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ON CERTAIN DOUBLE SIZED INTEGRAL OPERATORS OVER MULTIDIMENSIONAL CONES

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Abstract. We study certain double sized multidimensional integral operators where the two integrals are taken over cones, respectively, in $\mathbb{R}^M$ and $\mathbb{R}^N$. The boundedness and, in some cases, the compactness of these operators are studied between weighted Lebesgue spaces.

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1. Introduction

In 1920, Hardy [6] studied the boundedness of the Hardy operator

$$(Hf)(x) = \int_0^x f(t)\,dt$$

between suitable weighted Lebesgue spaces. During the last two decades, the corresponding Hardy inequality has drawn the attention of many mathematicians and tremendous amount of work has been done on this subject. A collection of such work can be found in the book [17] and also in a very recent book [16].

One of the directions in which this subject is progressing is to consider the boundedness of the Hardy operator in higher dimensional setting. Many people have contributed in this direction. Drábek, Heinig and Kufner [4] have considered the operator

$$(H_N f)(x) = \int_{B_N(x)} f(t)\,dt,$$

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where $B_N(x)$, $x \in \mathbb{R}^N$, is an N-dimensional ball centered at origin and with radius $|x|$.

Next, the two dimensional Hardy operator

$$(H_2f)(x,y) = \int_0^x \int_0^y f(s,t)\,dt\,ds$$

needs special attension. The boundedness of this operator between weighted Lebesgue spaces (with usual norm) has been characterized by Sawyer [18] (see subsection 2.1 below). As a variant, Appell and Kufner [1] considered Lebesgue spaces with mixed norm and obtained necessary and sufficient conditions for the boundedness of H_2. Very recently, the authors [12] have given the same treatment as Appell and Kufner to two different operators, namely, the Hardy-Steklov operator $(T_2f)(x,y) = \int_{a(x)}^{b(x)} \int_{a(y)}^{b(y)} f(s,t)\,dt\,ds$ and the geometric mean operator $(G_2f)(x,y) = exp\big(\frac{1}{xy}\int_0^x\int_0^y \ln f(s,t)\,dt\,ds\big)$.

Towards the further generalization of the subject, Heinig, Kufner and Persson [8] studied the so called double-sized multidimensional Hardy operator

$$(H_{M,N}f)(x,y) = \int_{B_M(x)} \int_{B_N(y)} f(s,t)\,dt\,ds, \quad x,s \in \mathbb{R}^M, \quad y,t \in \mathbb{R}^N.$$

In the present paper, we further investigate the operators of the type $H_{M,N}$. In fact, we consider the operators in which the integrals are taken over certain "cones" in $\mathbb{R}^N$ and note (*c.f. Remark 1*) that these operators are more general than $H_{M,N}$. Apart from studying the boundedness of these operators, we also study the compactness in certain cases. The paper is organized as follows.

In Section 2, we define and study a double sized multidimensional operator

$$(H_{E,F}f)(x,y) = \int_{S_{M_x}} \int_{S_{N_y}} f(s,t)\,dt\,ds, \quad x,s \in E, \quad y,t \in F,$$

where E, F are cones in $\mathbb{R}^M$, $\mathbb{R}^N$ and S_{M_x}, S_{N_y} are parts of E, F with "radii" not greater than $|x|$, $|y|$, respectively. In fact, we characterize the boundedness of $H_{E,F}$ in terms of the boundedness of H_2 and then, to obtain the precise criterion, we apply Sawyer's result. As a corollary, the boundedness of $H_{M,N}$ obtained by Heinig, Kufner and Persson [8] is achieved. Section 3 is devoted to the case of mixed norms. Here two other double sized multidimensional operators, namely, $T_{E,F}$ and $G_{E,F}$ are defined and for the boundedness of these operators, necessary as well as sufficient conditions are obtained. The compactness of the operator $T_{E,F}$ in Lebesgue spaces with mixed norm is studied in Section 4.

Throughout, we assume without any loss of generality that the functions involved (f, g, h, f_n, g_n or h_n in this paper) are non-negative. However, for the case of geometric mean operators G_2, $G_{E,F}$, we avoid the situation where the functions take value zero since the operators are not defined thereat.

2. Boundedness of $H_{E,F}$

2.1. **Sawyer's Result.** The following result of Sawyer [18] characterizes the boundedness of the two dimensional operator $H_2 : L^p((0,\infty)\times(0,\infty), W) \to L^q((0,\infty)\times(0,\infty), W_0)$ defined by

$$(H_2 g)(x,y) = \int_0^x \int_0^y g(s,t)\, dt\, ds.$$

Theorem *A. Let $1 < p \leq q < \infty$ and W, W_0 be weight functions on $(0,\infty)\times(0,\infty)$. Then the inequality*

$$\Big(\int_0^\infty \int_0^\infty W_0(x,y)(H_2 g)^q(x,y)\, dy\, dx\Big)^{\frac{1}{q}} \leq$$
$$\leq C\Big(\int_0^\infty \int_0^\infty W(x,y)\, g^p(x,y)\, dy\, dx\Big)^{\frac{1}{p}}, \quad g \geq 0, \tag{1}$$

holds for some constant $C > 0$ if and only if the following three conditions are satisfied:

$$C_0 \coloneqq \sup_{x>0,y>0} \Big(\int_x^\infty \int_y^\infty W_0(s,t)\, dt\, ds\Big)^{\frac{1}{q}} \Big(\int_0^x \int_0^y W^{1-p'}(s,t)\, dt\, ds\Big)^{\frac{1}{p'}} < \infty \tag{2}$$

$$\int_0^x \int_0^y \Big(\int_0^s \int_0^t W^{1-p'}(\rho,\tau)\, d\tau\, d\rho\Big)^q W_0(s,t)\, dt\, ds \leq$$
$$\leq C_0^q \Big(\int_0^x \int_0^y W^{1-p'}(s,t)\, dt\, ds\Big)^{\frac{q}{p}} \tag{3}$$

and

$$\int_x^\infty \int_y^\infty \Big(\int_s^\infty \int_t^\infty W_0(\rho,\tau)\, d\tau\, d\rho\Big)^{p'} W^{1-p'}(s,t)\, dt\, ds \leq$$

$$\leq C_0^{p'} \left(\int_x^\infty \int_y^\infty W_0(s,t)\; dt\, ds \right)^{\frac{p'}{q'}}, \tag{4}$$

where p' and q' are conjugate exponents to p and q, respectively.

Towards the first aim of this paper, we characterize the boundedness of a higher dimensional analogue of the operator H_2. This is the so called "double sized multidimensinal operator" which is obtained by replacing the two integrals in H_2 by integrals over certain 'cones' in $\mathbb{R}^M$ and $\mathbb{R}^N$.

2.2. **A double sized multidimensional operator.** Let Σ^{M-1} be the unit ball in $\mathbb{R}^M$, that is, $\Sigma^{M-1} = \{x \in \mathbb{R}^M : |x| = 1\}$. Let B_M be a measurable subset of Σ^{M-1} and $E \subset \mathbb{R}^M$ be a spherical cone, that is,

$$E = \big\{x \in \mathbb{R}^M : x = s\sigma,\, 0 \leq s < \infty,\, \sigma \in B_M\big\}.$$

Let S_{M_x}, $x \in \mathbb{R}^M$ denote the part of E with 'radius'$\leq |x|$, i.e.,

$$S_{M_x} = \big\{y \in \mathbb{R}^M : y = s\sigma,\, 0 \leq s < |x|,\, \sigma \in B_M\big\}.$$

Further, we denote by $\alpha S_M, \alpha > 0$, the part of E with 'radius' $\leq \alpha$. Note that $E = \underset{\alpha>0}{\cup} \alpha S_M$. For $x \in E \setminus \{0\}$, we denote by $|S_{M_x}|$, the volume of S_{M_x}. The symbols B_N, F, S_{N_y} and $|S_{N_y}|$ are defined similarly for an N-dimensional setting.

