inequality (1.3), of the same order as the third term of that equation. We consider a simply supported shell with uniformly distributed meridional stresses P_1 applied to its edges. The initial state is assumed to be momentless. On the basis of the corresponding solution and equalities (1.2) and (1.3) we obtain the following approximate expressions:

$$T_1^0 = P_1\Big[1 + \frac{\delta_0}{r}\big(\xi^2(r/\ell)^2 - 1\big)\Big], \quad T_2^0 = -2P_1\delta_0 r/\ell^2. \tag{1.5}$$

Taking into account the fact that

$$\big|\xi^2(r/\ell)^2 - 1\big|\frac{\partial^2 w}{\partial\xi^2} \ll 2\Big(\frac{r}{\ell}\Big)^2\frac{\partial^2 w}{\partial\varphi^2} \tag{1.6}$$

equation (1.4) takes the form

$$\begin{aligned}&\varepsilon\frac{\partial^8 w}{\partial\varphi^8} + \frac{E_1}{E_2}\Big(\frac{\partial^4 w}{\partial\xi^4} + 4\delta\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2} + 4\delta^2\frac{\partial^4 w}{\partial\varphi^4}\Big) - \\ &-\frac{P_1}{E_2 h}\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4} + \frac{2P_1\delta}{E_2 h}\frac{\partial^6 w}{\partial\varphi^6} + \frac{\rho r^2}{E_2}\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial\varphi^4}\Big) = 0\end{aligned} \tag{1.7}$$

Consider first the case $P_1 = \text{const}$. The expression

$$\begin{aligned}w &= A\cos\lambda_m\xi\sin n\varphi\sin\omega t,\\ \lambda_m &= m\pi\tau/L \;\; (m = 2i+1,\, i = 0,1,2,\ldots)\end{aligned} \tag{1.8}$$

satisfies the adopted boundary conditions. Substituting (1.8) in (1.7), for determination of eigen frequencies we obtain the following equality:

$$\frac{\rho r^2}{E_2}\omega^2 = \varepsilon n^4 + \frac{E_1}{E_2}\big(\lambda_m^4 n^{-4} + 4\delta\lambda_m^2 n^{-2} + 4\delta^2\big) + \frac{P_1}{E_2 h}\big(\lambda_m^2 - 2\delta n^2\big). \tag{1.9}$$

Introduce the dimensionless parameters

$$\alpha_1 = E_1/E, \quad \alpha_2 = E_2/E, \quad p = -P_1/Eh. \tag{1.10}$$

Then equation (1.9) takes the form

$$\omega^2 = \frac{E}{\rho r^2}\Big[\alpha_2\varepsilon n^4 + \alpha_1\big(\lambda_m^4 n^{-4} + 4\delta\lambda_m^2 n^{-2} + 4\delta^2\big) - p\big(\lambda_m^2 - 2\delta n^2\big)\Big]. \tag{1.11}$$

It is clear that for $p = 0$ and $\delta > 0$ the value $m = 1$ corresponds to the lower frequency. We can also show that this condition holds likewise for $\delta < 0$ if we take into account inequalities (1.2), (1.3) and the fact that $\omega^2 > 0$. Therefore first of all we will consider the forms of oscillations for which only one half-wave $m = 1$ locates along the shell length and n circumferential waves. Under pressure $p > 0$, and under tension $p < 0$.

To represent expression (1.11) (for $m = 1$) in dimensionless form we introduce dimensionless quantities θ, $\overline{P}$ and the following notation:

$$\theta = (\alpha_2/\alpha_1)^{1/4}N, \quad N = n^2/n_0^2, \quad \overline{P} = P/\sqrt{\alpha_1\alpha_2}, \quad P = p/p_*,$$
$$n_0^2 = \lambda_1\varepsilon^{-1/4}, \quad p_* = 2\varepsilon^{1/2}, \quad \delta_*^v = (\alpha_1/\alpha_2)^{1/4}\delta_*, \quad \delta_* = \varepsilon_*^{-1/2}\delta,$$
$$\varepsilon_* = (1-\nu^2)^{-1/2}\frac{h}{\tau}\Big(\frac{\tau}{L}\Big)^2, \quad \lambda_1 = \pi r/L, \quad \omega_*^2 = 2\lambda_1^2\varepsilon^{1/2}\frac{E}{\rho r^2}, \tag{1.12}$$

where ω_* and p_* are, respectively, the lower frequency and the critical loading for an isotropic cylindrical shell of middle length ([2], [7]). As a result, equality (1.11) can be written in the following dimensionless form:

$$\omega^2/\omega_*^2 = 0,5\sqrt{\alpha_1\alpha_2}\big[\theta^2 + \theta^{-2} + 2,37\delta_*^v\theta^{-1} + 1,404\delta_*^{v2} - \\ -2\overline{P}(1 - 1,185\delta_*^v\theta)\big]. \tag{1.13}$$

The lower frequency (for $\omega^2 > 0$) is defined from the condition $\omega^2(N)' = 0$. Hence we get

$$-1,185\delta_*^v\overline{P} = \theta - 1,185\delta_*^v\theta^{-2} - \theta^{-3} \tag{1.14}$$

or

$$\theta^4 + 1,185\delta_*^v\overline{P}\theta^3 - 1,185\delta_*^v\theta - 1 = 0. \tag{1.15}$$

