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TWO-WEIGHTED INEQUALITY FOR SINGULAR INTEGRALS IN LEBESGUE SPACES, ASSOCIATED WITH THE LAPLACE-BESSEL DIFFERENTIAL OPERATOR

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Abstract. In this paper, the authors establish several general theorems for the boundedness of singular integrals, associated with the Laplace-Bessel differential operator on a weighted Lebesgue space.

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1. Introduction

The classical Calderon and Zygmund singular integral operators are an important technical tool in harmonic analysis, theory of functions and partial differential equations. The maximal functions, singular integrals, potentials and related topics associated with the Laplace-Bessel differential operator

$$\Delta_{B_n} = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + B_n, \qquad B_n = \frac{\partial^2}{\partial x_n^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n}, \quad \gamma > 0,$$

which is known as an important differential operator in analysis and its applications, have been the research areas many mathematicans such as B. Muckenhoupt and E. Stein [22], I. Kipriyanov and M. Klyuchantsev [21, 20], K. Trimeche [30], L. Lyakhov [25, 26], K. Stemlak [27, 28], A. D. Gadjiev and I. A. Aliev [5, 6, 7, 8], V. S. Guliyev [9, 10, 11, 12] and others.

In this paper we consider the generalized shift operator, generated by Laplace-Bessel differential operator Δ_{B_n} by means of which singular integrals (B_n singular integral) are investigated.

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The singular integral operators that have been considered by Mihlin [23], and Calderon and Zygmund [3] are playing an important role in the theory Harmonic Analysis and in particular, in the theory partial differential equations. Klyuchantsev [20] and Kipriyanov and Klyuchantsev [21] have firstly introduced and investigated by the boundedness in L_p-spaces of multidimensional singular integrals generated by the Δ_{B_n} Laplace-Bessel differential operator (B_n singular integral). Aliev and Gadziev [8] have studied the boundedness of B_n singular integrals in weighted L_p-spaces with radial weights.

In this paper, the author establishes the boundedness of B_n singular integral operators in weighted L_p spaces on $\mathbb{R}^n_+$. Sufficient conditions on weighted functions ω and ω_1 are given so that B_n singular integral operator is bounded from the weighted Lebesgue spaces $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ into $L_{p,\omega_1,\gamma}(\mathbb{R}^n_+)$.

2. Notations and Background

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space, $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$ are vectors in $\mathbb{R}^n$, $(x, \xi) = x_1\xi_1 + \dots + x_n\xi_n$, $|x| = \sqrt{(x,x)}$. Let $\mathbb{R}^n_+ = \{x = (x_1, \dots x_n) : \ x_n > 0\}$, $\gamma > 0$. $E(x,r) = \{y \in R^n_+ : \ |x-y| < r\}$, $\Sigma_+ = \{x \in \mathbb{R}^n_+ : \ |x| = 1\}$.

For measurable set $E \subset \mathbb{R}^n_+$ let $|E|_\gamma = \int_E x_n^\gamma dx$, then $|E(0,r)|_\gamma = \omega(n,\gamma) r^{n+\gamma}$, where $\omega(n,\gamma) = |E(0,1)|_\gamma$.

An almost everywhere positive and locally integrable function $\omega : \ \mathbb{R}^n_+ \to \mathbb{R}$ will be called a weight. We shall denote by $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ the set of all measurable function f on $\mathbb{R}^n_+$ such that the norm

$$\|f\|_{L_{p,\omega,\gamma}} \equiv \|f\|_{p,\omega,\gamma} = \left(\int\limits_{\mathbb{R}^n_+} |f(x)|^p \omega(x) x_n^\gamma dx \right)^{1/p}, \qquad 1 \le p < \infty$$

is finite. For $\omega = 1$ the space $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ is denoted by $L_{p,\gamma}(\mathbb{R}^n_+)$, and the norm $\|f\|_{L_{p,\omega,\gamma}}$ by $\|f\|_{L_{p,\gamma}}$.

The operator of generalized shift (B_n shift operator) is defined by the following way (see [24], [20]):

$$T^y f(x) = C_\gamma \int\limits_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_n y_n \cos\alpha_n + y_n^2}\right) \sin^{\gamma-1} \alpha d\alpha,$$

where $C_\gamma = \pi^{-\frac{1}{2}} \Gamma\left(\gamma + \frac{1}{2}\right) \Gamma^{-1}(\gamma)$.

Note that this generalized shift operator is closely connected with Δ_{B_n} Laplace-Bessel differential operator (see [24], [20]).