We, now, define a double sized multidimensional operator

$$H_{E,F} : L^p(E \times F, w) \to L^q(E \times F, w_0)$$

by

$$(H_{E,F}f)(x,y) = \int_{S_{M_x}} \int_{S_{N_y}} f(s,t)\; dt\, ds, \quad x \in E, \quad y \in F.$$

2.3. **The Main Results.** Now, we prove the following result which characterizes the boundedness of the operator $H_{E,F}$. In fact, the characterization is obtained in terms of the boundedness of the operator H_2.

Theorem 1. *Let $1 < p,\, q < \infty$, E, F, S_{M_x}, S_{Ny} be as defined above and w_0, w be weight functions on $E \times F$. Then the inequality*

$$\left(\int_E \int_F w_0(x,y)\, (H_{E,F}f)^q(x,y)\; dy\, dx \right)^{\frac{1}{q}} \leq$$

$$\leq C \left(\int_E \int_F w(x,y)\, f^p(x,y)\; dy\, dx \right)^{\frac{1}{p}}, \quad f \geq 0, \tag{5}$$

holds if and only if (1) *holds with*

$$W_0(x_0,y_0)=\int\limits_{B_M}\int\limits_{B_N} w_0(x_0x',y_0y')x_0^{M-1}y_0^{N-1}\,dy'\,dx',\quad x_0,y_0>0,\tag{6}$$

$$W(x_0,y_0)=\Bigg(\int\limits_{B_M}\int\limits_{B_N} w^{1-p'}(x_0x',y_0y')x_0^{M-1}y_0^{N-1}\,dy'\,dx'\Bigg)^{1-p},\tag{7}$$

$$x_0,\ \ y_0>0.$$

Moreover, the constants in (1) *and* (5) *are the same.*

Proof. Suppose (1) holds. Fix a locally integrable function $f:E\times F\to\mathbb{R}$. Define

$$g(x_0,y_0)=\int\limits_{B_M}\int\limits_{B_N} f(x_0x',y_0y')x_0^{M-1}y_0^{N-1}\,dy'\,dx',\quad x_0,y_0>0\tag{8}$$

so that on making the changes of variables $x=x_0x'$ where $x\in E$, $x_0=|x|\in(0,\infty)$, $x'\in B_M$ and similarly $y=y_0y'$, $s=s_0s'$, $t=t_0t'$, we have

$$\begin{aligned}(H_{E,F}f)(x,y)&=\int\limits_{S_{M_x}}\int\limits_{S_{N_y}} f(s,t)\,dt\,ds=\\ &=\int\limits_0^{x_0}\int\limits_0^{y_0}\int\limits_{B_M}\int\limits_{B_N} f(s_0s',t_0t')s_0^{M-1}t_0^{N-1}\,dt'\,ds'\,dt_0\,ds_0=\\ &=\int\limits_0^{x_0}\int\limits_0^{y_0} g(s_0,t_0)\,dt_0\,ds_0=(H_2g)(x_0,y_0).\end{aligned}\tag{9}$$

Now making variable transformation $x=x_0x'$, $y=y_0y'$, using (6), (9) and then using (1), we have

$$\Bigg(\int\limits_E\int\limits_F w_0(x,y)(H_{E,F}f)^q(x,y)\,dy\,dx\Bigg)^{\frac{1}{q}}=$$

$$=\Bigg(\int\limits_0^{\infty}\int\limits_0^{\infty}\int\limits_{B_M}\int\limits_{B_N} w_0(x_0x',y_0y')(H_2g)^q(x_0,y_0)x_0^{M-1}y_0^{N-1}\,dy'\,dx'dy_0\,dx_0\Bigg)^{\frac{1}{q}}=$$

$$=\Bigg(\int\limits_0^{\infty}\int\limits_0^{\infty} W_0(x_0,y_0)(H_2g)^q(x_0,y_0)\,dy_0\,dx_0\Bigg)^{\frac{1}{q}}\le$$

$$\le C\Bigg(\int\limits_0^{\infty}\int\limits_0^{\infty} W(x_0,y_0)g^p(x_0,y_0)\,dy_0\,dx_0\Bigg)^{\frac{1}{p}}.$$

Next, we use (8), apply Hölders inequality for the inner integral and then use (7) to get

$$\Big(\int_E\int_F w_0(x,y)(H_{E,F}f)^q(x,y)\ dy\,dx\Big)^{\frac{1}{q}} \leq$$

$$\leq C\Big(\int_0^\infty\int_0^\infty W(x_0,y_0)\Big(\int_{B_M}\int_{B_N} f(x_0x',y_0y')x_0^{M-1}y_0^{N-1}dy'\,dx'\Big)^p dy_0\,dx_0\Big)^{\frac{1}{p}} \leq$$

$$\leq C\Big(\int_0^\infty\int_0^\infty w(x_0,y_0)\Big(\int_{B_M}\int_{B_N} f^p(x_0x',y_0y')w(x_0x',y_0y')x_0^{M-1}y_0^{N-1}\ dy'\,dx'\Big)\times$$

$$\times\Big(\int_{B_M}\int_{B_N} w^{1-p'}(x_0x',y_0y')x_0^{M-1}y_0^{N-1}\ dy'\,dx'\Big)^{\frac{p}{p'}}\ dy_0\,dx_0\Big)^{\frac{1}{p}} =$$

$$= C\Big(\int_E\int_F w(x,y)f^p(x,y)\ dy\,dx\Big)^{\frac{1}{p}}.$$

Thus (5) holds. □

To prove the converse, suppose (5) holds and fix a locally integrable function $g:(0,\infty)\times(0,\infty)\to\mathbb{R}$. Define $f:E\times F\to\mathbb{R}$ by

$$f(x_0x',y_0y') = g(x_0,y_0)W^{p'-1}(x_0,y_0)w^{1-p'}(x_0x',y_0y'), \qquad (10)$$
$$x_0,y_0>0,\quad x'\in B_M,\quad y'\in B_N.$$

Then, using (7)

$$\int_{B_M}\int_{B_N} f(x_0x',y_0y')x_0^{M-1}y_0^{N-1}\ dy'\,dx' = g(x_0,y_0).$$

Therefore, as in the first part of the proof, we have

$$\Big(\int_0^\infty\int_0^\infty W_0(x_0,y_0)(H_2g)^q(x_0,y_0)\ dy_0\,dx_0\Big)^{\frac{1}{q}} =$$

$$= \Big(\int_E\int_F w_0(x,y)(H_{E,F}f)^q(x,y)\ dy\,dx\Big)^{\frac{1}{q}}.$$

Now, using (5) and then making use of (7) and (10), we have

$$\Big(\int_0^\infty\int_0^\infty W_0(x_0,y_0)(H_2g)^q(x_0,y_0)\ dy_0\,dx_0\Big)^{\frac{1}{q}} \leq$$

$$\leq C\Big(\int\limits_E\int\limits_F w(x,y)f^p(x,y)\;dy\,dx\Big)^{\frac{1}{p}} =$$

$$= C\Big(\int\limits_0^\infty\int\limits_0^\infty\int\limits_{B_M}\int\limits_{B_N} w(x_0x',y_0y')f^p(x_0x',y_0y')x_0^{M-1}y_0^{N-1}\,dy'\,dx'\,dy_0\,dx_0\Big)^{\frac{1}{p}} =$$

$$= C\Big(\int\limits_0^\infty\int\limits_0^\infty g^p(x_0,y_0)W^{p'}(x_0,y_0)\Big)\times$$

$$\times\Big(\int\limits_{B_M}\int\limits_{B_N} w^{1-p'}(x_0x',y_0y')x_0^{M-1}y_0^{N-1}\;dy'\,dx'\,dy_0\,dx_0\Big)^{\frac{1}{p}} =$$

$$= C\Big(\int\limits_0^\infty\int\limits_0^\infty W(x_0,y_0)g^p(x_0,y_0)\;dy_0\,dx_0\Big)^{\frac{1}{p}}.$$

Thus, (1) holds.

