ON THE NUMBER OF REPRESENTATIONS OF POSITIVE INTEGERS BY A DIRECT SUM OF QUATERNARY QUADRATIC FORMS WITH DISCRIMINANT $19^2$

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Abstract. Explicit exact formulas are obtained for the number of representations of positive integers by direct sums of reduced quadratic forms $F_2 = x_1^2 + 5x_2^2 + x_3x_4 + 5x_4^2$ and $\Phi_2 = x_1^2 + 2x_2^2 + 3x_3^2 + 6x_4^2 - x_1x_3 + 2x_2x_4 + 3x_3x_4$. Further let $F_4 = F_2 \oplus F_2$ and $\Phi_4 = \Phi_2 \oplus \Phi_2$.

§ 1. Preliminaries

1.1. Let $F_2 = x_1^2 + x_1x_2 + 5x_2^2 + x_3x_4 + 5x_4^2$ and $\Phi_2 = x_1^2 + 2x_2^2 + 3x_3^2 + 6x_4^2 + x_1x_2 - x_1x_3 + 2x_2x_4 + 3x_3x_4$ be positive reduced quaternary quadratic forms with discriminant $19^2$. Further let $F_4 = F_2 \oplus F_2$ and $\Phi_4 = \Phi_2 \oplus \Phi_2$.

In the present paper we obtain explicit exact formulas for the arithmetical function $r(n, Q)$ when $Q = F_4, F_2 \oplus F_2, \Phi_4$.

1.2. In this paper the notations, definitions and some results from [1] will be mostly used.

Let

$$Q = Q(x_1, x_2, \ldots, x_f) = \sum_{1 \leq r \leq s \leq f} b_{rs}x_rx_s$$

be a positive quadratic form in $f$ ($f$ is even) variables with integral coefficients $b_{rs}$. Further let $D$ be the determinant of the quadratic form

$$2Q = \sum_{r, s=1}^{f} a_{rs}x_rx_s \quad (a_{rr} = 2h_{rr}, \ a_{rs} = a_{sr} = b_{rs}, \ r < s);$$

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the algebraic cofactors of elements \(a_{rs}\) in \(D\); \(\Delta\) the discriminant of the form \(Q\), i.e., \(\Delta = (-1)^{f/2}D\); \(\delta = \gcd\) of \((\frac{d}{\Delta}, A_{rs})\) \((r, s = 1, \ldots, f)\); \(N = \frac{\Delta}{\delta}\) the level of the form \(Q\); \(\chi(d)\) the character of the form \(Q\), i.e., \(\chi(d) = 1\) if \(\Delta\) is a perfect square, but if \(\Delta\) is not a perfect square and \(2 \nmid \Delta\), then \(\chi(d) = (\frac{d}{\Delta})\) for \(d > 0\) and \(\chi(d) = (-1)^{f/2} \chi(-d)\) for \(d < 0\) (here \((\frac{d}{\Delta})\) is the generalized Jacobi symbol). A positive quadratic form in \(f\) variables of level \(N\) and character \(\chi\) is called a quadratic form of type \((-\frac{f}{2}, N, \chi)\).

In what follows \(q\) is an odd prime and \(z = \exp(2\pi i\tau), \text{Im}\tau > 0\).

As is well known to each positive quadratic form \(Q\) there corresponds the theta-series
\[
\vartheta(\tau; Q) = 1 + \sum_{n=1}^{\infty} r(n; Q)z^n.
\]
(1.1)

We shall formulate the well-known results in the form of the following lemmas.

**Lemma 1** ([1], pp. 874, 875, 817; see also [2], p. 15). If \(Q\) is an arbitrary primitive positive quadratic form of type \((-k, q, 1)\), \(2 \mid k, k > 2\), then \(\Delta = q^2\ell (1 \leq \ell \leq k - 1)\) and it corresponds one and the same Eisenstein series
\[
E(\tau; Q) = 1 + \sum_{n=1}^{\infty} (\alpha \sigma_{k-1}(n)z^n + \beta \sigma_{k-1}(n)z^{qn}),
\]
where
\[
\alpha = \frac{i^k q^{k-\ell} - i^k}{p_k q^{k} - 1}, \quad \beta = \frac{1}{p_k q^k - 1}, \quad \rho_k = (-1)^{k/2} \frac{(k-1)!}{(2\pi)^k} \zeta(k).
\]
(1.2)

**Lemma 2** ([1], pp. 874, 875, 895). If \(Q\) is a primitive quadratic form of type \((-k, q, 1)\), \(2 \mid k, \) then the difference \(\vartheta(\tau; Q) - E(\tau; Q)\) is a cusp form of type \((-k, q, 1)\).

