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# THE MAXIMAL OPERATOR ON SPACES OF HOMOGENOUS TYPE

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Abstract. We study the boundedness of the maximal operator in the spaces $L^{p(\cdot)}(\Omega)$ over a measurable subset of a space of homogenous type with an exponent $p(x)$ satisfying the Dini–Lipschitz condition.

**რეზიუმე.** შესწავლილია მაქსიმალური ოპერატორის შემოსაზღვრულობის საკითხი ერთგვაროვანი ტიპის სივრცის ზომად ქვესიმრავლეზე განსაზღვრულ ცვლადმაჩვენებლიან $L^{p(\cdot)}(\Omega)$ სივრცეზე, როცა $p(x)$ მაჩვენებელი აკმაყოფილებს დინი–ლიფშიცის პირობას.

In the last years the Lebesgue spaces $L^{p(\cdot)}$ with variable exponent have become an object of intensive investigation, see [1]-[4] for basic properties of the spaces $L^{p(\cdot)}$. One of the main results in this field was Diening's theorem [1] about the boundedness of the Hardy-Littlewood maximal operator in $L^{p(\cdot)}(\Omega)$ when $\Omega$ is a mesuareble bounded set in $R^n$, under certain conditions on $p$. Later Diening has extended this result to the whole $R^n$ with the aditional assumption that $p$ is constant outside of a fixed ball. Our goal was to expand this research on the spaces of homogenous type. We have to mention that for that case our Theorem 1 independently was proved by P. Harjulento, P. Hástó amd M. Pere [5] for bounded metric measure spaces.

We start with the definition of the space of homogenous type (see e.g. [6]).

**Definition 1.** A space of homogenous type (SHT in following) $(X,\rho,\mu)$ is a topological space with a measure $\mu$ such that the space of compactly supported continuous functions is dense in $L^1(X,\mu)$ and there exists a non-negative real-valued function $\rho : X \times X \to R^1$ satisfying:

(i) $\rho(x,x) = 0$ for all $x \in X$.

(ii) $\rho(x,y) > 0$ for all $x \neq y$, $x,y \in X$.

(iii) There is a constant $a_0 > 0$ such that $\rho(x,y) \le a_0\rho(y,x)$ for all $x,y \in X$.

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(iv) There is a constant $a_1 > 0$ such that $\rho(x,y) \le a_1(\rho(x,z)+\rho(z,y))$ for all.

(v) For every neighbourhood $V$ of $x \in X$ there is $r > 0$ such that the ball $B(x,r) = \{y \in X : \rho(x,y) < r\}$ is contained in $V$.

(vi) Balls $B(x,r)$ are measurable for every $x \in X$ and every $r > 0$.

(vii) There is a constant $b > 0$ such that $\mu B(x,2r) \le b\mu B(x,r) < \infty$ for every $x \in X$ and every $r > 0$.

One can find many interesting examples of SHT in [6].

Suppose that $\Omega$ is a mesuareble bounded set in $X$ and $p$ is a measurable function on $\Omega$ such that $1 \le p(x) \le \overline{p} < \infty$, $x \in \Omega$. By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions $f(x)$ on $\Omega$ such that

$$\big|f\big|_{p(\cdot)} = \int\limits_{\Omega} \big|f(x)\big|^{p(x)}\,d\mu(x) < \infty.$$

$L^{p(\cdot)}(\Omega)$ is a Banach space with the norm:

$$\big\|f\big\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \Big|\frac{f}{\lambda}\Big|_{p(\cdot)} \le 1\Big\}.$$

In this paper we consider the Hardy-Littlewood maximal operator,

$$Mf(x) = \sup_{r>0} \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)\cap\Omega} \big|f(y)\big|\,d\mu(y).$$

We will use also the following notations:

$p_B = \operatorname*{essinf}_{x\in B} p(x)$, $\overline{p}_B = \operatorname*{esssup}_{x\in B} p(x)$, $\underline{p}_\Omega = \underline{p}$, $\overline{p}_\Omega = \overline{p}$,

$M_r f(x) = \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)\cap\Omega} |f(y)|d\mu(y)$. We also assume that $\rho$ is a metric.