Combining Theorems A and 1, and using the definitions of W_0, W in (2), (3) and (4), we immediately obtain the precise necessary and sufficient conditions for the boundedness of $H_{E,F}$.

Corollary 1. *Let $1 < p \leq q < \infty$ and all the assertions of Theorem 1 are satisfied. Then the inequality (5) holds if and only if the following three conditions are satisfied:*

$$A := \sup_{\alpha>0,\beta>0}\Big(\int\limits_{E\setminus\alpha S_M}\int\limits_{F\setminus\beta S_N} w_0(x,y)\;dy\,dx\Big)^{\frac{1}{q}}\times$$

$$\times\Big(\int\limits_{\alpha S_M}\int\limits_{\beta S_N} w^{1-p'}(x,y)\;dy\,dx\Big)^{\frac{1}{p'}} < \infty,$$

$$\int\limits_{S_{M_x}}\int\limits_{S_{N_y}}\Big(\int\limits_{S_{M_\xi}}\int\limits_{S_{N_\eta}} w^{1-p'}(\rho,\tau)\,d\tau\,d\rho\Big)^q w_0(\xi,\eta)d\eta\,d\xi \leq$$

$$\leq A^q\Big(\int\limits_{S_{M_x}}\int\limits_{S_{N_y}} w^{1-p'}(\xi,\eta)\,d\eta\,d\xi\Big)^{\frac{q}{p}},$$

$$\int\limits_{E\setminus S_{M_x}}\int\limits_{F\setminus S_{N_y}}\Big(\int\limits_{E\setminus S_{M_\xi}}\int\limits_{F\setminus S_{N_\eta}} w_0(\rho,\tau)\,d\tau\,d\rho\Big)^{p'} w^{1-p'}(\xi,\eta)\,d\eta\,d\xi \leq$$

$$\leq A^{p'}\Bigg(\int\limits_{E\setminus S_{M_x}}\int\limits_{F\setminus S_{N_y}} w_0(\xi,\eta)\,d\eta\,d\xi\Bigg)^{\frac{p'}{q'}},$$

for every $x \in E$, $y \in F$.

Remark 1. As a special case, if we take $E = \mathbb{R}^M$ and $F = \mathbb{R}^N$, then $S_{M_x} = B_M(x)$ and $S_{N_y} = B_N(y)$, where $B_M(x)$ and $B_N(y)$ are the balls in $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively, centered at origin with radii $|x|$ and $|y|$, respectively. We immediately obtain the following corollary to Theorem 1.

Corollary 2. *Let* $1 < p \leq q < \infty$ *and let* $w_0 = w_0(x,y)$ *and* $w = w(x,y)$ *be weight functions on* $\mathbb{R}^M \times \mathbb{R}^N$. *Then, the inequality*

$$\Bigg(\int\limits_{\mathbb{R}^M}\int\limits_{\mathbb{R}^N} w_0(x,y)\Bigg(\int\limits_{B_M(x)}\int\limits_{B_N(y)} f(s,t)\;dt\,ds\Bigg)^q dy\,dx\Bigg)^{\frac{1}{q}} \leq$$

$$\leq C\Bigg(\int\limits_{\mathbb{R}^M}\int\limits_{\mathbb{R}^N} w(x,y)f^p(x,y)\;dy\,dx\Bigg)^{\frac{1}{p}},\quad f \geq 0,$$

holds for some constant $C > 0$ *if and only if the following three conditions are satisfied :*

$$C_1 := \sup_{\alpha>0,\beta>0}\Bigg(\int\limits_{|x|\geq\alpha}\int\limits_{|y|\geq\beta} w_0(x,y)\,dy\,dx\Bigg)^{\frac{1}{q}}\times$$

$$\times\Bigg(\int\limits_{|x|\leq\alpha}\int\limits_{|y|\leq\beta} w^{1-p'}(x,y)dy\,dx\Bigg)^{\frac{1}{p'}}<\infty,$$

$$\int\limits_{B_M(x)}\int\limits_{B_N(y)}\Bigg(\int\limits_{B_M(\xi)}\int\limits_{B_N(\eta)} w^{1-p'}(\rho,\tau)\,d\tau\,d\rho\Bigg)^q w_0(\xi,\eta)\,d\eta\,d\xi \leq$$

$$\leq C_1^q\Bigg(\int\limits_{B_M(x)}\int\limits_{B_N(y)} w^{1-p'}(\xi,\eta)\;d\eta\,d\xi\Bigg)^{\frac{q}{p}},$$

$$\int\limits_{\mathbb{R}^M\setminus B_M(x)}\int\limits_{\mathbb{R}^N\setminus B_N(y)}\Bigg(\int\limits_{\mathbb{R}^M\setminus B_M(\xi)}\int\limits_{\mathbb{R}^N\setminus B_N(\eta)} w_0(\rho,\tau)d\tau\,d\rho\Bigg)^{p'} w^{1-p'}(\xi,\eta)\,d\eta\,d\xi \leq$$

$$\leq C_1^{p'}\Bigg(\int\limits_{\mathbb{R}^M\setminus B_M(x)}\int\limits_{\mathbb{R}^N\setminus B_N(y)} w_0(\xi,\eta)\,d\eta\,d\xi\Bigg)^{\frac{p'}{q'}}$$

for every $x \in \mathbb{R}^M$, $y \in \mathbb{R}^N$.

Remark 2. Corollary 2 characterizes the boundedness of another (but special case of $H_{E,F}$) double sized multidimensional operator

$$(H_{M,N}f)(x,y) = \int\limits_{B_M(x)} \int\limits_{B_N(y)} f(s,t)\,dt\,ds.$$

The boundedness of $H_{M,N}$ has already been obtained by Heinig, Kufner and Persson [8]. However, their characterization is not in terms of the characterization of H_2. Thus, Theorem 1 extends their result and also the approach in our case is quite different than what those authors used.

Remark 3. Theorem 1 is true for all p, q with $1 < p$, $q < \infty$ but Corollary 1 has the restriction that $1 < p \le q < \infty$. This is because we are applying Theorem 1 which demands this restriction. Thus, if one obtains a result (corresponding to Theorem A) that characterizes the boundedness of the operator H_2 for the case $q < p$, then by applying Theorem 1, a characterization for the boundedness of $H_{E,F}$ (and also $H_{M,N}$) can be obtained for this case.

3. The Case of Mixed Norm

3.1. **Mixed normed spaces.** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces of real functions over $\Omega_1 \subset \mathbb{R}^M$ and $\Omega_2 \subset \mathbb{R}^N$, respectively. We denote by $[X,Y]$ the space of all real functions f on $\Omega_1 \times \Omega_2$ such that

(i) the function $f(x,\cdot)$ is in Y for each $x \in \Omega_1$;

(ii) the function $g(x) = \|f(x,\cdot)\|_Y$ is in X.

The space $[X,Y]$ is usually equipped with the norm

$$\|f\|_{[X,Y]} = \|g\|_X = \|x \to \|f(x,\cdot)\|_Y\,\|_X.$$

We shall consider the case when X and Y are weighted Lebesgue spaces. Certain basic properties of the corresponding spaces have been studied by Benedek and Panzone [2].

3.2. **The operator $T_{E,F}$.** We define the operator

$$T_{E,F} : \big[L^{p_1}(E,U), L^{p_2}(F,V)\big] \to \big[L^{q_1}(E,u), L^{q_2}(F,v)\big]$$

by

$$(T_{E,F}f)(x,y) = \int\limits_{b(|x|)S_M \setminus a(|x|)S_M} \int\limits_{d(|y|)S_N \setminus c(|y|)S_N} f(s,t)\,dt\,ds, \quad x \in E,\ y \in F,$$

where a, b, c, d are strictly increasing differenciable functions on $[0,\infty]$ satisfying

$$a(0)=b(0)=0,\ \ a(x) < b(x)\ \text{ for }\ 0 < x < \infty,\ \ a(\infty) = b(\infty) = \infty, \qquad (11)$$

$$c(0) = d(0) = 0,\ \ c(x) < d(x)\ \text{ for }\ 0 < x < \infty,\ \ c(\infty) = d(\infty) = \infty, \qquad (12)$$

S_M, S_N, E, F are as defined in Section 2, U, u are weight functions on E and V, v are weight functions on F.