**Lemma 3** ([1], p. 853, Theorem 33). The homogeneous quadratic polynomials in \(f\) variables \(\varphi_{rs} = x_r x_s - \frac{1}{4} \frac{d}{\Delta_{rs}} 2Q\) \((r, s = 1, 2, \ldots, f)\) are spherical functions of second order with respect to the positive quadratic form \(Q\) in \(f\) variables.

**Lemma 4** ([1], p. 855). If \(Q\) is a quadratic form of type \((-\frac{f}{2}, N, \chi)\) and \(P_\nu\) is the spherical function of order \(\nu\) with respect to \(Q\), then the generalized multiple theta-series
\[
\vartheta(\tau; Q, P_\nu) = \sum_{n=1}^{\infty} \left( \sum_{Q=n} \right) P_\nu(z^n).
\]
is a cusp form of type $-\left(\frac{f}{2} + \nu\right), N, \chi$.

Lemma 5 ([1], p. 846). If the quadratic forms $Q_1$ and $Q_2$ have the same level $N$ and characters $\chi_1(d)$ and $\chi_2(d)$ respectively, then the quadratic form $Q_1 \oplus Q_2$ will have the level $N$ and the character $\chi_1(d)\chi_2(d)$.

§ 2. SOME AUXILIARY RESULTS

2.1. For the quadratic form $F_2$ we have: $D = \Delta = 19^2$, $A_{11} = 10 \cdot 19$, $\delta = 19$, $N = 19$, $\chi(d) = 1$. Hence, if in Lemma 3 we put $f = 4$, $Q = F_2$, $r = s = 1$, then the polynomials

$$\varphi_{11} = x_1^2 - \frac{5}{19} F_2$$

will be spherical function of second order with respect to $F_2$.

For the quadratic form $\Phi_2$ we have: $D = \Delta = 19^2$, $A_{11} = 12 \cdot 19$, $A_{22} = 6 \cdot 19$, $A_{33} = 4 \cdot 19$, $\delta = 19$, $N = 19$, $\chi(d) = 1$. Hence, if in Lemma 3 we put $f = 4$, $Q = \Phi_2$, $r = s = 1$, then the polynomials

$$\varphi_{11} = x_1^2 = \frac{6}{19} \Phi_2, \quad \varphi_{22} = x_2^2 - \frac{3}{19} \Phi_2, \quad \varphi_{33} = x_3^2 - \frac{2}{19} \Phi_2$$

will be spherical functions of second order with respect to $\Phi_2$.

$F_2$ and $\Phi_2$ are quadratic forms of type $(-2, 19, 1)$.

2.2. It is easy to verify that the equation $F_2 = n$

(a) has four integral solutions for $n = 1$: $x_1 = \pm 1$, $x_2 = x_3 = x_4 = 0$;

(b) has no integral solutions for $n = 2$;

(c) has four integral solutions for $n = 3$;

(d) has four integral solutions for $n = 4$: $x_1 = \pm 2$, $x_2 = x_3 = x_4 = 0$.

Hence, according to (1.1), we get

$$\vartheta(\tau; F_2) = 1 + 4z + 4z^2 + 4z^4 + \cdots$$

The quadratic form $\Phi_2$ is discovered by Shavgulidze, who in ([3], p. 202) shows that the equation $\Phi_2 = n$

(a) has two integral solutions for $n = 1$: $x_1 = \pm 1$, $x_2 = x_3 = x_4 = 0$;

(b) has four integral solutions for $n = 2$: $x_1 = \pm 1$, $x_2 = \mp 1$, $x_3 = x_4 = 0$;

(c) has four integral solutions for $n = 3$: $x_3 = \pm 1$, $x_1 = x_2 = x_4 = 0$;

(d) has four integral solutions for $n = 4$: $x_1 = \pm 2$, $x_2 = x_3 = x_4 = 0$;
(d) has eight integral solutions for \( n = 4 \): \( x_1 = \pm 2, x_2 = x_3 = x_4 = 0; x_1 = \pm 2, x_2 = \mp 1, x_3 = x_4 = 0; x_1 = x_2 = \pm 1, x_3 = x_4 = 0; x_1 = x_3 = \pm 1, x_2 = \mp 1, x_4 = 0. \)