**Theorem 1.** *Let $\Omega$ be a bounded set in a homogenous type space $X$ and $p : \Omega \to [1,\infty)$ satisfy the following conditions:*

a) $1 < \underline{p} \le \overline{p} < \infty$ (1)

b) *there exists a positive number $c_0$, such that for every pair $x$, $y \in \Omega$*

$$\big|p(x) - p(y)\big| \le \frac{c_0}{\log\frac{1}{\rho(x,y)}}, \quad \textit{when} \quad \rho(x,y) < \frac{1}{2}. \tag{2}$$

*Then the maximal operator $M$ is bounded in $L^{p(\cdot)}$ space.*

We need some lemmas to prove this theorem.

**Lemma 1.** *Let $\Omega$ be a bounded set and $r_0 > 0$. Then there exist positive numbers $s$, $\alpha$ and $\beta$, such that*
a) $\mu B(x,r) \leq \beta r^s$, *when $x \in \Omega$ and $r \geq r_0$;*
b) $\mu B(x,r) \geq \alpha r^s$, *when $x \in \Omega$ and $r < r_0$.*

*Proof.* Let us suppose that $r_0 = 1$ and $k(x) = \mu B(x,1)$, $x \in \Omega$. As $\Omega$ is a bounded set there exists a ball $B_0$, such that $\Omega \subset B_0$. We can suppose that $B_0$ is sufficiently large so that $B(x,1) \subset 2B_0$ for every $x \in \Omega$. Consequently

$$k(x) \leq \mu(2B_0) = K < \infty.$$

For the other side, it's easy to show that there exists a natural number $N$, independent of $x$, such that $\Omega \subset B(x,N)$. Thus $\mu\Omega \leq \mu B(x,N) \leq c_1 \mu B(x,1)$ and $k(x) \geq \dfrac{\mu\Omega}{c_1} = k > 0$. So we have

$$0 < k \leq k(x) \leq K < \infty, \quad x \in \Omega.$$

Using the doubling condition several times we get:

$$\mu B(x,2^n) \leq c^n \mu B(x,1) = k(x)(2^n)^{\log_2 c} \leq K(2^n)^s,$$
$$\mu B\Big(x, \frac{1}{2^n}\Big) \geq c^{-n} \mu B(x,1) = k(x)\big(2^{-n}\big)^{\log_2 c} \geq k(2^{-n})^s,$$

where $s = \log_2 c$.
Now let $r \geq r_0 = 1$. There exists a natural number $n$, such that $2^{n-1} \leq r < 2^n$. Then

$$\mu B(x,r) \leq \mu B(x,2^n) \leq K(2^n)^s \leq K(2r)^s = K2^s r^s = \beta r^s.$$

In the same way, if $r < r_0 = 1$, there exists a natural number $n$, such that $2^{-n} \leq r < 2^{-n+1}$ and

$$\mu B(x,r) \geq \mu B(x,2^{-n}) \geq k\Big(\frac{r}{2}\Big)^s = k2^{-s} r^s = \alpha r^s.$$

In the case of an arbitrary $r_0$ the proof is analogous. The lemma is proved. □

**Lemma 2.** *Let $\Omega$ be a bounded set and the condition* (2) *holds. Then there exists a positive number $c_1$, such that for every ball $B$*

$$\big(\mu B\big)^{\underline{p}_B - \overline{p}_B} \leq c_1 \tag{3}$$

*when $\mu(\Omega \cap B) > 0$.*

*Proof.* As $\Omega$ is a bounded and (2) holds, it is obvious that $1 \leq \underline{p} \leq \underline{p}_B \leq \overline{p}_B \leq \overline{p} < \infty$. If $\mu B \geq 1$ then $(\mu B)^{\underline{p}_B - \overline{p}_B} \leq 1$ and (3) holds. Let $\mu B < 1$ and $\operatorname{diam}(B) \geq \frac{1}{2}$. As we have seen in the proof of Lemma 1, in this case $\mu B \geq k > 0$. Hence, $(\mu B)^{\underline{p}_B - \overline{p}_B} \leq k^{\underline{p} - \overline{p}}$. Thus we can assume that