Remark 4. The operator $T_{E,F}$ is a higher dimensional analogue of the two dimensional operator

$$(T_2 f)(x,y) = \int\limits_{a(x)}^{b(x)} \int\limits_{c(y)}^{d(y)} f(s,t)\,dt\,ds,$$

where a, b, c, d satisfy (11) and (12). In [12], the authors have studied the necessary as well as sufficient conditions for the boundedness of the operator T_2 when $c \equiv a$ and $d \equiv b$.

Remark 5. The operator T_2, in some sense, is a generalized version of H_2. However, H_2 can not be obtained from T_2 since for that the natural choice of the functions a, b, c, d would be $a \equiv c \equiv 0$ and $b(x) = d(x) = x$, $x \in (0,\infty)$ but then this choice does not satisfy (11) and (12). Thus T_2 is different than H_2. For the same reason $T_{E,F}$ is also a generalized version of (but different than) $H_{E,F}$.

In what follows, we obtain necessary as well as sufficient conditions for the boundedness of $T_{E,F}$ between suitable weighted Lebesgue spaces with mixed norm. The boundedness of $H_{E,F}$ (with mixed norm) can be studied similarly.

Remark 6. As mentioned in Remark 4, the boundedness of T_2 has been studied in [12]. The proof for boundedness of the operator $T_{E,F}$ can be obtained on the same lines with some obvious modifications and therefore we only state the results.

Theorem 2. *Let E and F be two cones in $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively. Suppose that $1 < p_i < \infty$, $0 < q_i < \infty$, $i = 1,2$; U, u are weight functions on E and V, v are weight functions on F. Then, a necessary condition for the validity of the inequality*

$$\begin{aligned} &\left(\int\limits_E u(x)\left(\int\limits_F v(y)(T_{E,F}f)^{q_2}(x,y)\;dy\right)^{\frac{q_1}{q_2}} dx\right)^{\frac{1}{q_1}} \le \\ &\le C\left(\int\limits_E U(x)\left(\int\limits_F V(y) f^{p_2}(x,y)\;dy\right)^{\frac{p_1}{p_2}} dx\right)^{\frac{1}{p_1}}, \quad f \ge 0, \end{aligned} \tag{13}$$

is the validity of at least one of the following inequalities:

$$\left(\int\limits_E u(x)(T_E g)^{q_1}(x)\;dx\right)^{\frac{1}{q_1}} \le C_1\left(\int\limits_E U(x) g^{p_1}(x)\;dx\right)^{\frac{1}{p_1}} \tag{14}$$

$$\left(\int\limits_F v(y)(T_F h)^{q_2}(y)\ dy\right)^{\frac{1}{q_2}} \le C_2\left(\int\limits_F V(y)h^{p_2}(y)\ dy\right)^{\frac{1}{p_2}}, \tag{15}$$

where $T_E : L^{p_1}(E,U) \to L^{q_1}(E,u)$ *and* $T_F : L^{p_2}(F,V) \to L^{q_2}(F,v)$ *are defined, respectively, by*

$$(T_E g)(x) = \int\limits_{b(|x|)S_M \setminus a(|x|)S_M} g(s)\, ds, \quad x \in E,$$

and

$$(T_F h)(y) = \int\limits_{d(|y|)S_N \setminus c(|y|)S_N} h(t)\, dt, \quad y \in F.$$

Theorem 3. *Let the assertions of Theorem* 2 *be satisfied and assume in addition that either* $p_1 \le p_2 \le q_1$ *or* $p_1 \le q_2 \le q_1$. *Then, a sufficient condition for the validity of the inequality* (13) *is that both* (14) *and* (15) *hold.*

Remark 7. Theorems 2 and 3 give the necessary and the sufficient conditions for the boundedness of $T_{E,F}$ in terms of the boundedness of T_E and T_F. The necessary and sufficient conditions for the boundedness of such operators have already been established by Sinnamon [19]. In fact, in Sinnamon's results, E and F are taken to be certain 'star-shaped regions' in $\mathbb{R}^N$ but those results also hold for cones in $\mathbb{R}^N$. Thus, using those, the precise conditions for the boundedness of $T_{E,F}$ can be obtained. For the sake of conciseness, we present only the precise sufficient conditions.

Corollary 3. *Let* E *and* F *be two cones in* $\mathbb{R}^M$ *and* $\mathbb{R}^N$, *respectively. Suppose* U, u *be weight functions on* E *and* V, v *be weight functions on* F. *Then, the inequality* (13) *holds if the following hypothesis are satisfied:*
Case (a) : *The conditions*

$$\sup_{\substack{0<t\le s\\ a(s)\le b(t)}} B_1(s,t) =$$

$$= \sup_{\substack{0<t\le s\\ a(s)\le b(t)}} \left(\int\limits_{b(t)S_M \setminus a(s)S_M} U^{1-p_1'}\right)^{\frac{1}{p_1'}} \left(\int\limits_{sS_M \setminus tS_M} u\right)^{\frac{1}{q_1}} < \infty \tag{16}$$

and

$$\sup_{\substack{0<t\le s\\ c(s)\le d(t)}} B_2(s,t) =$$

$$= \sup_{\substack{0<t\le s\\ c(s)\le d(t)}} \left(\int\limits_{d(t)S_N \setminus c(s)S_N} V^{1-p_2'}\right)^{\frac{1}{p_2'}} \left(\int\limits_{sS_N \setminus tS_N} v\right)^{\frac{1}{q_2}} < \infty \tag{17}$$

hold and the exponents satisfy either

$$1 < p_1 \le p_2 \le q_1 < \infty, \quad p_2 \le q_2 < \infty$$

or

$$1 < p_1 \le q_2 \le q_1 < \infty, \quad 1 < p_2 \le q_2.$$

Case (b) : *The conditions* (16),

$$\Bigg(\int_0^\infty \int\limits_{tS_N \setminus d^{-1}(c(t))S_N} \Bigg(\int\limits_{d(|x|)S_N \setminus c(t)S_N} V^{1-p_2'}\Bigg)^{\frac{r_2}{p_2'}} \times \\ \times \Bigg(\int\limits_{tS_N \setminus S_{N_x}} v\Bigg)^{\frac{r_2}{p_2}} v(x)\ dx\rho(t)\,dt\Bigg)^{\frac{1}{r_2}} < \infty, \tag{18}$$

and

$$\Bigg(\int_0^\infty \int\limits_{c^{-1}(d(t))S_N \setminus tS_N} \Bigg(\int\limits_{d(t)S_N \setminus c(|x|)S_N} V^{1-p_2'}\Bigg)^{\frac{r_2}{p_2'}} \times \\ \times \Bigg(\int\limits_{S_{N_x} \setminus tS_N} v\Bigg)^{\frac{r_2}{p_2}} v(x)\ dx\rho(t)\,dt\Bigg)^{\frac{1}{r_2}} < \infty, \tag{19}$$

hold with $\dfrac{1}{r_2} = \dfrac{1}{q_2} - \dfrac{1}{p_2}$ *and the exponents satisfy either*

$$1 < p_1 \le p_2 \le q_1 < \infty, \quad 0 < q_2 < p_2$$

or

$$1 < p_1 \le q_2 \le q_1 < \infty, \quad q_2 < p_2 < \infty.$$

The function ρ *in* (18) *and* (19) *is the normalizing function defined as follows: Fix* $M_0 = d^{-1}(1)$ *and define*

$$M_{k+1} = c^{-1}(d(M_k)) \ \textit{if}\ k \ge 0 \ \textit{ and }\ M_k = d^{-1}(c(M_{k+1})) \ \textit{if}\ k < 0.$$

The normalizing function ρ *is defined by*

$$\rho(t) = \sum_{k\in\mathbb{Z}} \chi_{(M_k,M_{k+1})}(t)\frac{d}{dt}(d^{-1}\circ c)^k(t),$$

where $(d^{-1}\circ c)^k$ *denotes* k *times repeated composition.*

Remark 8. In view of Remark 1, we can obtain a special case of Corollary 3 when $E = \mathbb{R}^M$, $F = \mathbb{R}^N$. In this case the operator $T_{E,F}$ takes the form

$$(T_{M,N}f)(x,y) = \int\limits_{a(|x|)\le|s|\le b(|x|)} \int\limits_{c(|y|)\le|t|\le d(|y|)} f(s,t)\,dt\,ds.$$

The precise sufficient conditions for the boundedness of $T_{M,N}$ are given in the following corollary. Corresponding necessary conditions can be stated similarly.