Note that for $\overline{P} = 0$ we obtain the following well-known equation:

$$\theta^4 - 1,185\delta_*^v - 1 = 0, \tag{1.16}$$

whose roots have been obtained explicitly in [8]. Moreover, from (1.15) for $\delta_* = 0$ we get the equality $\theta^4 - 1 = 0$ whose positive root is $\theta = 1$ ($N = (\alpha_1/\alpha_2)^{1/4}$). Consequently, for the orthotropic cylindrical shell of middle length the lower frequency is realized for $N = (\alpha_1/\alpha_2)^{1/4}$ independently of P. For an isotropic case, all the above-said corresponds completely with [9]. Moreover, from equation (1.15) for $\overline{P} = 1$ ($P = \alpha_1^{1/2}\alpha_2^{1/2}$) we find that the positive root $\theta = 1$ does not depend on δ_*^v. For $\omega = 0$, equality (1.13) yields

$$\overline{P} = \frac{\theta^2 + \theta^{-2} + 2,37\delta_*^v\theta^{-1} + 1,404\delta_*^{v2}}{2(1 - 1,185\delta_*^v\theta)}. \tag{1.17}$$

As is known, the least value $\overline{P}$ is called a critical load. In particular, for $\delta_* = 0$, $\theta = 1$ from (1.17) we obtain the well-known formula of critical compressive force for a cylindrical orthotropic shell $\overline{P} = 1$ ([2]). The least value $\overline{P}$ ($\overline{P} > 0$), depending on θ, is realized for $\overline{P}'_\theta = 0$. This implies that

$$2(\theta - \theta^{-3} - 1,185\delta_*^v\theta^{-2})(1 - 1,185\delta_*^v) = \\ = -1,185\delta_*^v(\theta^2 + \theta^{-2} + 2,37\delta_*^v\theta^{-1} + 1,404\delta_*^{v2}). \tag{1.18}$$

More simplified equation (1.18) is of the fifth degree; it is impossible to obtain its roots explicitly. Therefore we suggest somewhat different way of finding a positive root of that equation. Denote a positive root of equation

(1.18) by θ_*. The value $\theta = \theta_*$ corresponds to a number of transversal waves for which is realized critical load of loss of stability P_*. Substituting equality (1.18) in (1.17), we obtain

$$-1,185\delta^v\overline{P}_* = \theta_* - 1,185\delta_*^v\theta^{-2} - \theta^{-3}. \tag{1.19}$$

It is not difficult to notice that equality (1.19) follows likewise from (1.14) for $\omega = 0$. Consequently, the values $\overline{P}$ and θ, satisfying equality (1.14), for which the expression (1.13) vanishes, are the critical values $\overline{P}_*$ and θ_*.

By virtue of equality (1.15) for $\overline{P} = 0$, we obtain equation (1.16) whose positive root $\theta = \theta_0$ corresponds to the lower frequency of the unloaded shell ([8]), while for $\overline{P} = \overline{P}_*$ equation (1.19), whose root is $\theta = \theta_*$, corresponds to $\omega = 0$. Thus as $\overline{P}$ varies in the interval

$$0 \le \overline{P} \le \overline{P}_* \tag{1.20}$$

the lower frequency varies in the interval $[\omega(\theta_0, \overline{P} = 0), 0]$.

Reasoning analogous to that of [7] enables us to show that when $\overline{P}$ varies in the interval (1.20) for $\delta_* \le 0$, the value θ realizing the lower frequency $\omega(\theta, \overline{P})$ is in the interval

$$\theta_0 \le \theta \le \theta_*. \tag{1.21}$$

For clarity we pass to the values $N = \theta(\alpha_1/\alpha_2)^{1/4}$. In particular, for $\delta_* = 0$ inequalities (1.20) and (1.21) take the form $0 \le \overline{P} \le 1$, $\theta_0 = \theta_* = 1$ (or $0 \le P \le \alpha_1^{1/2}\alpha_2^{1/2}$, $N_0 = N_* = (\alpha_1/\alpha_2)^{1/4}$).

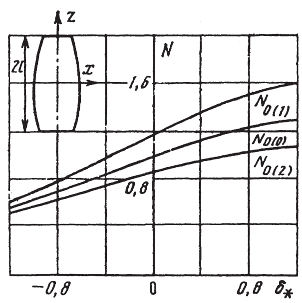

Fig. 1

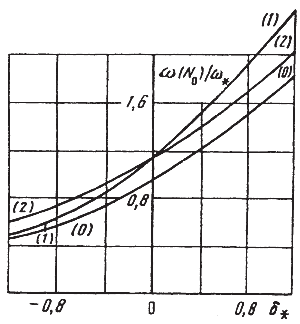

Fig. 2

Figs. 1 and 2 show the dependencies $N_0 = n^2/n_0^2$, $\omega(N_0)/\omega$ on the parameter δ_* for the cases $\alpha_1 = 1$, $\alpha_2 = 1(0)$, $\alpha_1 = 2$, $\alpha_2 = 1(1)$, $\alpha_1 = 1$, $\alpha_2 = 2(2)$ (ω_* and n_0 are, respectively, the lower frequency and a number of waves for the cylindrical isotropic midlength shell which are defined by equalities (1.12)); corresponding curves are denoted by $N_{0(i)}$ and (i) ($i = 0, 1, 2$). It is not difficult to see that for convex shells the influence of the elastic axial parameter is greater than of the elastic circumferential one, while for concave shells the situation is opposite.