Hence, according to (1.1), we get
\[
\vartheta(\tau; \Phi_2) = 1 + 2z + 4z^2 + 4z^3 + 8z^4 + \cdots.
\]

Further it is obvious that
\[
\begin{align*}
\vartheta(\tau; F_2) &= \vartheta^2(\tau; F_2) = 1 + 8z + 24z^2 + 32z^3 + 24z^4 + \cdots, \\
\vartheta(\tau; F_2 \oplus \Phi_2) &= \vartheta(\tau; F_2)\vartheta(\tau; \Phi_2) = \\
&= 1 + 6z + 16z^2 + 28z^3 + 44z^4 + \cdots, \\
\vartheta(\tau; \Phi_4) &= \vartheta^2(\tau; \Phi_2) = 1 + 4z + 12z^2 + 24z^3 + 48z^4 + \cdots,
\end{align*}
\]
since \( F_4 = F_2 \oplus F_2 \) and \( \Phi_4 = \Phi_2 \oplus \Phi_2 \).

§ 3. Formulas for \( r(n; F_4), r(n; F_2 \oplus \Phi_2) \) and \( r(n; \Phi_4) \)

**Theorem 1.** The system of generalized fourfold theta-series

\[
\begin{align*}
\vartheta(\tau; F_2, \varphi_{11}) &= \frac{1}{19} \sum_{n=1}^{\infty} \left( \sum_{F_2=n} 19x_1^2 - 5n \right) z^n, \\
\vartheta(\tau; \Phi_2, \varphi_{11}) &= \frac{1}{19} \sum_{n=1}^{\infty} \left( \sum_{\Phi_2=n} 19x_1^2 - 6n \right) z^n, \\
\vartheta(\tau; \Phi_2, \varphi_{22}) &= \frac{1}{19} \sum_{n=1}^{\infty} \left( \sum_{\Phi_2=n} 19x_2^2 - 3n \right) z^n, \\
\vartheta(\tau; \Phi_2, \varphi_{33}) &= \frac{1}{19} \sum_{n=1}^{\infty} \left( \sum_{\Phi_2=n} 19x_3^2 - 2n \right) z^n
\end{align*}
\]
is a basis of the space \( S_4(19, 1) \) (the space of cusp forms of type \( -4, 19, 1 \)).

**Proof.** As is said above, \( F_2 \) is a quadratic form of type \( -2, 19, 1 \) and \( \varphi_{11} = x_1^2 - \frac{1}{19} F_2 \) is the spherical function of second order with respect to \( F_2 \). Hence, by Lemma 4, the theta-series (3.1) is a cusp form of type \( -4, 19, 1 \).

Taking into account the solutions of the equation \( F_2 = n \) in the Subsection 2.2, we get:
\[
\vartheta(\tau; F_2, \varphi_{11}) = \frac{1}{19} \left\{ (19 \cdot 2 - 5 \cdot 4)z + (19 - 5 \cdot 2)4z^2 + \\
+ (19 \cdot 4 \cdot 2 - 5 \cdot 4 \cdot 4)z^4 + \cdots \right\} = \\
= \frac{18}{19}z + \frac{36}{19}z^2 + \frac{72}{19}z^4 + \cdots.
\]

II. As is said above \( \Phi_2 \) is a quadratic form of type \( -2, 19, 1 \) and \( \varphi_{11} = x_1^2 - \frac{6}{19} \Phi_2, \varphi_{22} = x_2^2 - \frac{3}{19} \Phi_2, \varphi_{33} = x_3^2 - \frac{2}{19} \Phi_2 \) are the spherical functions
of second order with respect to $\Phi_2$. Hence, by Lemma 4 the theta-series (3.2)–(3.4) are cusp forms of type $(-4, 19, 1)$.

Taking into account the solutions of the equation $\Phi_2 = n$ in Subsection 2.2, we get:

$$\vartheta(\tau; \Phi_2, \varphi_{11}) = \frac{1}{19} \left\{ (19 - 6)2z + (19 \cdot 2 - 6 \cdot 2 \cdot 4)z^2 + 
(19 \cdot 2 - 6 \cdot 3 \cdot 4)z^3 + 
(19 \cdot 4 + 19 \cdot 4 - 6 \cdot 4 \cdot 8)z^4 + \cdots \right\} = 
= \frac{26}{19} z - \frac{10}{19} z^2 - \frac{34}{19} z^3 + \frac{188}{19} z^4 + \cdots \quad (3.6)$$