The translation operator T^y generated the corresponding B_n-convolution

$$(f \otimes g)(x) = \int\limits_{\mathbb{R}^n_+} f(y)[T^y f(x)] y_n^\gamma dy,$$

for which the Young inequality

$$\|f \otimes g\|_{L_{r,\gamma}} \leq \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \leq p, q, r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

Lemma 1. [8] *Let $1 \leq p \leq \infty$. Then*

$$\|T^y f(\cdot)\|_{L_{p,\gamma}} \leq \|f\|_{L_{p,\gamma}}, \quad \forall y \in \mathbb{R}^n_+. \tag{1}$$

The main goal of this paper is to establish weighted L_p-estimates for the norms of the singular integral operator generated by a generalized shift operator (B_n singular integral operator) :

$$Tf(x) = p.v. \int\limits_{\mathbb{R}^n_+} \frac{\Omega(\theta)}{|y|^{n+\gamma}} [T^y f(x)] y_n^\gamma dy =$$

$$= \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^n_+ \setminus E(0,\varepsilon)} \frac{\Omega(\theta)}{|y|^{n+\gamma}} [T^y f(x)] y_n^\gamma dy = \lim_{\varepsilon \to 0} T_\varepsilon f(x), \tag{2}$$

where $\theta = y/|y|$, and the characteristic $\Omega(\theta)$ belong to some function space on the hemisphere $S_+ = \{x \in \mathbb{R}^n_+ \ : \ |x| = 1\}$ and satisfying the "cancellation" condition

$$\int\limits_{S_+} \Omega(\theta) \theta_n^\gamma d\sigma(\theta) = 0 \tag{3}$$

($d\sigma(\theta)$ is the area element of the sphere $|\theta| = 1$). The existence of the limit (2) for all $x \in \mathbb{R}^n_+$ and for Schwartz test functions $f(x)$ can be proved in the standard way if we take into account the well-known estimate $|T^y f(x) - f(x)| \leq c(x)|y|$.

The theorem below is known about the behavior of the B_n singular integral operator T in $L_{p,\gamma}$ (see [20, 21]).

Theorem 1. *Suppose that the characteristic $\Omega(\theta)$ of the B_n singular integral (2) satisfies the conditions*

$$\int\limits_{S_+} \Omega(\theta) \theta_n^\gamma d\sigma(\theta) = 0, \qquad \left(\int\limits_{S_+} |\Omega(\theta)|^q \theta_n^\gamma d\sigma(\theta) \right)^{\frac{1}{q}} = R_q < \infty \tag{4}$$

for some $q > 1$. Then

$$\|Tf\|_{L_{p,\gamma}} \leq CR_q \|f\|_{L_{p,\gamma}}, \qquad 1 < p < \infty.$$

3. Weighted Estimates For The B_n Singular Integral Operator

The aim of this paper is the following assertion about the behavior of the B_n singular integral operator (2) in weighted spaces.

We establish the boundedness in weighted L_p spaces for the B_n singular integrals.

Theorem 2. *Suppose that the characteristic $\Omega(\theta)$ of the B_n singular integral (2) satisfies the conditions*

$$\int_{S_+} \Omega(\theta)\theta_n^{\gamma} d\sigma(\theta) = 0, \qquad \sup_{\theta\in S_+} |\Omega(\theta)| < \infty. \tag{5}$$

Moreover, let $\omega(x)$, $\omega_1(x)$ be weight functions on $\mathbb{R}^n_+$ and the following three conditions are satisfied:

(a) *there exist $b > 0$ such that*

$$\sup_{|x|/4<|y|\le 4|x|} \omega_1(y) \le b\,\omega(x) \quad \textit{for a.e. } x \in \mathbb{R}^n_+,$$

$$(b) \quad \mathcal{A} \equiv \sup_{r>0} \Big(\int_{\mathbb{R}^n_+ \setminus E(0,2r)} \omega_1(x)|x|^{-(n+\gamma)p} x_n^{\gamma} dx \Big) \Big(\int_{E(0,r)} \omega^{1-p'}(x) x_n^{\gamma} dx \Big)^{p-1} < \infty,$$

$$(c) \quad \mathcal{B} \equiv \sup_{r>0} \Big(\int_{E(0,r)} \omega_1(x) x_n^{\gamma} dx \Big) \Big(\int_{\mathbb{R}^n_+ \setminus E(0,2r)} \omega^{1-p'}(x)|x|^{-(n+\gamma)p'} x_n^{\gamma} dx \Big)^{p-1} < \infty,$$

where $1 < p < \infty$, $pp' = p + p'$. Then,

i) *There exists a constant K_1, independent of f and ε such that for all $f \in L_{p,\omega}(\mathbb{R}^n_+)$*

$$\int_{\mathbb{R}^n_+} |T_\varepsilon f(x)|^p \omega_1(x) x_n^{\gamma} dx \le K_1 \int_{\mathbb{R}^n_+} |f(x)|^p \omega(x) x_n^{\gamma} dx. \tag{6}$$

ii) *The limit $\lim_{\varepsilon\to 0} T_\varepsilon f$, which will be denoted by Tf, exists in the sense of $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$, and*

$$\int_{\mathbb{R}^n_+} |Tf(x)|^p \omega_1(x) x_n^{\gamma} dx \le K_1 \int_{\mathbb{R}^n_+} |f(x)|^p \omega(x) x_n^{\gamma} dx. \tag{7}$$

Moreover, condition (a) can be replaced by the condition

(a') *there exist $b > 0$ such that*

$$\omega_1(x) \Big(\sup_{|x|/4\le|y|\le 4|x|} \frac{1}{\omega(y)} \Big) \le b \quad \textit{for a.e., } x \in \mathbb{R}^n.$$

Proof. We note that the coefficients c_k in the estimates below depend in general on the parameters n, p and γ, but not the function f and the parameter $\varepsilon > 0$. We first prove part $i)$ of the theorem. Part $ii)$ follows from part $i)$. Without loss of generality we assume that $f(x)$ is an infinitely differentiable function, because such functions are dense in $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$.