$\operatorname{diam}(B) < \frac{1}{2}$. Let us choose a pair $u$, $\nu \in \Omega \cap B$, so that $0 \le \frac{1}{2}(\underline{p}_B - \overline{p}_B) \le p(u) - p(\nu)$. Since $\rho(u,\nu) < \frac{1}{2}$,

$$\big|p(u) - p(\nu)\big| \le c_0 \log^{-1} \frac{1}{\rho(u,\nu)}.$$

Hence,

$$e^{c_0} \ge \big(\rho(u,\nu)\big)^{-|p(u)-p(v)|} \ge \big(\rho(u,v)\big)^{\frac{1}{2}(\underline{p}_B - \overline{p}_B)}.$$

By Lemma 1 $\rho(u,\nu) \le 2r(B) \le c(\mu B)^{\frac{1}{s}}$. So,

$$e^{2c_0} \ge \big(\rho(u,\nu)\big)^{\underline{p}_B - \overline{p}_B} \ge c\big(\mu B\big)^{\frac{1}{s}(\underline{p}_B - \overline{p}_B)},$$

or

$$\big(\mu, B\big)^{\underline{p}_B - \overline{p}_B} \le c e^{2c_0 s} = c_1.$$

and the lemma is proved. □

**Lemma 3.** *Let $\Omega$ be a bounded set and the conditions* (1) *and* (2) *hold. Then there exists a positive number c such that for every $f \in L^{p(\cdot)}(\Omega)$, with $\|f\|_{p(\cdot)} \le 1$*

$$\big|M_r f(x)\big|^{p(x)} \le c\big(1 + M_r(|f(\cdot)|^{p(\cdot)}(x))\big), \quad r > 0 \tag{4}$$

$$\big|M f(x)\big|^{p(x)} \le c\big(1 + M(|f(\cdot)|^{p(\cdot)}(x))\big). \tag{5}$$

*Proof.* Let $r \ge \frac{1}{2}$:

$$\big|M_r f(x)\big|^{p(x)} = \Big(\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)\cap\Omega} \big|f(y)\big|\, d\mu(y)\Big)^{p(x)} \le$$

$$\le \Big(\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} \big(1 + \big|f(y)\big|^{p(y)}\big)\, d\mu(y)\Big)^{p(x)}.$$

As we have seen while proving the Lemma 1 $\mu B(x,1) \ge k$. Hence,

$$\mu B(x,r) \ge \mu B\Big(x, \frac{1}{2}\Big) \ge c\mu B(x,1) \ge ck,$$

and

$$\big|M_r f(x)\big|^{p(x)} \le \Big(\frac{1}{ck} + 1\Big)^{p(x)} < \Big(\frac{1}{ck} + 1\Big)^{\overline{p}}.$$

Now let $0 < r < \frac{1}{2}$:

$$\big|M_r f(x)\big|^{p(x)} = \Big(\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)\cap\Omega} \big|f(y)\big|\, d\mu(y)\Big)^{p(x)} \le$$

$$\le \Big(\frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} \big|f(y)\big|^{\underline{p}_{B(x,r)}}\, d\mu(y)\Big)^{\frac{p(x)}{\underline{p}_{B(x,r)}}} \le$$

$$\leq \left( \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)\cap\Omega} \left(1 + \left|f(y)\right|^{p(y)}\right) d\mu(y) \right)^{\frac{p(x)}{\underline{p}_{B(x,r)}}} \leq$$

$$\leq \left(\mu B(x,r)\right)^{\frac{p(x)}{\underline{p}_B(x,r)}} \left( \int\limits_{B(x,r)\cap\Omega} \left|f(y)\right|^{p(y)} d\mu(y) + \mu B(x,r) \right)^{\frac{p(x)}{\underline{p}_{B(x,r)}}}.$$

Since $\mu B(x,r) \leq \mu B(x,1) \leq k$,

$$\int\limits_{B(x,r)\cap\Omega} \left|f(y)\right|^{p(y)} d\mu(y) + \mu B(x,r) \leq 1 + K.$$