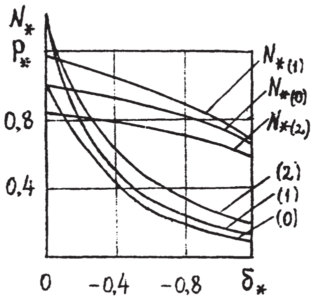

Fig. 3

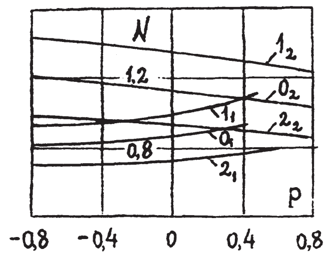

Fig. 4

Dependencies $N_* = n_*^2/n_0^2$ and $P_* = p_*^2/p_0^2$ on the parameter $\delta_* < 0$ for the cases $i = 0, 1, 2$ can be found in Fig. 3. The curves are denoted respectively by $N_{*(i)}$ and (i). It can be easily seen that for the concave shells the main part belongs to the elastic circumferential parameter compared to the axial one.

On the basis of equality (1.14) we can construct dependencies $N(P)$ which realize minimal frequency of the prestressed shell for different values of δ_*. Towards this end, we fix α_1, α_2, δ_* and for the given θ from the interval (1.21) we find P by formula (1.14). Fig. 4 displays the dependence $N(P)$ for the cases $i = 0, 1, 2$ (for $\delta_* = -0,4$ and $\delta_* = 0,4$) which are denoted respectively by i_1 and i_2. For comparison, in Fig. 5 we can see the curves of dependence of dimensionless lower frequency $\omega(N,P)/\omega_*$ on P for the cases under consideration which are also denoted by i_1 and i_2. Moreover,

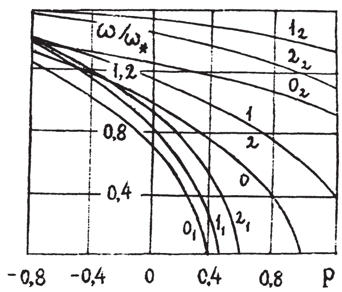

Fig. 5

Fig. 5 presents diagrams of dependence ω/ω_* on P for the cylindrical shell ($\delta_*=0$) for the cases $i=0,1,2$. On the basis of these curves it is not difficult to notice that if the influence of orthotropy parameters for the cylindrical shell is practically the same, the role of elastic circumferential parameter compared with the elastic axial one is much more greater for convex shells, whereas for concave shells the situation is opposite.

In the case of tensile force $P<0$, and equalities (1.13) and (1.14) take the form

$$\omega^2/\omega_*^2 = 0,5\alpha_1^{1/2}\alpha_2^{1/2}\big[\theta^2+\theta^{-2}+2,37\delta_*^v\theta^{-1}+1,404\delta_*^{v2}+ \\ +2|P|(1+1,185\delta_*^v\theta)\big], \tag{1.22}$$

$$1,185\delta_*|P| = \theta - 1,185\delta_*\theta^{-2}-\theta^{-3}. \tag{1.23}$$

Analogously, on the basis of formulas (1.22) and (1.23) we can construct corresponding dependencies. In Figs. 4 and 5, on the left from the coordinate axis we can see the curves of dependencies ω/ω_* and N on $P<0$ for the cases $0,1,2$ (for $\delta_*=-0,4$ and $\delta_*=0,4$).

Let us find under which conditions the inequality $n^2\gg\lambda_1^2$ used for the above-stated theory, is valid. It is seen in Fig.4 that in the worst case for $\delta_*=-0,4$, $i_1=2_1$ when contractive load P varies in the interval $0\le P\le P_*$, the value N is in the interval $0,753\le N\le 0,795$, whereas for large values of δ_* the boundaries of the interval increase. This implies that when the action of loading does not exceed the value of contractive critical loading for an isotropic cylindrical shell, the condition $n^2\gg\lambda_1^2$ is valid, and hence the above-given formulas are true. Moreover, it is not difficult to notice that for $\delta_*=-0,4$, under tensile loading $0\le|P|\le 0,8$ the value N decreases negligibly. Consequently, for the tensile loading in the intervals under consideration this inequality is likewise valid. The fulfilment of these conditions along with the conditions (1.1) and (1.2) justifies the validity of application of the theory of shallow shells and of the above-described investigation.