Note that

$$T_\varepsilon f(x) = \int\limits_{\mathbb{R}^n_+} \chi_\varepsilon(y) \frac{\Omega(y/|y|)}{|y|^{n+\gamma}} [T^y f(x)] y_n^\gamma dy = \int\limits_{\mathbb{R}^n_+} T^y \Big[\chi_\varepsilon(x) \frac{\Omega(x/|x|)}{|x|^{n+\gamma}}\Big] f(y) y_n^\gamma dy,$$

where χ_ε is the characteristic function of the set $\mathbb{R}^n_+ \setminus E_+(0,\varepsilon)$.

For simplicity we let $K_\varepsilon(x) = \chi_\varepsilon(x) \frac{\Omega(x/|x|)}{|x|^{n+\gamma}}$.

For $k \in Z$ we define $E_k = \{x \in R^n : 2^k < |x| \le 2^{k+1}\}$, $E_{k,1} = \{x \in R^n : |x| \le 2^{k-1}\}$, $E_{k,2} = \{x \in R^n : 2^{k-1} < |x| \le 2^{k+2}\}$, $E_{k,3} = \{x \in R^n : |x| > 2^{k+2}\}$. Then $E_{k,2} = E_{k-1} \cup E_k \cup E_{k+1}$ and the multiplicity of the covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Given $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$, we write

$$\begin{aligned}
|T_\varepsilon f(x)| &= \sum_{k\in Z} |T_\varepsilon f(x)| \chi_{E_k}(x) \le \sum_{k\in Z} |T_\varepsilon f_{k,1}(x)|\, \chi_{E_k}(x) + \\
&+ \sum_{k\in Z} |T_\varepsilon f_{k,2}(x)|\, \chi_{E_k}(x) + \sum_{k\in Z} |T_\varepsilon f_{k,3}(x)|\, \chi_{E_k}(x) \equiv \\
&\equiv T_{1,\varepsilon} f(x) + T_{2,\varepsilon} f(x) + T_{3,\varepsilon} f(x), \qquad (8)
\end{aligned}$$

where χ_{E_k} is the characteristic function of the set E_k, $f_{k,i} = f\chi_{E_{k,i}}$, $i = 1,2,3$.

First we shall estimate $\|T_{1,\varepsilon} f\|_{L_{p,\omega_1,\gamma}}$. Note that for $x \in E_k$, $y \in E_{k,1}$ we have $|y| \le 2^{k-1} \le |x|/2$. Moreover, $E_k \cap \operatorname{supp} f_{k,1} = \emptyset$ and $|x-y| \ge |x|/2$. Hence by (5)

$$\begin{aligned}
T_{1,\varepsilon} f(x) &\le K_0 \sum_{k\in Z} \Bigg(\int\limits_{\mathbb{R}^n_+} T^y |x|^{-n-\gamma}\ |f_{k,1}(y)| y_n^\gamma dy \Bigg) \chi_{E_k} \le \\
&\le K_0 \int\limits_{\{y\in\mathbb{R}^n_+ :\ |y|\le|x|/2\}} |x-y|^{-n-\gamma} |f(y)|\ y_n^\gamma dy \le \\
&\le 2^{n+\gamma} K_0 |x|^{-n-\gamma} \int\limits_{\{y\in\mathbb{R}^n_+ :\ |y|\le|x|/2\}} |f(y)|\ y_n^\gamma dy
\end{aligned}$$

for any $x \in E_k$, where $K_0 = \sup\limits_{\theta\in S_+} |\Omega(\theta)|$. Hence we have

$$\int\limits_{\mathbb{R}^n_+} \big|T_{1,\varepsilon} f(x)\big|^p \omega_1(x)\ x_n^\gamma dx \le$$

$$\le \left(2^{n+\gamma}K_0\right)^p \int\limits_{\mathbb{R}^n_+} \Bigg(\int\limits_{\{y\in\mathbb{R}^n_+:\ |y|<|x|/2\}} \big|f(y)\big|\ y_n^\gamma dy\Bigg)^p |x|^{-(n+\gamma)p}\omega_1(x)\ x_n^\gamma dx.$$

Since $\mathcal{A}<\infty$, the Hardy inequality

$$\int\limits_{\mathbb{R}^n_+} \omega_1(x)\big|x\big|^{-(n+\gamma)p}\Bigg(\int\limits_{\{y\in\mathbb{R}^n_+:\ |y|<|x|/2\}} \big|f(y)\big|\ y_n^\gamma dy\Bigg)^p\ x_n^\gamma dx \le$$

$$\le C\int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p\omega(x)\ x_n^\gamma dx$$

holds and $C\le c'\mathcal{A}$, where c' depends only on n and p. In fact the condition $\mathcal{A}<\infty$ is necessary and sufficient for the validity of this inequality (see [1], [16]). Hence, we obtain

$$\int\limits_{\mathbb{R}^n_+} |T_{1,\varepsilon}f(x)|^p\omega_1(x)\ x_n^\gamma dx \le c_1\int\limits_{\mathbb{R}^n_+} |f(x)|^p\omega(x)\ x_n^\gamma dx, \tag{9}$$

where c_1 is independent of f and ε.