Thus,

$$\left|M_r f(x)\right|^{p(x)} \leq$$

$$\leq \left(\mu B(x,r)\right)^{-\frac{p(x)}{p_{B(x,r)}}} \left(1+K\right)^{\frac{\overline{p}}{\underline{p}}} \left( \frac{1}{1+K} \int\limits_{B(x,y)\cap\Omega} \left|f(y)\right|^{p(y)} d\mu(y) + \frac{\mu B(x,r)}{1+K} \right)^{\frac{\overline{p}}{\underline{p}}} \leq$$

$$\leq \left(\mu B(x,r)\right)^{\frac{p(x)}{\underline{p}_{B(x,r)}}} \left(1+K\right)^{\frac{\overline{p}}{\underline{p}}} \left( \frac{1}{1+K} \int\limits_{B(x,y)\cap\Omega} \left|f(y)\right|^{p(y)} d\mu(y) + \frac{\mu B(x,r)}{1+K} \right) =$$

$$= \left(\mu B(x,r)\right)^{1-\frac{p(x)}{\underline{p}_{B(x,r)}}} \left(1+K\right)^{\frac{\overline{p}}{\underline{p}}-1} \left( \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)\cap\Omega} \left|f(y)\right|^{p(y)} d\mu(y) + 1 \right) \leq$$

$$\leq \left(\mu B(x,r)\right)^{\frac{\underline{p}_{B(x,r)} - \overline{p}_{B(x,r)}}{\overline{p}_{B(x,r)}}} (1+K)^{\frac{\overline{p}}{\underline{p}}-1} \left(1 + M_r\left(\left|f(\cdot)\right|^{p(\cdot)}(x)\right)\right).$$

By virtue of Lemma 2, $(\mu B(x,r))^{\underline{p}_B - \overline{p}_B} \leq c_1$. Hence,

$$\left(\mu B(x,r)\right)^{\frac{\underline{p}_B - \overline{p}_B}{\underline{p}_B}} \leq \max\left(1, c_1^{\frac{1}{\underline{p}}}\right)$$

and

$$\left(M_r f(x)\right)^{p(x)} \leq c\left(1 + M_r\left(\left|f(\cdot)\right|^{p(\cdot)}(x)\right)\right).$$

The inequality (4) is proved. Taking the supremum by $r$ we obtain (5). $\square$

*Proof of Theorem* 1. First of all we should mention that if $f \in L^{p(\cdot)}(\Omega)$ then $f \in L^1(\Omega)$, as $\Omega$ is bounded. Define a function $q$ by the quality $q(x) = \frac{p(x)}{\underline{p}}$. It is obvious that $1 \leq q(x) \leq p(x) \leq \overline{p} < \infty$ and there exists a positive number $A$, such that $\|f\|_{q(\cdot)} \leq A\|f\|_{p(\cdot)}$ for every $f \in L^{p(\cdot)}(\Omega)$. Let $\|f\|_{p(\cdot)} \leq \frac{1}{A}$. Then $\|f\|_{q(\cdot)} \leq 1$ and by virtue of Lemma 3

$$\left|Mf\right| = \left\|(Mf)^{q(\cdot)}\right\|_{L^{\underline{p}}(\Omega)}^{\underline{p}} \leq \left\|c\left(M\left(|f(\cdot)|^{q(\cdot)}\right) + 1\right)\right\|_{L^{\underline{p}}(\Omega)}^{\underline{p}} \leq$$

$$\leq c\Big(c'\left\|\left|f(\cdot)\right|^{q(\cdot)}\right\|_{L^{\underline{p}}(\Omega)} + \left\|1\right\|_{L^{\underline{p}}(\Omega)}\Big)^{\underline{p}} \leq c\Big(c'\left\|\left|f(\cdot)\right|^{q(\cdot)}\right\|_{L^{\underline{p}}(\Omega)} + \left\|1\right\|_{L^{\underline{p}}(\Omega)}\Big)^{\underline{p}} =$$

$$= c\big(c'\big(|f|_{p(\cdot)}\big)^{\frac{1}{\underline{p}}} + \|1\|_{L^{\underline{p}}(\Omega)}\big)^{\underline{p}} < c_1 < \infty.$$

Thus, $|Mf|_{p(\cdot)} < c_1$ when $|f|_{p(\cdot)} < \frac{1}{A}$. Then $|Mf|_{p(\cdot)} < c_2$ when $|f|_{p(\cdot)} < \frac{1}{A}$. But this means that the operator $M$ is bounded in $L^{p(\cdot)}$. The theorem is proved. □