Consider next the value $m>1$. Using notation (1.12), formula (1.11) can be represented as follows:

$$\omega^2/\omega_*^2 = 0,5\sqrt{\alpha_1\alpha_2}m^2\big[Q^2+Q^{-2}+2,37\delta_*^vQ^{-1}m^{-1}+1,404\delta_*^{v2}-m^{-2} \\ -2\overline{P}(1-1,185\delta_*^vQm^{-1})\big] \tag{1.24}$$

where

$$Q=\theta/m,\quad \theta=(\alpha_2/\alpha_1)^{1/4}N,\quad m=2i+1\quad (i=0,1,2,\ldots). \tag{1.25}$$

For $\delta_*=0$, formula (1.24) takes the form

$$\omega^2/\omega_*^2 = 0,5\sqrt{\alpha_1\alpha_2}m^2\big(Q^2+Q^{-2}-2\overline{P}\big). \tag{1.26}$$

For $\omega=0$ we obtain

$$\overline{P}=0,5(Q^2+Q^{-2}). \tag{1.27}$$

It is not difficult to see that for any m the least value of (1.27) is realized for $Q = 1$, and the critical compressive force $\overline{P} = 1$ ($P = \sqrt{\alpha_1\alpha_2}$). The least value of (1.26), depending on Q (for $\overline{P} < 1$), is likewise realized for $Q = 1$, and (1.24) takes the form

$$\omega/\omega_* = \sqrt{\alpha_1\alpha_2}m^2(1-\overline{P}) \begin{cases} \overline{P} < 1 \\ \delta_* = 0 \end{cases} .$$

Consequently, the least value of frequency (for $\overline{P} < 1$) is realized for $m = 1$.

Consider now the expression for finding a critical load for $\delta_* \neq 0$. In the sequel, taking into account inequalities (1.2), we will restrict ourselves to the consideration of the case $|\delta_*| \leq 1$. The relation (1.24) vanishes if

$$\overline{P} = \frac{Q^2 + Q^{-2} + 2,37\delta_*^v Q^{-1}m^{-1} + 1,404\delta_*^{v2} - m^{-2}}{2(1 - 1,185\delta_*^v\theta m^{-1})}. \tag{1.28}$$

Consider first the case $\delta_* < 0$. The expression (1.28) for $\delta_* = -|\delta_*|$ takes the form

$$\overline{P} = \frac{Q^2 + Q^{-2} - |\delta_*^v|m^{-1}(2,37Q^{-1} - 1,404\delta_*^v - m^{-1})}{2(1 + 1,185|\delta_*^v|Qm^{-1})}. \tag{1.29}$$

Now we define how $\overline{P}$ varies together with the variation of m. As m increases, the denominator of (1.27) decreases. As for the nominator (for $Q \leq 1$, $|\delta_*^v| \leq 1$), the third term $|\delta_*^v|m^{-1}(2,37Q^{-1} - 1,404|\delta_*^v|m^{-1})$ decreases as m increases, hence the nominator increases. This implies that the expression (1.29), depending on m, will be minimal for $m = 1$. On the other hand, simple numerical calculations show that the least value $\overline{P}$ (for $|\delta_*| \leq 1$) is realized for $m = 1$ and $Q \leq 1$.

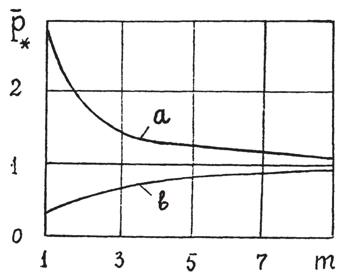

Fig. 6

In particular, Fig. 6 displays least values of $\overline{P}$ (depending on Q) for fixed m (for $\delta_*^v = -0,4$) which are given in the form of the curve b.

Consider the case $\delta_* > 0$. Now the formula for finding the critical stress has the form (1.28). It is easily seen that as m increases, the nominator of (1.28) decreases and the denominator increases. Thus the value P decreases. Therefore the least critical load defined by means of (1.28) is attainable for sufficiently large m; therefore we can neglect the terms corresponding to the deviation of the shell from the cylindrical form, and the corresponding critical load $\overline{P}_{kp}$ will practically be equal to the critical compressive loading for the cylindrical orthotropic shell $\overline{P}_* \approx 1$ $(Q = 1)$. This result is verified in [10]. The least values of $\overline{P}$ (depending on Q) for fixed m (for $\delta_*^v = 0,4$) are presented in Fig. 6 in the form of the curve a. As $\delta_*^v \to 0$, the both curves a and b merge with the straight line $\overline{P}_* = 1$.

Consider now the expression (1.24) and find how it depends on m. We denote $m = x$ and write the expression for the derivative ω^2 with respect to x.

$$(\omega^2)'_x = 2\Big\{x\big[0,5(Q^2 + Q^{-2}) - \overline{P}\big] + 0,5925\delta_*^v(Q^{-1} + PQ)\Big\}. \tag{1.30}$$

The function $0,5(Q^2 + Q^{-2})$ has the least value for $Q = 1$ and is equal to unity; note that the second summand (for $\delta_*^v > 0$) is positive. Thus for $\overline{P} \le 1$ the value of the derivative is more than zero. This implies that the function ω^2 increases monotonically and attains its minimum for $m = 1$.