Corollary 4. *Let U, u be weight functions on $\mathbb{R}^M$ and V, v be weight functions on $\mathbb{R}^N$. Then, the inequality*

$$\Bigg(\int\limits_{\mathbb{R}^M} u(x)\Bigg(\int\limits_{\mathbb{R}^N} v(y)(T_{M,N}f)^q(x,y)\,dy\Bigg)^{\frac{q_1}{q_2}} dx\Bigg)^{\frac{1}{q_1}} \leq$$

$$\leq C\Bigg(\int\limits_{\mathbb{R}^M} U(x)\Bigg(\int\limits_{\mathbb{R}^N} V(y)f^{p_2}(x,y)\,dy\Bigg)^{\frac{p_1}{p_2}} dx\Bigg)^{\frac{1}{p_1}}, \quad f \geq 0,$$

holds if the following hypothesis are satisfied:
Case (a) : *The conditions*

$$\sup_{\substack{0<t\leq s\\ a(s)\leq b(t)}} \Bigg(\int\limits_{a(s)\leq|x|\leq b(t)} U^{1-p_1'}(x)\,dx\Bigg)^{\frac{1}{p_1'}}\Bigg(\int\limits_{t\leq|x|\leq s} u(x)\,dx\Bigg)^{\frac{1}{q_1}} < \infty, \tag{20}$$

$$\sup_{\substack{0<t\leq s\\ c(s)\leq d(t)}} \Bigg(\int\limits_{c(s)\leq|x|\leq d(t)} V^{1-p_2'}(x)\,dx\Bigg)^{\frac{1}{p_2'}}\Bigg(\int\limits_{t\leq|x|\leq s} v(x)\,dx\Bigg)^{\frac{1}{q_2}} < \infty \tag{21}$$

hold and the exponents satisfy either

$$1 < p_1 \leq p_2 \leq q_1 < \infty, \quad p_2 \leq q_2 < \infty$$

or

$$1 < p_1 \leq q_2 \leq q_1 < \infty, \quad 1 < p_2 \leq q_2.$$

Case (b) : *The conditions* (20),

$$\Bigg(\int\limits_0^\infty \int\limits_{d^{-1}(c(t))\leq|x|\leq t} \Bigg(\int\limits_{c(t)\leq|y|\leq d(|x|)} V^{1-p_2'}(y)\,dy\Bigg)^{\frac{r_2}{p_2'}} \times$$

$$\times\Bigg(\int\limits_{|x|\leq|y|\leq t} v(y)\,dy\Bigg)^{\frac{r_2}{p_2}} v(x)\,dx\rho(t)\,dt\Bigg)^{\frac{1}{r_2}} < \infty$$

and

$$\Bigg(\int\limits_0^\infty \int\limits_{t\leq|x|\leq c^{-1}(d(t))} \Bigg(\int\limits_{c(|x|)\leq|y|\leq d(t)} V^{1-p_2'}(y)\,dy\Bigg)^{\frac{r_2}{p_2'}} \times$$

$$\times\Bigg(\int\limits_{t\leq|y|\leq|x|} v(y)\,dy\Bigg)^{\frac{r_2}{p_2}} v(x)\,dx\rho(t)\,dt\Bigg)^{\frac{1}{r_2}} < \infty$$

hold with $\frac{1}{r_2} = \frac{1}{q_2} - \frac{1}{p_2}$*,* ρ *being the normalizing function, defined in Corollary* 3*, and the exponents satisfy either*

$$1 < p_1 \le p_2 \le q_1 < \infty, \quad 0 < q_2 < p_2$$

or

$$1 < p_1 \le q_2 \le q_1 < \infty, \quad q_2 < p_2 < \infty.$$

3.3. **The operator** $G_{E,F}$**.** We define the double sized multidimensional geometric mean operator

$$G_{E,F} : \big[L^{p_1}(E,U), L^{p_2}(F,V)\big] \to \big[L^{q_1}(E,u), L^{q_2}(F,v)\big]$$

by

$$(G_{E,F}f)(x,y) = \exp\Big(\frac{1}{|S_{M_x}|\,|S_{N_y}|}\int\limits_{S_{M_x}}\int\limits_{S_{N_y}} \ln f(s,t)\ dt\, ds\Big),$$

where $|S_{M_x}|$ and $|S_{N_y}|$ denote the volumes of S_{M_x} and S_{N_y}, respectively.

Remark 9. The operator $G_{E,F}$ is a higher dimensional analogue of the two dimensional geometric mean operator $G_2 : [L^{p_1}(U), L^{p_2}(V)] \to [L^{q_1}(u), L^{q_2}(v)]$ defined by

$$(G_2 f)(x,y) = \exp\left(\frac{1}{xy}\int\limits_0^x\int\limits_0^y \ln f(s,t)\, dt\, ds\right).$$

The boundedness property of G_2 in the context of Lebesgue spaces with usual norm has been studied in [7], [11], [15] while in the context of mixed norm in [12].

In the following two theorems, we give, respectively, the necessary and the sufficient condition for the boundedness of $G_{E,F}$. Since the proofs are on similar lines as those of G_2, we only state the theorems.

Theorem 4. *Let* $0 < p_i,\ q_i < \infty,\ i = 1, 2$*;* U, u *be weight functions on* E *and* V*,* v *be weight functions on* F*. Then a necessary condition for the inequality*

$$\Big(\int\limits_E u(x)\Big(\int\limits_F v(y)(G_{E,F}f)^{q_2}(x,y)\ dy\Big)^{\frac{q_1}{q_2}}\ dx\Big)^{\frac{1}{q_1}} \le$$

$$\le C\Big(\int\limits_E U(x)\Big(\int\limits_F V(y)f^{p_2}(x,y)\ dy\Big)^{\frac{p_1}{p_2}}\ dx\Big)^{\frac{1}{p_1}}, \quad f \ge 0, \tag{22}$$

to hold is that either of the following two inequalities holds

$$\Big(\int\limits_E u(x)(G_E g)^{q_1}(x)\ dx\Big)^{\frac{1}{q_1}} \le C_1\Big(\int\limits_E U(x)g^{p_1}(x)\ dx\Big)^{\frac{1}{p_1}}, \quad g \ge 0, \tag{23}$$

$$\bigg(\int\limits_F v(y)(G_F h)^{q_2}(x)\,dx\bigg)^{\frac{1}{q_2}} \le C_2\bigg(\int\limits_F V(y)h^{p_2}(y)\,dy\bigg)^{\frac{1}{p_2}},\ \ h\ge 0, \qquad (24)$$

where G_E and G_F are defined respectively by

$$(G_E g)(x) = \exp\bigg(\frac{1}{|S_{M_x}|}\int\limits_{S_{M_x}} \ln g(s)\,ds\bigg),$$

$$(G_F h)(x) = \exp\bigg(\frac{1}{|S_{N_x}|}\int\limits_{S_{N_x}} \ln h(t)\,dt\bigg).$$

Theorem 5. *Let the hypothesis of Theorem* 4 *be satisfied and assume in addition that either $p_1 \le p_2 \le q_1$ or $p_1 \le q_2 \le q_1$. Then, a sufficient condition for the validity of* (22) *is the validity of both* (23) *and* (24).