Next we estimate $\|T_{3,\varepsilon}f\|_{L_{p,\omega_1,\gamma}}$. As is easy to verify, for $x\in E_k$, $y\in E_{k,3}$ we have $|y|>2|x|$ and $|x-y|\ge|y|/2$. Since $E_k\cap suppf_{k,3}=\emptyset$, for $x\in E_k$ by (5) we obtain

$$T_{3,\varepsilon}f(x)\le K_0 \int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>2|x|\}} T^y|x|^{-n-\gamma}|f(y)|\ y_n^\gamma dy \le$$

$$\le 2^{n+\gamma}K_0 \int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>2|x|\}} |f(y)||x-y|^{-n-\gamma}\ y_n^\gamma dy \le$$

$$\le 2^{n+\gamma}K_0 \int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>2|x|\}} |f(y)||y|^{-n-\gamma}\ y_n^\gamma dy.$$

Hence we have

$$\int\limits_{\mathbb{R}^n_+} |T_{3,\varepsilon}f(x)|^p\omega_1(x)\ x_n^\gamma dx \le$$

$$\le \left(2^{n+\gamma}K_0\right)^p \int\limits_{\mathbb{R}^n_+} \Bigg(\int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>|2x|\}} |f(y)||y|^{-n-\gamma}\ y_n^\gamma dy\Bigg)^p \omega_1(x)\ x_n^\gamma dx.$$

Since $\mathcal{B}<\infty$, the Hardy inequality

$$\int\limits_{\mathbb{R}^n_+} \omega_1(x)\Bigg(\int\limits_{\{y\in\mathbb{R}^n_+:\ |y|>|2x|\}} |f(y)||y|^{-n-\gamma}\ y_n^\gamma dy\Bigg)^p x_n^\gamma dx \le C\int\limits_{\mathbb{R}^n_+} |f(x)|^p\omega(x)x_n^\gamma dx$$

holds and $C \le c'\mathcal{B}$, where c' depends only on n and p. In fact the condition $\mathcal{B} < \infty$ is necessary and sufficient for the validity of this inequality (see [1], [16]). Hence, we obtain

$$\int_{\mathbb{R}^n_+} |T_{3,\varepsilon} f(x)|^p \omega_1(x)\ x_n^\gamma dx \le c_2 \int_{\mathbb{R}^n_+} |f(x)|^p \omega(x)\ x_n^\gamma dx, \tag{10}$$

where c_2 is independent of f and ε.

Finally, we estimate $\|T_{2,\varepsilon} f\|_{L_{p,\omega_1,\gamma}}$. By the $L_{p,\gamma}(\mathbb{R}^n_+)$ boundedness of T_ε and condition (a) we have

$$\int_{\mathbb{R}^n_+} |T_{2,\varepsilon} f(x)|^p \omega_1(x)\ x_n^\gamma dx = \int_{\mathbb{R}^n_+} \Big(\sum_{k\in Z} |T_\varepsilon f_{k,2}(x)|\, \chi_{E_k}(x)\Big)^p \omega_1(x) x_n^\gamma dx =$$

$$= \int_{\mathbb{R}^n_+} \Big(\sum_{k\in Z} |T_\varepsilon f_{k,2}(x)|^p\, \chi_{E_k}(x)\Big) \omega_1(x) x_n^\gamma dx = \sum_{k\in Z} \int_{E_k} |T_\varepsilon f_{k,2}(x)|^p\, \omega_1(x)\ x_n^\gamma dx \le$$

$$\le \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int_{\mathbb{R}^n_+} |T_\varepsilon f_{k,2}(x)|^p\ \ x_n^\gamma dx \le c_2 \sum_{k\in Z} \sup_{x\in E_k} \omega_1(x) \int_{\mathbb{R}^n_+} |f_{k,2}(x)|^p\ \ x_n^\gamma dx =$$

$$= c_2 \sum_{k\in Z} \sup_{y\in E_k} \omega_1(y) \int_{E_{k,2}} |f(x)|^p\ x_n^\gamma dx.$$

Since, for $x \in E_{k,2}$, $2^{k-1} < |x| \le 2^{k+2}$, we have by condition (a)

$$\sup_{y\in E_k} \omega_1(y) = \sup_{2^{k-1}<|y|\le 2^{k+2}} \omega_1(y) \le \sup_{|x|/4<|y|\le 4|x|} \omega_1(y) \le b\,\omega(x)$$

for almost all $x \in E_{k,2}$. Therefore

$$\int_{\mathbb{R}^n_+} |T_{2,\varepsilon} f(x)|^p\, \omega_1(x) x_n^\gamma dx \le$$

$$\le c_2 b \sum_{k\in Z} \int_{E_{k,2}} |f(x)|^p \omega(x) x_n^\gamma dx \le c_3 \int_{\mathbb{R}^n_+} |f(x)|^p \omega(x) x_n^\gamma dx, \tag{11}$$

where $c_3 = 3c_2 b$, since the multiplicity of covering $\{E_{k,2}\}_{k\in Z}$ is equal to 3.