**Theorem 2.** *Let $X$ be a homogenous type space, $p : X \to [1,\infty)$ satisfy the conditions* (1) *and* (2) *and $p$ be a constant outside some ball. Then the operator $M$ is bounded in $L^{p(\cdot)}(\Omega)$:*

$$\|Mf\|_{L^{p(\cdot)}(X)} < c\|f\|_{L^{\underline{p}(\cdot)}(x)}.$$

*Proof.* Let us suppose that $p(x) = p_0$ when $x \notin B_0 = B(x_0, R)$ and $B_1 = B(x_0, 2r)$. Let $\varphi(x) = f(x)1_{B_1}(x)$ and $\psi(x) = f(x)1_{X\setminus B_1}(x)$ . It is obvious that $Mf \le M\varphi + M\psi$. We are going to show that $|Mf|_{p(\cdot)} < c < \infty$ when $|f|_{p(\cdot)} \le 1$. We will do it separately for $|M\varphi|_{p(\cdot)}$ and $|M\psi|_{p(\cdot)}$. We start with $|M\varphi|_{p(\cdot)}$. Let $x \in B_1$. Then

$$M\varphi(x) = \sup_r \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} \big|\varphi(y)\big| d\mu(y) =$$

$$\sup_r \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)\cap B_1} \big|f(y)\big| d\mu(y) = M_{B_1} f(x)$$

and by virtue of Theorem 1

$$\int\limits_{B_1} \big(M\varphi(x)\big)^{p(x)} d\mu(x) = \int\limits_{B_1} \big(M_{B_1} f(x)\big)^{p(x)} d\mu(x) < c_1 \tag{6}$$

as $\|f\|_{p(\cdot),B_1} \le \|f\|_{p(\cdot)} \le 1$.

Now let $x \in X \setminus B_1$ and $B(x,r) \cap B_0 = \varnothing$:

$$M_r\varphi(x) = \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)\cap(B_1\setminus B_0)} \big|\varphi(y)\big| d\mu(y) =$$

$$= \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)} \big|f(y)\cdot 1_{B_1\setminus B_0}(y)\big| d\mu(y) \le M\big(f\cdot 1_{B_1\setminus B_0}\big)(x) \tag{7}$$

Now suppose that $x \in X \setminus B_1$ and $B(x,r) \cap B_0 \ne \varnothing$. It is not difficult to check that in this case $B_1 \subset B(x, 9r)$. Let $h = |f|_{B_1} \cdot 1_{B_1}$, where $|f|_{B_1} = \frac{1}{\mu B_1}\int_{B_1} |f| d\mu$. It is obvious that $h \in L^{p_0}(X)$ and

$$\|h\|_{L^{p_0}(x)} = \big(\mu B_1\big)^{\frac{1}{p_0}} |f|_{B_1} = \big(\mu B_1\big)^{\frac{1}{p_0}-1} \int\limits_{B_1} |f| d\mu(y) \le$$

$$\leq (\mu B_1)^{\frac{1}{p_0}-1}\Big(\int\limits_{B_1}|f|^{p_0}d\mu\Big)^{\frac{1}{p_0}}(\mu B_1)^{\frac{1}{q_0}} \leq \Big(\int\limits_{B_1}|f(y)|^{p(y)}d\mu(y)\Big)^{\frac{1}{p_0}} \leq 1. \quad (8)$$

Now we estimate $M_r\varphi(x)$:

$$M_r\varphi(x) = \frac{1}{\mu B(x,r)}\int\limits_{B(x,r)}|\varphi(y)|d\mu(y) \leq \frac{1}{\mu B(x,r)}\int\limits_{B_1}|f(y)|d\mu(y) \leq$$

$$\leq \frac{c}{\mu B(x,9r)}\int\limits_{B_1}|f(y)|d\mu(y) \leq \frac{c}{\mu B(x,9r)}\mu B_1\frac{1}{\mu B_1}\int\limits_{B_1}|f(y)|d\mu(y) =$$

$$= \frac{c}{\mu B(x,9r)}\int\limits_{B_1}|h(y)|d\mu(y) = \frac{c}{\mu B(x,9r)}\int\limits_{B(x,9r)}|h(y)|d\mu(y) \leq$$

$$\leq CMh(x). \quad (9)$$

From (7) and (9) follows that

$$M\varphi(x) \leq M\big(f\cdot 1_{B_1\setminus B_0}\big)(x) + cMh(x), \quad x \in X\setminus B_1.$$