The least value ω^2 with respect to Q (for fixed m) can be defined by means of the condition

$$(\omega^2)_Q = \sqrt{\alpha_1\alpha_2}m^2\big(Q - Q^{-3} - 1,185\delta_*^v Q^{-2}m^{-1} + 1,185\delta_*^v\overline{P}m^{-1}\big) = 0 \tag{1.31}$$

whence

$$\begin{gathered} Q^4 + bQ^3 + dQ + \ell = 0, \quad b = 1,185\delta_*^\circ\overline{P}, \\ d = -1,185\delta_*^\circ, \quad \ell = -1, \quad \delta_*^\circ = \delta_*^v m^{-1}. \end{gathered} \tag{1.32}$$

The roots of equation (1.32) coincide with those of the two quadratic equations

$$Q^2 + (b + B_{1,2})Q/2 + \Big(y_1 + \frac{by_1 - d}{B_{1,2}}\Big) = 0, \quad B_{1,2} = \pm\sqrt{8y_1} - b^2. \tag{1.33}$$

The roots of equations (1.33) have the form

$$Q_{1,2} = \frac{-(\sqrt{8\gamma_1} + b)}{4} \pm \sqrt{-\frac{(by_1 - d)}{\sqrt{8\gamma_1}} + \frac{(b\sqrt{8\gamma_1} - 4\gamma_2)}{8}} \tag{1.34}$$

$$\gamma_1 = y_1 + \frac{b^2}{8},$$

$$_{3,4} = \frac{(\sqrt{8\gamma_1} + b)}{4} \pm \sqrt{\frac{(by_1 - d)}{\sqrt{8\gamma_1}} + \frac{(b\sqrt{8\gamma_1} + 4\gamma_2)}{8}} \tag{1.35}$$

$$\gamma_2 = y_1 - \frac{b^2}{4},$$

where y_1 is any real root of the cubic equation

$$y^3 + 3py + 2q = 0, \quad 3p = \frac{(bd - 4\ell)}{4}, \quad 2q = \frac{-(b^2\ell + d)}{8}, \tag{1.36}$$

$$p = \frac{1}{3}\Big(1 - \frac{1,185^2\delta_*^{\circ 2}\overline{P}}{4}\Big), \quad q = -\frac{1,185^2\delta_*^{\circ 2}(1-\overline{P})}{16}. \tag{1.37}$$

Next, we simplify the expression (1.37) for p assuming that

$$\frac{1,185^2\delta_*^{\circ 2}\overline{P}}{4} \ll 1 \quad \big(|\delta_*^\circ| \le 0,5,\ \overline{P} \le 0,5\big). \tag{1.38}$$

Then $p = \frac{1}{3}$, and

$$D = p^3 + q^2 = \frac{1}{3^3} + \frac{1,185^4\delta_*^{\circ 4}\overline{P}}{16^2} > 0, \quad \delta_*^\circ = \delta_*^v/m. \tag{1.39}$$

Since the discriminant of the equation $D > 0$, the equation has only one real root

$$y_1 = \big(-q + \sqrt{D}\big)^{1/3} + \big(-q - \sqrt{D}\big)^{1/3}.$$

Substituting the values of q and D, we obtain

$$y_1 = \frac{1}{\sqrt{3}}\Big[\Big(0,456\delta_*^{\circ 2}(1-\overline{P}^2) + \sqrt{1 + 0,208\delta_*^{\circ 4}(1-\overline{P}^2)^2}\Big)^{1/3} +$$

$$+\Big(0,456\delta_*^{\circ 2}(1-\overline{P}^2) - \sqrt{1 + 0,208\delta_*^{\circ 4}(1-\overline{P}^2)^2}\Big)^{1/3}\Big]. \tag{1.40}$$

Taking

$$0,104\delta_*^{\circ 4}(1-\overline{P}^2) \ll 1 \tag{1.41}$$

performing series expansion of the expressions contained in (1.40) and neglecting the values of the second order of smallness, we find that

$$y_1 \approx 0,1755\delta_*^{\circ 2}(1-\overline{P}^2). \tag{1.42}$$

Note that if the condition (1.38) is fulfilled, then the condition (1.41) is all the more so. Substituting the values y_1, b and d in (1.34) and (1.35) and taking into account the fact that only positive values Q are of interest (since $n^2 > 0$), we find that for $\delta_*^\circ < 0$ $(d > 0)$ the positive root is only the root of Q_1, whereas for $\delta_*^\circ > 0$ $(d < 0)$ there takes place the root of Q_3. As a result, we have

$$Q = \sqrt{1 + 0,1755\delta_*^{\circ 2}\overline{P}(1-\overline{P}^2) - 0,0877\delta_*^{\circ 2}(1 + 2\overline{P} - 3\overline{P}^2) -}$$

$$-0,2962|\delta_*^\circ|(1-\overline{P}) \quad (\delta_* < 0), \tag{1.43}$$

$$Q = \sqrt{1 + 0,1755\delta_*^{\circ 2}\overline{P}(1-\overline{P}^2) - 0,0877\delta_*^{\circ 2}(1 + 2\overline{P} - 3\overline{P}^2) +}$$

$$+0,2962\delta_*^\circ(1-\overline{P}) \quad (\delta_* > 0). \tag{1.44}$$

The above equalities for $P = 0$ yield

$$\begin{aligned} \theta &= \sqrt{1 - 0,0877\delta_*^2/m^2} - 0,2962|\delta_*^v|/m \quad (\delta_* < 0), \\ \theta &= \sqrt{1 - 0,0877\delta_*^2/m^2} + 0,2962\delta_*^v/m \quad (\delta_* > 0). \end{aligned} \tag{1.45}$$