Inequalities (3), (9), (10), (11) imply (2) which completes the proof of the first part of the theorem. Now let us proceed to the second part. We prove that the limit $\lim\limits_{\varepsilon\to 0} T_\varepsilon f = Tf$ exists in the sense of $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ and hence the estimate (7) hold for Tf. It suffices to prove that the limit exists for functions that have compact support, are smooth, and are even with respect to the variable x_n. Indeed, representing any function f in $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ in the form of a sum $f = f_1 + f_2$, where f_1 is a function that has compact support, is smooth, and is even with respect to x_n and f_2 is such that $\|f\|_{L_{p,\omega,\gamma}}$ is sufficiently small, we have from the equality $T_\varepsilon f = T_\varepsilon f_1 + T_\varepsilon f_2$ and (6) that

$$\|T_\varepsilon f - T_\varepsilon f_1\|_{L_{p,\omega_1,\gamma}} \le c\|f\|_{L_{p,\omega,\gamma}} \le \delta,$$

where δ is a sufficiently small number.

Therefore, it suffices to prove the existence of the limit $\lim\limits_{\varepsilon\to 0} T_\varepsilon f = Tf$ (in the sense of $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$) for smooth compacly supported functions that is even with respect to the variable x_n. Taking $f(x)$ as such a function and using the "cancellation" condition (5), we have

$$T_{\varepsilon_2} f(x) - T_{\varepsilon_1} f(x) = \int\limits_{\{y\in\mathbb{R}^n_+ : \varepsilon_1<|y|<\varepsilon_2\}} \frac{\Omega(y/|y|)}{|y|^{n+\gamma}} [T^y f(x)] y_n^\gamma dy =$$

$$= \int\limits_{\{y\in\mathbb{R}^n_+ : \varepsilon_1<|y|<\varepsilon_2\}} \frac{\Omega(y/|y|)}{|y|^{n+\gamma}} \left[T^y f(x) - f(x)\right] y_n^\gamma dy,$$

where $x \in \mathbb{R}^n_+$.

By using the Taylor-Delsarte formula [24] for $T^y f(x)$ is not hard to show that

$$\|T^y f(x) - f(x)\|_{L_{p,\omega,\gamma}} \le c|y|.$$

Therefore,

$$\|T_{\varepsilon_2} f - T_{\varepsilon_1} f\|_{L_{p,\omega_1,\gamma}} \le \int\limits_{\{y\in\mathbb{R}^n_+ : \varepsilon_1<|y|<\varepsilon_2\}} \frac{c|y|}{|y|^{n+\gamma}} y_n^\gamma dy \le$$

$$= c(\varepsilon_2 - \varepsilon_1).$$

Since the space $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ is complete, this implies that the limit $\lim\limits_{\varepsilon\to 0} T_\varepsilon f = Tf$ exists and belongs to $L_{p,\omega,\gamma}(\mathbb{R}^n_+)$. Thus the proof is complete. □

Theorem 3. *Suppose that the characteristic $\Omega(\theta)$ of the B_n singular integral* (2) *satisfies the conditions* (5). *Moreover, let $p \in (1,\infty)$, $\omega(t)$ be a weight function on $(0,\infty)$, $\omega_1(t)$ be a positive increasing function on $(0,\infty)$ and $\omega(|x|)$, $\omega_1(|x|)$ be satisfied the conditions (a), (b).*

Then there exists a constant $c > 0$, such that for all $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$

$$\int\limits_{\mathbb{R}^n_+} \left|Tf(x)\right|^p \omega_1\big(|x|\big) x_n^\gamma dx \le c \int\limits_{\mathbb{R}^n_+} \left|f(x)\right|^p \omega\big(|x|\big) x_n^\gamma dx. \qquad (12)$$

Proof. Suppose that $f \in L_{p,\omega,\gamma}(\mathbb{R}^n_+)$ and ω_1 are positive increasing functions on $(0,\infty)$ and $\omega(|x|)$, $\omega_1(|x|)$ be satisfied the conditions (a), (b).

Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(0+) + \int\limits_0^t \psi(\lambda)d\lambda,$$

where $\omega_1(0+) = \lim_{t\to 0}\omega_1(t)$ and $\omega_1(t) \geq 0$ on $(0,\infty)$. In fact there exists a sequence of increasing absolutely continuous fuctions ϖ_n, such that $\varpi_n(t) \leq \omega_1(t)$ and $\lim\limits_{n\to\infty} \varpi_n(t) = \omega_1(t)$ for any $t \in (0,\infty)$ (see [4],[13] for details).