Remark 10. The precise necessary and the sufficient conditions for the inequality (22) can now be obtained using any known criterion for the boundedness of single sized geometric mean operators G_E and G_F, e.g., one may use the criterion given in [5].

4. Compactness of $T_{E,F}$ with Mixed Norm

4.1. **Preliminaries.** Let X be a normed linear space and let X^* denote the conjugate space of X. A sequence $\{x_n\}$ in X is said to be strongly convergent (or simply convergent), written $x_n \to x$, if there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. A sequence $\{x_n\}$ in X is said to converge weakly to $x \in X$, written $x_n \xrightarrow{w} x$, if $f(x_n) \to f(x)$, for each $f \in X^*$. We say that a sequence $\{f_n\}$ in X^* is weak* convergent to $f \in X^*$, written $f_n \xrightarrow{w^*} f$, if $f_n(x) \to f(x)$ for each $x \in X$. The strong convergence implies the weak convergence which in turn implies the weak* convergence. The implications in the reverse direction do not hold in general. However, if X is reflexive then weak* convergence implies the weak convergence.

For a bounded linear operator A between two normed linear spaces X and Y, we denote by A^*, the conjugate of A acting between the conjugate spaces Y^* and X^*.

It can be seen that the operators $T_{E,F}$ and

$$T^*_{E,F} : \big[L^{q_1'}(E, u^{1-q_1'}), L^{q_2'}(F, v^{1-q_2'})\big] \to \big[L^{p_1'}(E, U^{1-p_1'}), L^{p_2'}(F, V^{1-p_2'})\big]$$

defined by

$$(T^*_{E,F} g)(x,y) = \int\limits_{a^{-1}(|x|)S_M \setminus b^{-1}(|x|)S_M}\ \int\limits_{c^{-1}(|y|)S_N \setminus d^{-1}(|y|)S_N} f(s,t)\,dt\,ds,$$

$$x \in E,\ \ y \in F$$

are mutually conjugate, i.e.,

$$\langle T_{E,F}f, g\rangle = \langle f, T^*_{E,F}g\rangle.$$

Similarly, the operators T_E and $T^*_E : L^{q'_1}(E, u^{1-q'_1}) \to L^{p'_1}(E, U^{1-p'_1})$ defined by

$$(T^*_E f)(x) = \int\limits_{a^{-1}(|x|)S_M \setminus b^{-1}(|x|)S_M} f(s)\, ds, \quad x \in E$$

and the operators T_F and $T^*_F : L^{q'_2}(F, v^{1-q'_2}) \to L^{p'_2}(F, V^{1-p'_2})$ defined by

$$(T^*_F f)(y) = \int\limits_{c^{-1}(|y|)S_N \setminus d^{-1}(|y|)S_N} f(t)\, dt, \quad y \in F$$

are also mutually conjugate. The proofs of the results presented in this section require certain well known assertions which we collect in the following theorems.

Theorem *B*. *Let X and Y be Banach spaces.*

(a) *A bounded linear operator $A : X \to Y$ is compact if and only if its conjugate $A^* : Y^* \to X^*$ is compact.*

(b) *If $A : X \to Y$ is compact and $\{x_n\}$ is a sequence in X such that $x_n \xrightarrow{w} x$, for some $x \in X$, then $Ax_n \to Ax$.*

(c) *An operator $A : X \to Y$ is compact if $A^* : Y^* \to X^*$ is weak *-norm sequentially continuous, i.e., for each sequence $\{f_n\}$ in Y^* with $\{f_n\} \xrightarrow{w^*} f$ for some $f \in Y^*$, we have $\|A^*(f_n) - A^*f\| \to 0$.*

Theorem *C*. *Let Ω be any domain in $\mathbb{R}^N$.*

(a) *Let $1 \le p < \infty$. A sequence $\{f_n\}$ in $L^p(\Omega)$ converges weakly to $f \in L^p(\Omega)$ if and only if*

(i) $\sup \|f_n\| < \infty$

(ii) $\int\limits_M f_n(t)\, dt \to \int\limits_M f(t)\, dt$, *for every measurable subset $M \subset \Omega$.*

(b) *Let $1 < p < \infty$. A sequence $\{f_n\}$ in $L^p(\Omega)$ is weakly null if*

(i) $\sup \|f_n\| < \infty$

(ii) $f_n(s) \to 0$, *for almost all $s \in \Omega$.*

These results can be found in any standard book on functional analysis, e.g., [3], [9], [10]. In particular, Theorem B(c) is taken from [3, p15] and C(b) from [10].

4.2. **The compactness results.** We prove below two theorems which give, respectively, the necessary and the sufficient conditions for the compactness of $T_{E,F}$.

Theorem 6. *Let E and F be two cones in $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively, $E = \cup_{\alpha>0}\alpha S_M$ and $F = \cup_{\alpha>0}\alpha S_N$. Suppose that $1 < p_i,\ q_i < \infty$, $i = 1, 2$; U, u are weight functions on E and V, v are weight functions on F. Then, a* *necessary condition for the compactness of* $T_{E,F} : [L^{p_1}(E,U), L^{p_2}(F,V)] \to [L^{q_1}(E,u), L^{q_2}(F,v)]$ *is the compactness of at least one of the operators* $T_E : L^{p_1}(E,U) \to L^{q_1}(E,u)$ *and* $T_F : L^{p_2}(F,V) \to L^{q_2}(F,v)$.

Proof. In view of Theorem B (c), it suffices to show that at least one of the operators $T^*_E : L^{q'_1}(E, u^{1-q'_1}) \to L^{p'_1}(E, U^{1-p'_1})$ and $T^*_F : L^{q'_2}(F, v^{1-q'_2}) \to L^{p'_2}(F, V^{1-p'_2})$ is w^*-norm sequentially continuous. Fix two w*-null sequences $\{g_n\}$ and $\{h_n\}$ in $L^{q'_1}(E, u^{1-q'_1})$ and $L^{q'_2}(F, v^{1-q'_2})$, respectively. Since the spaces $L^{q'_1}(E, u^{1-q'_1})$ and $L^{q'_2}(F, v^{1-q'_2})$ are reflexive, $\{g_n\}$ and $\{h_n\}$ are weakly null sequences. Define $f_n(s,t) = g_n(s)h_n(t)$, $s \in E$, $t \in F$, $n \in \mathbb{N}$. Then $\{f_n\}$ is a weakly null sequence in $[L^{q'_1}(E, u^{1-q'_1}),\ L^{q'_2}(F, v^{1-q'_2})]$. Therefore, by Theorem B ((a) and (b)), we have

$$\left\|T^*_{E,F}\, f_n\right\|_{[L^{p'_1}(E,U^{1-p'_1}), L^{p'_2}(F,V^{1-p'_2})]} \to 0.$$

Moreover, since

$$(T^*_{E,F} f_n)(x,y) = (T^*_E g_n)(x)(T^*_F h_n)(y),$$

we have

$$\left\|T^*_{E,F}\, f_n\right\|_{[L^{p'_1}(E,U^{1-p'_1}), L^{p'_2}(F,V^{1-p'_2})]} =$$

$$= \left(\int\limits_E U^{1-p'_1}(x) \left(\int\limits_F V^{1-p'_2}(y) (T^*_{E,F} f_n)^{p'_2}(x,y)\, dy \right)^{\frac{p'_1}{p'_2}} dx \right)^{\frac{1}{p'_1}} =$$

$$= \left(\int\limits_E U^{1-p'_1}(x) (T^*_E g_n)^{p'_1}(x)\, dx \right)^{\frac{1}{p'_1}} \left(\int\limits_F V^{1-p'_2}(y) (T^*_F h_n)^{p'_2}(y)\, dy \right)^{\frac{1}{p'_2}} =$$

$$= \left\|T^*_E g_n\right\|_{L^{p'_1}(E,U^{1-p'_1})} \cdot \left\|T^*_F h_n\right\|_{L^{p'_2}(F,V^{1-p'_2})}.$$