Taking into consideration (8) we get:

$$\int\limits_{X\setminus B_1}\big(M\varphi(x)\big)^{p(x)}d\mu(x) = \int\limits_{X\setminus B_1}\big(M\varphi(x)\big)^{p_0}d\mu(x) \leq$$

$$\leq c\int\limits_{X\setminus B_1}\big(M\big(f\cdot 1_{B_1\setminus B_0}\big)(x)\big)^{p_0}d\mu(x) + c\int\limits_{X\setminus B_1}\big(Mh(x)\big)^{p_0}d\mu(x) \leq$$

$$\leq c\int\limits_{X\setminus B_1}\big(\big(f\cdot 1_{B_1\setminus B_0}\big)(x)\big)^{p_0}d\mu(x) + c\int\limits_{X\setminus B_1}\big(h(x)\big)^{p_0}d\mu(x) \leq$$

$$\leq c\int\limits_{X\setminus B_1}|f(x)|^{p(x)}d\mu(x) + c \leq c_2 < \infty. \quad (10)$$

Combining (6) and (10) we have:

$$\int\limits_{X}\big(M\varphi(x)\big)^{p(x)}d\mu(x) \leq c_1 + c_2 < \infty. \quad (11)$$

Now we start to estimate $|M\psi|_{p(\cdot)}$. Let $x \in B_0$. If $r < R$ then $B(x,r)\cap(X\setminus B_1) = \varnothing$ and, therefore, $M_r\psi(x) = 0$. So we can suppose that $r \geq R$. Then:

$$M_r\psi(x) = \frac{1}{\mu B(x,r)}\int\limits_{B(x,r)\cap(X\setminus B_1)}|f(y)|d\mu(y) \leq$$

$$\le \frac{1}{\mu B(x,r)} \int\limits_{B(x,r)\cap(X\setminus B_1)} \big(1+\big|f(y)\big|^{p_0}\big)\,d\mu(y) \le$$

$$\le \frac{\mu B(x,r)+1}{\mu B(x,r)} = 1+\frac{1}{\mu B(x,r)}.$$

As $x\in B_0$ and $r\ge R$, by virtue of Lemma 1 $\mu B(x,r)\ge k>0$ and

$$M\psi(x)\le 1+\frac{1}{k}=m,$$

which immediately gives

$$\int\limits_{B_0} \big(M\psi(x)\big)^{p(x)}\,d\mu(x) \le m^{\overline{p}}\mu B_0 = c_3<\infty. \tag{12}$$

As a last step we are going to estimate $\int_{X\setminus B_0} \big(M\psi(x)\big)^{p(x)}\,d\mu(x)$:

$$\int\limits_{X\setminus B_0} \big(M\psi(x)\big)^{p(x)}\,d\mu(x) = \int\limits_{X\setminus B_0} \big(M\psi(x)\big)^{p_0}\,d\mu(x) \le \int\limits_{X} \big(M\psi(x)\big)^{p_0}\,d\mu(x) \le$$

$$\le c\int\limits_{X} \big|\psi(x)\big)^{p_0}\,d\mu(x) \le c\int\limits_{X} \big|f(x)\big|^{p(x)}\,d\mu(x) \le c_4<\infty. \tag{13}$$

From (12) and (13) follows that

$$\int\limits_{X} \big(M\psi(x)\big)^{p(x)}\,d\mu(x) \le c_3+c_4 \tag{14}$$

and (11) and (14) gives

$$\int\limits_{X} \big(Mf(x)\big)^{p(x)}\,d\mu(x) \le$$

$$\le 2^{\overline{p}}\Bigg(\int\limits_{X} \big(M\varphi(x)\big)^{p(x)}\,d\mu(x) + \int\limits_{X} \big(M\psi(x)\big)^{p(x)}\,d\mu(x)\Bigg) \le$$

$$\le 2^{\overline{p}}(c_1+c_2+c_3+c_4)=c.$$

Thus, $|Mf|_{p(\cdot)}\le c$ when $\|f\|_{p(\cdot)}\le 1$, which signifies the boundedness of the operator $M$ and the proof of Theorem 2 is completed. $\square$

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