We have

$$\int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p \omega_1\big(|x|\big) x_n^\gamma dx = \omega_1(0+) \int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p x_n^\gamma dx +$$

$$+ \int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p \bigg(\int\limits_0^{|x|} \psi(\lambda)d\lambda \bigg) x_n^\gamma dx = J_1 + J_2.$$

If $\omega_1(0+) = 0$, then $J_1 = 0$. If $\omega_1(0+) \neq 0$ by the boundedness of T in $L_{p,\gamma}(\mathbb{R}^n_+)$ thanks to (a)

$$J_1 \leq \|T\|^p \omega_1(0+) \int\limits_{\mathbb{R}^n_+} |f(x)|^p x_n^\gamma dx \leq$$

$$\leq \|T\|^p \int\limits_{\mathbb{R}^n_+} |f(x)|^p \omega_1(|x|) x_n^\gamma dx \leq b\,\|T\|^p \int\limits_{\mathbb{R}^n_+} |f(x)|^p \omega(|x|) x_n^\gamma dx.$$

After changing the order of integration in J_2 we have

$$J_2 = \int\limits_0^\infty \psi(\lambda) \bigg(\int\limits_{|x|>\lambda} \big|Tf(x)\big|^p x_n^\gamma dx \bigg) d\lambda \leq$$

$$\leq 2^{p-1} \int\limits_0^\infty \psi(\lambda) \bigg(\int\limits_{|x|>\lambda} \big|T(f\chi_{\{|x|>\lambda/2\}})(x)\big|^p x_n^\gamma dx + \int\limits_{|x|>\lambda} \big|T(f\chi_{\{|x|\leq\lambda/2\}})(x)\big|^p x_n^\gamma dx \bigg) d\lambda =$$

$$= J_{21} + J_{22}.$$

Using the boundeedness of T in $L_{p,\gamma}(\mathbb{R}^n_+)$ and condition (a) we have

$$J_{21} \leq \big\|T\big\|^p \int\limits_0^\infty \psi(t) \bigg(\int\limits_{\{y\in\mathbb{R}^n_+ : |y|>\lambda/2\}} \big|f(y)\big|^p y_n^\gamma dy \bigg) dt =$$

$$= \big\|T\big\|^p \int\limits_{\mathbb{R}^n_+} \big|f(y)\big|^p \bigg(\int\limits_0^{2|y|} \psi(\lambda)d\lambda \bigg) y_n^\gamma dy \leq$$

$$\le \|T\|^p \int_{\mathbb{R}^n_+} |f(y)|^p \omega_1(2|y|) y_n^\gamma dy \le b\,\|T\|^p \int_{\mathbb{R}^n_+} |f(y)|^p \omega(|y|) y_n^\gamma dy.$$

Let us estimate J_{22}. For $|x| > \lambda$ and $|y| \le \lambda/2$ we have $|x|/2 \le |x-y| \le 3|x|/2$, and so

$$J_{22} \le c_4 \int_0^\infty \psi(\lambda) \left(\int_{\{x\in\mathbb{R}^n_+ : |x|>\lambda\}} \left(\int_{\{y\in\mathbb{R}^n_+ : |y|\le\lambda/2\}} T^y |x|^{-n-\gamma} |f(y)| y_n^\gamma dy \right)^p x_n^\gamma dx \right) d\lambda \le$$

$$\le c_5 \int_0^\infty \psi(\lambda) \left(\int_{\{x\in\mathbb{R}^n_+ : |x|>\lambda\}} \left(\int_{\{y\in\mathbb{R}^n_+ : |y|\le\lambda/2\}} |f(y)| y_n^\gamma dy \right)^p |x|^{-(n+\gamma)p} x_n^\gamma dx \right) d\lambda =$$

$$= c_6 \int_0^\infty \psi(\lambda) \lambda^{-(n+\gamma)(p-1)} \left(\int_{\{y\in\mathbb{R}^n_+ : |y|\le\lambda/2\}} |f(y)| y_n^\gamma dy \right)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda) \lambda^{-(n+\gamma)(p-1)} \left(\int_{\{y\in\mathbb{R}^n_+ : |y|\le\lambda/2\}} |f(y)| y_n^\gamma dy \right)^p d\lambda \le$$

$$\le C \int_{\mathbb{R}^n_+} |f(y)|^p \omega(|y|) y_n^\gamma dy$$

for $p \in (1,\infty)$ is characterized by the condition $C \le c'\mathcal{A}'$ ([1], [16], see also [2], [17]), where

$$\mathcal{A}' \equiv \sup_{\tau>0} \left(\int_{2\tau}^\infty \psi(t) t^{-(n+\gamma)(p-1)} d\tau \right) \left(\int_{E(0,\tau)} \omega^{1-p'}(x) x_n^\gamma dx \right)^{p-1} < \infty.$$

Note that

$$\int_{2t}^\infty \psi(\tau) \tau^{-(n+\gamma)(p-1)} d\tau =$$

$$= (n+\gamma)(p-1) \int_{2t}^\infty \psi(\tau) d\tau \int_\tau^\infty \lambda^{-1-(n+\gamma)(p-1)} d\lambda =$$

$$= (n+\gamma)(p-1) \int_{2t}^\infty \lambda^{-1-(n+\gamma)(p-1)} d\lambda \int_{2t}^\lambda \psi(\tau) d\tau \le$$

$$\leq (n+\gamma)(p-1)\int\limits_{2t}^{\infty}\lambda^{-1-(n+\gamma)(p-1)}\omega_1(\lambda)d\lambda =$$

$$= \frac{(n+\gamma)(p-1)}{\omega(n,\gamma)}\int\limits_{\mathbb{R}^n_+\setminus E(0,2t)}\omega_1(|x|)|x|^{-(n+\gamma)p}x_n^{\gamma}dx.$$