It follows, now, that at least one of the two sequences $\|T^*_E g_n\|_{L^{p'_1}(E,U^{1-p'_1})}$ and $\|T^*_F h_n\|_{L^{p'_2}(F,V^{1-p'_2})}$ converge to 0. $\square$

Theorem 7. *Let all the assertions of Theorem 6 be satisfied. Then, a sufficient condition for the compactness of* $T_{E,F} : [L^{p_1}(E,U), L^{p_2}(F,V)] \to [L^{q_1}(E,u), L^{q_2}(F,v)]$ *is the compactness of both the operators* $T_E : L^{p_1}(E,U) \to L^{q_1}(E,u)$ *and* $T_F : L^{p_2}(F,V) \to L^{q_2}(F,v)$.

Proof. It suffices to show that the operator $T^*_{E,F} : [L^{q'_1}(E, u^{1-q'_1}), L^{q'_2}(F, v^{1-q'_2})] \to [L^{p'_1}(E, U^{1-p'_1}), L^{p'_2}(F, V^{1-p'_2})]$ is w^*-norm sequentially continuous. Let $\{f_n\}$ be any w^*-null sequence in $[L^{q'_1}(E, u^{1-q'_1}), L^{q'_2}(F, v^{1-q'_2})]$ and since it is a reflexive space, $\{f_n\}$ is weakly null in it. For each $s \in E$, define $h_{n,s} : F \to \mathbb{R}$ by

$$h_{n,s}(t) = f_n(s,t), \quad s \in E, \quad t \in F, \quad n \in \mathbb{N}.$$

Then, for each $s \in E$, $\{h_{n,s}\}$ is a weakly null sequence in $L^{q_2'}(F, v^{1-q_2'})$, so that by the compactness of T_F and Theorem B ((a) and (b)), we have

$$\left\|T_F^* h_{n,s}\right\|_{L^{p_2'}(F,V^{1-p_2'})} \to 0, \quad s \in E. \tag{25}$$

Now define

$$g_n(s) = \left\|T_F^* h_{n,s}\right\|_{L^{p_2'}(F,V^{1-p_2'})}, \quad n \in \mathbb{N}, \quad s \in E.$$

Then, $\{g_n\}$ is a sequence in $L^{q_1'}(E, u^{1-q_1'})$ since using the boundedness of T_F^*, we have

$$\left(\int_E u^{1-q_1'}(s) g_n^{q_1'}(s)\, ds\right)^{\frac{1}{q_1'}} =$$

$$= \left(\int_E u^{1-q_1'}(s)\left(\int_F V^{1-p_2'}(y)(T_F^* h_{n,s})^{p_2'}(y)\, dy\right)^{\frac{q_1'}{p_2'}} ds\right)^{\frac{1}{q_1'}} \leq$$

$$\leq C\left(\int_E u^{1-q_1'}(s)\left(\int_F v^{1-q_2'}(y) h_{n,s}^{q_2'}(y)\, dy\right)^{\frac{q_1'}{q_2'}} ds\right)^{\frac{1}{q_1'}} =$$

$$= C\left(\int_E u^{1-q_1'}(s)\left(\int_F v^{1-q_2'}(y) f_n^{q_2'}(s,y)\, dy\right)^{\frac{q_1'}{q_2'}} ds\right)^{\frac{1}{q_1'}} < \infty.$$

Moreover, since $\{f_n\}$ is weakly null by Theorem C (a)

$$\sup_n \left\|g_n\right\|_{L^{q_1'}(E,u^{1-q_1'})} \leq \sup_n \left\|f_n\right\|_{[L^{q_1'}(E,u^{1-q_1'}),L^{q_2'}(F,v^{1-q_2'})]} < \infty$$

and by (25), $g_n(s) \to 0$, $s \in E$. Thus, by Theorem C(b), $\{g_n\}$ is weakly null. Now, by the compactness of T_E and Theorem B ((a) and (b)),

$$\left\|T_E^*\, g_n\right\|_{L^{p_1'}(E,U^{1-p_1'})} \to 0.$$

But, using the Minkowski's integral inequality, we have

$$\left\|T_{E,F}^* f_n\right\|_{[L^{p_1'}(E,U^{1-p_1'}),L^{p_2'}(F,V^{1-p_2'})]} =$$

$$= \left(\int_E U^{1-p_1'}(x)\left(\int_F V^{1-p_2'}(y)\left(\int\limits_{a^{-1}(|x|)S_M \setminus b^{-1}(|x|)S_M} \times\right.\right.\right.$$

$$\left.\left.\left.\times\left(\int\limits_{c^{-1}(|y|)S_N \setminus d^{-1}(|y|)S_N} f_n(s,t)\, dt\right) ds\right)^{p_2'} dy\right)^{\frac{p_1'}{p_2'}} dx\right)^{\frac{1}{p_1'}} \leq$$

$$\leq \left(\int_E U^{1-p_1'}(x)\left(\int\limits_{a^{-1}(|x|)S_M \setminus b^{-1}(|x|)S_M} \left(\int_F V^{1-p_2'}(y) \times\right.\right.\right.$$

$$\times\Big(\int\limits_{c^{-1}(|y|)S_N\setminus d^{-1}(|y|)S_N} f_n(s,t)\,dt\Big)^{p_2'}\,dy\Big)^{\frac{1}{p_2'}}\,ds\Big)^{p_1'}\,dx\Big)^{\frac{1}{p_1'}} =$$

$$= \left\|T_E^* g_n\right\|_{L^{p_1'}(E,U^{1-p_1'})}.$$

Now the compactness of $T_{E,F}$ follows by Theorem B(c). □

4.3. **The precise criterion.** The precise necessary and sufficient conditions for the compactness of $T_{E,F}$ can now be obtained by using any known criterion for the compactness of T_E and T_F. In [13], the authors characterize the compactness of T_E and T_F in a more general setting but the same holds for cones also. We state the result below:

Theorem *D. Let E be a cone in $\mathbb{R}^N$ and S be a part of E with radius 1. Suppose u, v are weight functions on E. Then, the operator $T_E : L^p(E,u) \to L^q(E,v)$ defined by*

$$(T_E f)(x) = \int\limits_{b(|x|)S\setminus a(|x|)S} f(t)\,dt, \quad x\in E$$

where a and b are functions as defined in Section 3, is compact if and only if either