For $m = 1$, $\alpha_1 = \alpha_2 = 1$, and we obtain the well-known formulas (see [8]). For $\overline{P} = 1$, equation (1.32) takes the form

$$Q^4 + 1,185\frac{\delta_*^v}{m}\theta^3 - 1,185\frac{\delta_*^v}{m}\theta - 1 = 0,$$

and we obtain

$$(Q^2 - 1)\Big(Q^2 + 1 - 1,185\frac{\delta_*^v}{m}\Big).$$

It is not difficult to see that the positive root of the above equation is $Q = 1$. For $\overline{P} = 1$, the expressions (1.43) and (1.44) likewise yield $Q = 1$.

Compare now, in particular for $\delta_*^v = 0,4$, the least value of frequency for $m = 1$, when $\overline{P} = 0$ and $\overline{P} = 1$:

$$\omega\big(\delta_*^v = 0,4,\ \overline{P} = 0,\ m = 1,\ Q = 1,148\big)/\omega_* = 1,2495$$
$$\omega\big(\delta_*^v = 0,4,\ \overline{P} = 1,\ m = 1,\ Q = 1\big)/\omega_* = 1,0297.$$

For sufficiently large m by formulas (1.43) and (1.44) we find that $Q \approx 1$, and on the basis of formula (1.24) we have

$$\omega^2/\omega_*^2 \approx 0,5\sqrt{\alpha_1\alpha_2}\big(Q^2 + Q^{-2} - 2\overline{P}\big)$$

i.e., we obtain the formula which practically coincides with formula (1.26) for cylindrical shells.

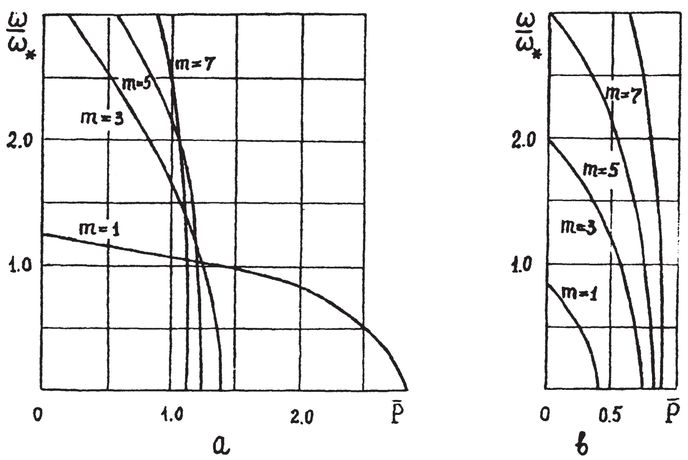

Fig. 7

For $\delta_*^v = 0,4$, Fig.7a displays changes of least frequencies which take place depending on $\overline{P}$ for different $m(m = 1,3,5,7)$. In is not difficult to notice that the lower frequency for $\overline{P} \leq 1$ is realized for $m = 1$, but as $\overline{P}$ tends to unity from above the lower frequency is realized by the form with sufficiently large m, corresponding to the form of stability loss. For $\delta_* < 0$, the expression (1.24) takes the form

$$\omega^2/\omega_*^2 = 0,5m^2\sqrt{\alpha_1\alpha_2}\Big[Q^2 + Q^{-2} - |\delta_*^v|m^{-1}\big(2,37\theta^{-1} - 1,404|\delta_*^v|m^{-1}\big) - \\ -2P\big(1 + 1,185|\delta_*^v Q|m^{-1}\big)\Big].$$

Taking into account the fact that the last and the third term of that expression (for $\theta \leq 1$, $|\delta_*^v| \leq 1$) decrease as m increases, this implies that the expression in square brackets increases and, moreover, the factor m^2 increases as well. Consequently, the lower frequency for $\delta_* < 0$ is realized for $m = 1$. This will be demonstrated below by our calculations.

Fig 7a displays changes of the lower frequencies depending on $\overline{P}$ for different m, for $\delta_* = -0,4$. It is easily seen that the lower frequency is realized for $m = 1$. Moreover, the diagrams in Fig.7b show that for $\delta_* < 0$, as m increases the curves tend to $\overline{P} = 1$ from below, whereas for $\delta_* > 0$ on the contrary from above (Fig 7a).

It should be noted that the results obtained for $m > 1$ hold for the values of Q, close to unity ($Q \approx 1$), i.e., when $n^2 \approx (\alpha_1/\alpha_2)^{1/4}\varepsilon^{-1/4}\lambda_m$. Therefore these results are valid only for sufficiently thin shells. When $(\alpha_1/\alpha_2)^{1/4}\varepsilon^{-1/4} \gg \lambda_m$, we have the relation $n^2 \gg \lambda_m^2$, and the given theory is valid.