Condition (b) of the theorem guarantees that $\mathcal{A}' \leq \frac{(n+\gamma)(p-1)}{\omega(n,\gamma)}\mathcal{A} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq c_7\int\limits_{\mathbb{R}^n_+}\big|f(x)\big|^p\omega\big(|x|\big)x_n^{\gamma}dx.$$

Combining the estimates of J_1 and J_2, we get (12) for $\omega_1(t) = \omega_1(0+) + \int\limits_0^t\psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (12). The theorem is proved. □

Example 1. *Let*

$$\omega(t) = \begin{cases} t^{(n+\gamma)(p-1)}\ln^p\frac{1}{t}, & for \quad t\in\left(0,\frac{1}{2}\right) \\ \left(2^{\beta-p+1}\ln^p 2\right)t^{\beta}, & for \quad t\in\left[\frac{1}{2},\infty\right) \end{cases},$$

$$\omega_1(t) = \begin{cases} t^{(n+\gamma)(p-1)}, & for \quad t\in\left(0,\frac{1}{2}\right) \\ 2^{\alpha-p+1}t^{\alpha}, & for \quad t\in\left[\frac{1}{2},\infty\right) \end{cases},$$

where $0 < \alpha \leq \beta < (n+\gamma)(p-1)$. *Then the pair* (ω,ω_1) *satisfies the condition of Theorem* 3.

Theorem 4. *Suppose that the characteristic* $\Omega(\theta)$ *of the* B_n *singular integral* (2) *satisfies the conditions* (5). *Moreover, let* $p\in(1,\infty)$, $\omega(t)$ *be a weight function on* $(0,\infty)$, $\omega_1(t)$ *be a positive decreasing function on* $(0,\infty)$ *and* $\omega(|x|)$, $\omega_1(|x|)$ *be satisfied the conditions* (a), (c). *Then inequality* (12) *is valid.*

Proof. Without loss of generality we can suppose that ω_1 may be represented by

$$\omega_1(t) = \omega_1(+\infty) + \int\limits_t^{\infty}\psi(\tau)d\tau,$$

where $\omega_1(+\infty) = \lim\limits_{t\to\infty}\omega_1(t)$ and $\omega_1(t)\geq 0$ on $(0,\infty)$. In fact there exists a sequence of decreasing absolutely continuous fuctions ϖ_n such that $\varpi_n(t)\leq\omega_1(t)$ and $\lim\limits_{n\to\infty}\varpi_n(t) = \omega_1(t)$ for any $t\in(0,\infty)$ (see [4],[13] for details).

We have

$$\int\limits_{\mathbb{R}^n_+}\big|Tf(x)\big|^p\omega_1\big(|x|\big)x_n^{\gamma}dx = \omega_1(+\infty)\int\limits_{\mathbb{R}^n_+}\big|Tf(x)\big|^p x_n^{\gamma}dx +$$

$$+\int\limits_{\mathbb{R}^n_+} \big|Tf(x)\big|^p \bigg(\int\limits_{|x|}^{\infty} \psi(\tau)d\tau\bigg) x_n^\gamma dx = I_1 + I_2.$$

If $\omega_1(+\infty) = 0$, then $I_1 = 0$. If $\omega_1(+\infty) \neq 0$, by the boundedness of T in $L_{p,\gamma}(\mathbb{R}^n_+)$ and condition (a) we have

$$J_1 \le \|T\| \omega_1(+\infty) \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p x_n^\gamma dx \le$$

$$\le \|T\| \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega_1\big(|x|\big) x_n^\gamma dx \le b\,\|T\| \int\limits_{\mathbb{R}^n_+} \big|f(x)\big|^p \omega(|x|) x_n^\gamma dx.$$

After changing the order of integration in J_2 we have

$$J_2 = \int\limits_0^\infty \psi(\lambda) \bigg(\int\limits_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}} \big|Tf(x)\big|^p x_n^\gamma dx \bigg) d\lambda \le$$

$$\le 2^{p-1} \int\limits_0^\infty \psi(\lambda) \bigg(\int\limits_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}} \big|T(f\chi_{\{|x|<2\lambda\}})(x)\big|^p x_n^\gamma dx +$$

$$+ \int\limits_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}} \big|T(f\chi_{\{|x|\ge 2\lambda\}})(x)\big|^p x_n^\gamma dx \bigg) d\lambda = J_{21} + J_{22}.$$