Case (a) : *$p \le q$ and the following conditions hold:*

$$\sup_{\substack{0<t\le s\\ a(s)\le b(t)}} \mathcal{S}(s,t) = \sup_{\substack{0<t\le s\\ a(s)\le b(t)}} \Big(\int\limits_{b(t)S\setminus a(s)S} u^{1-p'}\Big)^{\frac{1}{p'}}\Big(\int\limits_{sS\setminus tS} v\Big)^{\frac{1}{q}} < \infty$$

$$\lim_{s\to t^+}\mathcal{S}(s,t) = \lim_{s\to a^{-1}(b(t))^-}\mathcal{S}(s,t) = 0, \quad \text{for every } t>0$$

$$\lim_{t\to s^-}\mathcal{S}(s,t) = \lim_{t\to b^{-1}(a(s))^+}\mathcal{S}(s,t) = 0, \quad \text{for every } s>0$$

or

Case (b) : *$q<p$, $\dfrac{1}{r} = \dfrac{1}{q} - \dfrac{1}{p}$ and the following conditions hold :*

$$\Big(\int_0^\infty \int\limits_{tS\setminus b^{-1}(a(t))S}\Big(\int\limits_{b(|x|)S\setminus a(t)S} u^{1-p'}\Big)^{\frac{r}{p'}}\Big(\int\limits_{tS\setminus S_x} v\Big)^{\frac{r}{p}} v(x)\,dx\rho'(t)\,dt\Big)^{\frac{1}{r}} < \infty$$

$$\Big(\int_0^\infty \int\limits_{a^{-1}(b(t))S\setminus tS}\Big(\int\limits_{b(t)S\setminus a(|x|)S} u^{1-p'}\Big)^{\frac{r}{p'}}\Big(\int\limits_{S_x\setminus tS} v\Big)^{\frac{r}{p}} v(x)\,dx\rho'(t)\,dt\Big)^{\frac{1}{r}} < \infty,$$

where

$$\rho'(t) = \sum_{k\in\mathbb{Z}} \chi_{(M_k,M_{k+1})}(t)\frac{d}{dt}(b^{-1}\circ a)^k(t), \quad t\in(0,\infty).$$

Here $(b^{-1}\circ a)^k$ *denotes* k *times repeated composition and* $\{M_k\}_{k\in\mathbb{Z}}$ *is defined as*

$$M_0 = b^{-1}(1), \quad M_{k+1} = a^{-1}(b(M_k)) \ \text{ if } \ k \geq 0,$$
$$M_k = b^{-1}(a(M_{k+1})) \ \text{ if } \ k < 0.$$

Following corollary gives precise sufficient conditions for the compactness of $T_{E,F}$.

Corollary 5. *Let all the assertions of Theorem* 6 *be satisfied. Then, the operator* $T_{E,F} : [L^{p_1}(E,U), L^{p_2}(F,V)] \to [L^{q_1}(E,u), L^{q_2}(F,v)]$ *is compact if either*

Case (a) : $p_1 \leq q_1$, $p_2 \leq q_2$ *and the conditions* (16), (17)

$$\lim_{s\to t^+} B_1(s,t) = \lim_{s\to a^{-1}(b(t))^-} B_1(s,t) = 0, \quad \text{for every} \quad t > 0, \tag{26}$$

$$\lim_{t\to s^-} B_1(s,t) = \lim_{t\to b^{-1}(a(s))^+} B_1(s,t) = 0, \quad \text{for every} \quad s > 0, \tag{27}$$

$$\lim_{s\to t^+} B_2(s,t) = \lim_{s\to c^{-1}(d(t))^-} B_2(s,t) = 0, \quad \text{for every} \ t > 0, \tag{28}$$

$$\lim_{t\to s^-} B_2(s,t) = \lim_{t\to d^{-1}(c(s))^+} B_2(s,t) = 0, \quad \text{for every} \ s > 0 \tag{29}$$

are satisfied.

Case (b) : $p_1 \leq q_1$, $q_2 < p_2$ *and the conditions* (16), (18), (19), (26) *and* (27) *hold.*

Case (c) : $q_1 < p_1$, $p_2 \leq q_2$ *and the conditions* (17), (28), (29),

$$\left(\int_0^\infty \int_{tS_M\setminus b^{-1}(a(t))S_M} \left(\int_{b(|x|)S_M\setminus a(t)S_M} U^{1-p_1'}\right)^{\frac{r_1}{p_1'}} \times \right.$$
$$\left. \times \left(\int_{tS_M\setminus S_{M_x}} u\right)^{\frac{r_1}{p_1}} u(x)\ dx\rho'(t)\ dt\right)^{\frac{1}{r_1}} < \infty \tag{30}$$

and

$$\left(\int_0^\infty \int_{a^{-1}(b(t))S_M\setminus tS_M} \left(\int_{b(t)S_M\setminus a(|x|)S_M} U^{1-p_1'}\right)^{\frac{r_1}{p_1'}} \times \right.$$
$$\left. \times \left(\int_{S_{M_x}\setminus tS_M} u\right)^{\frac{r_1}{p_1}} u(x)\ dx\rho'(t)\ dt\right)^{\frac{1}{r_1}} < \infty \tag{31}$$

hold, where $\dfrac{1}{r_1} = \dfrac{1}{q_1} - \dfrac{1}{p_1}$ *and* ρ' *is as defined in Theorem D.*

Case (d) : $q_1 < p_1$, $q_2 < p_2$ *and the conditions* (18), (19), (30) *and* (31) *hold.*

Remark 11. As mentioned in Remark 8, we can derive a special case of the operator $T_{E,F}$ by taking $E = \mathbb{R}^M$, $F = \mathbb{R}^N$. The precise conditions for the compactness of this operator $T_{M,N}$ can then be stated similarly.

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References

1. J. Appell and A. Kufner, On the two dimensional Hardy operator in Lebesgue spaces with mixed norms. *Analysis* **15**(1995), No. 1, 91–98.

2. A. Benedek and R. Panzone, The spaces L^p with mixed norm. *Duke Math. J.* **28**(1961), 301–324.

3. J. Diestel, Sequences and series in Banach spaces. *Graduate Text in Mathematics,* 92*, Springer–Verlag, Berlin–Heidelberg–New York*, 1986.

4. P. Drábek, H. P. Heinig, and A. Kufner, Higher dimensional Hardy inequality. *General inequalities*, 7 (*Oberwolfach*, 1995), 3–16, *Internat. Ser. Numer. Math.*, 123, *Birkhauser, Basel*, 1997.

5. B. Gupta, P. Jain, L. E. Persson, and A. Wedestig, Weighted geometric mean inequalities over cones in $\mathbb{R}^N$. (*Submitted*).

6. G. H. Hardy, Note on a theorem of Hilbert. *Math. Z.* **6**(1920), 314–317.

7. H. Heinig, R. Kerman and M. Krbec, Weighted exponential inequalities. *Georgian Math. J.* **8**(2001), 69–86.

8. H. Heinig, A. Kufner, and L. E. Persson, Some generalizations and refinements of the Hardy inequality. *In: Recent Progress in Inequalities – A volume dedicated to Professor D. S. Mitrinovic*, 271–288, *Kluwer Publ.*, 1998.

9. P. K. Jain, O. P. Ahuja, and K. Ahmad, Functional Analysis. *New Age International* (*P*) *Ltd., New Delhi,* 1995.

10. P. K. Jain and V. P. Gupta, Lebesgue Measure and Integration. *New Age International* (*P*) *Ltd., New Delhi,* 1994.

11. P. Jain and R. Hassija, Some remarks on two dimensional Knopp type inequalities. *Appl. Math. Lett.* (to appear).

12. P. Jain, P. K. Jain, and B. Gupta, On certain weighted integral inequalities with mixed norm. *Italian J. Pure Appl. Math.* (to appear).

13. P. Jain, P. K. Jain, and B. Gupta, Compactness of Hardy type operators over star-shaped regions in $\mathbb{R}^N$. (*Submitted*).

14. P. Jain, L. E. Persson, and A. Wedestig, Multidimensional Cochran and Lee type inequalities with weights. *Proc. A. Razmadze Math. Inst.* **129**(2002), 17–27.

15. P. Jain, A. P. Singh, and P. K. Jain, A two dimensional Knopp inequality with weights. *In: Analysis and Applications*, 93–100, *Narosa Publishing House, New Delhi*, 2002.

16. A. Kufner and L. E. Persson, Weighted inequalities of Hardy Type. *World Sci., Singapore* (*to appear*).

17. B. Opic and A. Kufner, Hardy-Type Inequalities. *Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow,* 1990.

18. E. Sawyer, Weighed inequalities for the two dimensional Hardy operator. *Studia Math.* **82**(1985), 1–16.

19. G. Sinnamon, Hardy-type inequalities for a new class of integral operators. *In: Analysis of Divergence, Control and Management of Divergent Process Birkhauser*, 297–307, *Boston*, 1999.

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