It follows from the above-obtained qualitative results (for $\delta_* > 0$) that the forms of oscillations with $m > 1$, coinciding with the corresponding form of stability loss, take place for $\overline{P} > 1$, and the larger is m, the closer is the critical load $\overline{P}_{kp}$ to unity. When $\overline{P}$ varies in the interval $0 \leq \overline{P} \leq 1$, the lower frequency with the form of oscillation takes place when $m = 1$. For $\delta_* < 0$, the forms of oscillations with $m > 1$, coinciding with the forms of stability loss, take place for $\overline{P}$, larger than $\overline{P}_{kp}$ $(m = 1)$. The form of oscillations, corresponding to the lower frequency, and also the form of stability loss, corresponding to the least critical loading, are realized for $m = 1$.

2. Consider now the case

$$P_1 = P_0 + P_t \cos\Omega t. \tag{2.1}$$

A solution of equation (1.7) will be sought in the form

$$w = f_{mn}(t)\cos\lambda_m\xi\sin n\varphi. \tag{2.2}$$

Substituting the above-given solution in (1.7) and requiring for the latter to be satisfied for any ξ and φ, we obtain

$$\frac{d^2 f_{mn}}{dt^2}+\frac{E_2}{\rho r^2}\Big[\varepsilon n^4+\frac{E_1}{E_2}\big(\lambda_m^4 n^{-4}+4\delta\lambda_m^2 n^{-2}+4\delta^2\big)+ \\ +\frac{P_1(t)}{E_2 h}\big(\lambda_m^2-2\delta n^2\big)\Big]f_{mn}=0. \qquad (2.3)$$

Frequencies of eigen oscillations of the shell (for $P_1 = P_0$) can be defined from equation (2.3) by putting $f_{mn}(t) = C$ and expressed by formula (1.9). Since equation (2.3) is identical for all forms of oscillations, therefore we can omit the indices m and n.

Similarly to the above-said, we introduce dimensionless quantities (1.12), (1.25) and rewrite equation (2.3) as follows:

$$\frac{d^2 f}{dt^2}+0,5\omega_*^2 m^2\sqrt{\alpha_1\alpha_2}\big[Q^2+Q^{-2}+2,37\delta_*^v Q^{-1}m^{-1}+1,404\delta_*^{v2}m^{-2}- \\ -2\big(\overline{P}_0+\overline{P}_t\cos\Omega t\big)\big(1-1,185\delta_*^v Q m^{-1}\big)\big]f=0$$
$$\overline{P}_0=\overline{p}_0/p_{0*},\quad \overline{P}_t=\overline{p}_t/p_{0*}\quad \overline{P}_t=P_i\sqrt{\alpha_1\alpha_2}\quad (i=0,t). \qquad (2.4)$$

The critical stress $\overline{P}$ in the statical case ($P_t = 0,\ f = \text{const}$) is defined by means of equation (2.4) and formula (1.17). We reduce equation (2.4) to the standard Matye's equation

$$\frac{d^2 f}{dt^2}+\omega^2(Q)\big[1-2\mu(Q)\cos\Omega t\big]f=0, \qquad (2.5)$$

$$\omega^2(Q)=\overline{\omega}_0^2(Q)\Big(1-\frac{\overline{P}_0}{\overline{P}(Q)}\Big),\ \ \overline{\omega}_0^2(Q)= \\ =0,5\omega_*^2 m^2\sqrt{\alpha_1\alpha_2}\big(Q^2+Q^{-2}2,37\delta_*^v Q^{-1}m^{-1}+1,404\delta_*^{v2}m^{-2}\big), \qquad (2.6)$$

$$\mu(Q)=\frac{\overline{P}_t}{2[\overline{P}(Q)-\overline{P}_0]},\quad \omega_*^2=2\lambda_1^2\varepsilon^{1/2}E/\rho r^2, \qquad (2.7)$$

$$\overline{P}(Q)=\frac{Q^2+Q^{-2}+2,37\delta_*^v\theta^{-1}m^{-1}+1,404\delta_*^{v2}m^{-2}}{2(1-1,185\delta_*\theta m^{-1})} \qquad (2.8)$$

the value μ is commonly called the excitation coefficient. The solution of equation (2.5) have been investigated in a quite number of works where it is noted that under certain relations between μ, Ω, ω and $t\to\infty$ the solution of equation (2.5) increses unboundedly in the regions of instability.

Generalizing the obtained results [12] to the shell under consideration, we cite here the following formulas. To find regions of dynamical instability, we consider first the case $P_t \to 0$ ($\mu \to 0$). Thus we find that these regions of instability are near the frequencies

$$\Omega_*=2\omega(Q)/k\quad (k=1,2,3,\dots) \qquad (2.9)$$

Depending on number k, we can distinguish the first, second, third and so on regions of dynamical instability. The region of instability ($k = 1$) lying near $\Omega_* = 2\omega(Q)$, when $\omega(Q)$ takes the least value, is the most dangerous and is of great practical value. This region is called the principal region of dynamical instability.