Using the boundeedness of T in $L_p(\mathbb{R}^n)$ and condition (a) we obtain

$$J_{21} \le \|T\| \int\limits_0^\infty \psi(t) \bigg(\int\limits_{|y|<2\lambda} |f(y)|^p y_n^\gamma dy \bigg) dt =$$

$$= \|T\| \int\limits_{\mathbb{R}^n_+} |f(y)|^p \bigg(\int\limits_{|y|/2}^\infty \psi(\lambda)d\lambda \bigg) y_n^\gamma dy \le$$

$$\le \|T\| \int\limits_{\mathbb{R}^n_+} |f(y)|^p \omega_1(|y|/2) y_n^\gamma dy \le b\,\|T\| \int\limits_{\mathbb{R}^n_+} |f(y)|^p \omega(|y|) y_n^\gamma dy.$$

Let us estimate J_{22}. For $|x| < \lambda$ and $|y| \ge 2\lambda$ we have $|y|/2 \le |x-y| \le 3|y|/2$, and so

$$J_{22} \le$$

$$\le c_8 \int\limits_0^\infty \psi(\lambda) \bigg(\int\limits_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}} \bigg(\int\limits_{\{y\in\mathbb{R}^n_+:\ |y|\ge 2\lambda\}} T^y |x|^{-n-\gamma} \big|f(y)\big| y_n^\gamma dy \bigg)^p x_n^\gamma dx \bigg) d\lambda \le$$

$$\leq 2^n c_8 \int_0^\infty \psi(\lambda)\bigg(\int_{\{x\in\mathbb{R}^n_+:\ |x|<\lambda\}} \bigg(\int_{\{y\in\mathbb{R}^n_+:\ |y|\geq 2\lambda\}} \big|y\big|^{-n-\gamma}\big|f(y)\big|y_n^\gamma dy\bigg)^p x_n^\gamma dx\bigg) d\lambda =$$

$$= c_9 \int_0^\infty \psi(\lambda)\lambda^{n+\gamma}\bigg(\int_{\{y\in\mathbb{R}^n_+:\ |y|\geq 2\lambda\}} \big|y\big|^{-n-\gamma}\big|f(y)\big|y_n^\gamma dy\bigg)^p d\lambda.$$

The Hardy inequality

$$\int_0^\infty \psi(\lambda)\lambda^{n+\gamma}\bigg(\int_{\{y\in\mathbb{R}^n_+:\ |y|\geq 2\lambda\}} \big|y\big|^{-n-\gamma}\big|f(y)\big|y_n^\gamma dy\bigg)^p d\lambda \leq$$

$$\leq C \int_{\mathbb{R}^n_+} \big|f(y)\big|^p \omega\big(|y|\big) y_n^\gamma dy$$

for $p \in (1,\infty)$ is characterized by the condition $C \leq c\mathcal{B}'$ ([1], [16], see also [2], [17]), where

$$\mathcal{B}' \equiv \sup_{\tau>0}\bigg(\int_0^\tau \psi(t)t^{n+\gamma}d\tau\bigg)\bigg(\int_{\mathbb{R}^n_+\setminus E(0,2\tau)} \omega^{1-p'}\big(|x|\big)\big|x\big|^{-(n+\gamma)p'} x_n^\gamma dx\bigg)^{p-1} < \infty.$$

Note that

$$\int_0^\tau \psi(t)t^{n+\gamma}dt = (n+\gamma)\int_0^\tau \psi(t)dt \int_0^t \lambda^{n+\gamma-1}d\lambda =$$

$$= (n+\gamma)\int_0^\tau \lambda^{n+\gamma-1}d\lambda \int_\lambda^t \psi(\tau)d\tau \leq (n+\gamma)\int_0^\tau \lambda^{n+\gamma-1}\omega(\lambda)d\lambda =$$

$$= \frac{n+\gamma}{\omega(n,\gamma)} \int_{E(0,\tau)} \omega_1(|x|)x_n^\gamma dx,$$

Condition (c) of the theorem guarantees that $\mathcal{B}' \leq \frac{n+\gamma}{\omega(n,\gamma)}\mathcal{B} < \infty$. Hence, applying the Hardy inequality, we obtain

$$J_{22} \leq c_{10} \int_{\mathbb{R}^n_+} |f(x)|^p \omega(|x|) x_n^\gamma dx.$$

Combining the estimates of J_1 and J_2, we get (12) for $\omega_1(t) = \omega_1(+\infty) + \int_t^\infty \psi(\tau)d\tau$. By Fatou's theorem on passing to the limit under the Lebesgue integral sign, this implies (12). The theorem is proved. □

Example 2. *Let*

$$\omega(t) = \begin{cases} \frac{1}{t^{n+\gamma}} \ln^{\nu} \frac{1}{t}, & for \quad t < d \\ \left(d^{-n-\gamma-\alpha} \ln^{\nu} \frac{1}{d}\right) t^{\alpha}, & for \quad t \geq d \end{cases},$$

$$\omega_1(t) = \begin{cases} \frac{1}{t^{n+\gamma}} \ln^{\beta} \frac{1}{t}, & for \quad t < d \\ \left(d^{-n-\gamma-\lambda} \ln^{\beta} \frac{1}{d}\right) t^{\lambda}, & for \quad t \geq d \end{cases},$$

where $\beta < \nu \leq 0$, $-n-\gamma < \lambda < \alpha < 0$, $d = e^{\frac{\beta}{n+\gamma}}$. *Then the pair* (ω, ω_1) *satisfies the condition of Theorem* 4.

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