For P_t, different from zero, we obtain the formula for boundaries of the principal region of instability:

$$\Omega_* = 2\omega(Q)\sqrt{1 \pm \mu(Q)}. \tag{2.10}$$

If we take into account resistance forces, proportional to the first derivative with respect to time (with damping coefficient ε), then the formula for finding boundaries of the principal region of instability takes the form

$$\Omega_* = 2\omega(Q)\sqrt{1 \pm \sqrt{\mu^2(Q) - (\Delta/\pi)^2}}, \quad \Delta = 2\pi\varepsilon/\omega(Q) \tag{2.11}$$

where the terms containing higher degrees Δ/ε are omitted and the damping discriminant Δ is usually rather small compared to unity. The values of the expressions $\omega(Q)$, $P(Q)$, $\mu(Q)$ are defined by virtue of formulas (2.6), (2.7) and (2.8), in which m and Q correspond to the least value $\omega(Q)$. For $m = 1$, on the basis of formula (1.25), we have $Q = \theta = (\alpha_2/\alpha_1)^{1/4}N$. The corresponding values N and $\omega(Q)$, depending on α_1, α_2, P_0 and δ_*, are presented in Figs.4 and 5 (for the cases $\delta_* = -0,4;\ 0,4;\ i = 0,1,2$). It follows from (2.11) that the minimal value of the excitation (critical) coefficient for which undamped oscillations are still possible, can be defined by the equality

$$\mu_{*1} = \Delta/\pi \tag{2.12}$$

For the boundaries of the second region of instability ($k = 2$) we have the following formula:

$$\Omega_* = \omega(Q)\sqrt{1 + \mu^2(Q) \pm \sqrt{\mu^4(Q) - (\Delta/\pi)^2[1 - \mu^2(Q)]}}. \tag{2.13}$$

In this case the critical value of the excitation coefficient we define approximately by the equality $\mu_{*2}(Q) = (\Delta/\pi)^{1/2}$. Analogously, generalizing the obtained results [12], we can deduce formulas for the boundaries of the third region of instability which is realized practically very seldom.

On the basis of the above formulas and diagrams it is not difficult to define intervals of variation of exciting frequencies (depending on δ_*, P_0, P_t, α_1, α_2) which fall into the regions of dynamical instability. Thus, for example, for $\delta_* = 0,4$, $P_0 = 0,3$, $P_t = 0,05$, $\Delta = 0,01$, $\alpha_1 = 2$, $\alpha_2 = 1$ we find that the lower frequency is realized for $m = 1$, $\theta = 1,102$; $\omega(\theta) = 1,508\omega_*$, $P(\theta) = 4,445$, $\mu(\theta) = 0,00853$, $\mu_{*1} = 0,00318$, $\mu_{*2} = 0,0564$. Then by virtue of formula (2.11) we find that the values Ω contained in the interval $3,00\omega_* < \Omega < 3,028\omega_*$ are in the principal region of dynamical instability. Since $\mu(\theta) < \mu_{*2}$, this means that the second region of instability is unattainable.

For $\alpha_1, \alpha_2 = 2$, $\delta_* = 0,4$ and for the same values of external load the lower frequency is realized for $m = 1$, $\theta = 1,074$; $\omega(\theta) = 1,376\omega_*$, $P(\theta) = 3,611$, $\mu(\theta) = 0,00755$ and on the basis of formula (2.11) we find that the values Ω, contained in the interval $2,742\omega_* < \Omega < 2,761\omega_*$, fall into the principal region of instability. In this case the second region of instability is likewise unattainable.

For $\delta_* = -0,4$, $\alpha_1 = 2$, $\alpha_2 = 1$ and for the same values of external loading the lower frequency is realized for $m = 1$, $\theta = 0,880$; $\omega(\theta) = 0,574\omega_*$, $P(\theta) = 0,520$, $\mu(\theta) = 0,113$. Analogously, according to the above-said, we find that the values Ω, contained in the interval $1,081\omega_* < \Omega < 1,211\omega_*$, fall into the principal region of instability. Since $\mu(\theta) > \mu_{*2}$, the second region of instability is quite attainable. By virtue of formula (2.13) we find that the values Ω, contained in the interval $0,574\omega < \Omega < 0,581\omega$, fall into the second region of instability.

For $\alpha_1 = 1$, $\alpha_2 = 2$, $\delta_* = -0,4$ and for the same values of external loading we find that the values Ω contained in the interval $1,43\omega_* < \Omega < 1,516\omega_*$, fall into the principal region of instability, while the values Ω, contained in the interval $0,724\omega_* < \Omega < 0,731\omega_*$, fall into the second region of instability.

Comparing the principal regions of dynamical instability for different α_1, α_2, $\delta_* > 0$ we find that the elastic axial parameter α_1 is of more great importance compared with the elastic circumferential parameter α_2, whereas for $\delta_* < 0$ the situation is opposite.

The above formulas and diagrams allow one to define easily regions of instability for shells under consideration.

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Author's address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, Aleksidze St., Tbilisi 0193
Georgia