$$\vartheta(\tau; \Phi_2, \varphi_{22}) = \frac{1}{19} \left\{ (-3 \cdot 2)z + (19 - 3 \cdot 2)4z^2 + (-3 \cdot 3 \cdot 4)z^3 + 
(19 \cdot 6 - 3 \cdot 4 \cdot 8)z^4 + \cdots \right\} = 
= -\frac{6}{19} z + \frac{52}{19} z^2 - \frac{36}{19} z^3 + \frac{18}{19} z^4 + \cdots \quad (3.7)$$

$$\vartheta(\tau; \Phi_2, \varphi_{33}) = \frac{1}{19} \left\{ (-2 \cdot 2)z + (-2 \cdot 2 \cdot 4)z^2 + (19 - 2 \cdot 3 \cdot 4)z^3 + 
(19 \cdot 2 - 2 \cdot 4 \cdot 8)z^4 + \cdots \right\} = 
= -\frac{4}{19} z - \frac{16}{19} z^2 + \frac{52}{19} z^3 - \frac{26}{19} z^4 + \cdots \quad (3.8)$$

The system of theta-series (3.1)–(3.4) is linearly independent, since the determinant of fourth order, whose elements are coefficients in the expansions of (3.5)–(3.8), is different from zero. Thus the Theorem is proved, since $S_4(19, 1) = 4$ ([1], p. 899). □

**Theorem 2.**

$$r(n; F_4) = \frac{120}{181} \sigma_3^e(n) + \frac{43004}{9 \cdot 31 \cdot 181} \sum_{F_2 = n} (19x_1^2 - 5n) - 
- \frac{2592}{31 \cdot 181} \sum_{\Phi_2 = n} (19x_1^2 - 6n) - 
- \frac{2460}{31 \cdot 181} \sum_{\Phi_2 = n} (19x_2^2 - 3n) - 
- \frac{1948}{31 \cdot 181} \sum_{\Phi_2 = n} (19x_3^2 - 2n), \quad (I)$$

$$r(n; F_2 \oplus \Phi_2) = \frac{120}{181} \sigma_3^e(n) + \frac{476615}{9 \cdot 19 \cdot 31 \cdot 181} \sum_{F_2 = n} (19x_1^2 - 5n) - 
- \frac{22098}{19 \cdot 31 \cdot 181} \sum_{\Phi_2 = n} (19x_1^2 - 6n) -$$
\[
\sum_{\Phi_2=n} \Phi_2 = n (19x_2^2 - 3n) - \frac{23934}{19 \cdot 31 \cdot 181} \sum_{\Phi_2=n} (19x_2^2 - 3n) - \frac{11672}{19 \cdot 31 \cdot 181} \sum_{\Phi_2=n} (19x_2^2 - 2n),
\]

\[
(II)
\]

\[
r(n; \Phi_4) = \frac{120}{181} \sigma_3^*(n) + \frac{9785}{2 \cdot 31 \cdot 181} \sum_{\Phi_2=n} (19x_2^2 - 5n) - \frac{11600}{19 \cdot 31 \cdot 181} \sum_{\Phi_2=n} (19x_2^2 - 6n) - \frac{25967}{2 \cdot 19 \cdot 31 \cdot 181} \sum_{\Phi_2=n} (19x_2^2 - 3n) - \frac{10855}{2 \cdot 19 \cdot 31 \cdot 181} \sum_{\Phi_2=n} (19x_2^2 - 2n),
\]

\[
(III)
\]

where

\[
\sigma_3^*(n) = \begin{cases} 
\sigma_3(n) & \text{if } 19 \nmid n, \\
\sigma_3(n) + 19^2 \sigma_3 \left( \frac{n}{19} \right) & \text{if } 19 \mid n.
\end{cases}
\]

**Proof.** By Lemma 5, \(F_4, F_2 \oplus \Phi_2\) and \(\Phi_4\) are quadratic forms of type \((-4, 19, 1)\) and by Lemma 1, one and the same Eisenstein series corresponds to them. For \(k = 4\) and \(\ell = 2\) from (1.2) we have

\[
\alpha = \frac{1}{\rho_4} \frac{1}{19^2 - 1} = \frac{1}{\rho_4} \frac{1}{19^2 + 1}, \quad \beta = \frac{1}{\rho_4} \frac{19^4 - 19^2}{19^4 - 1} = \frac{1}{\rho_4} \frac{19^2}{19^2 + 1},
\]

where \(\rho_4 = \frac{1}{240} ([1], \text{p. 823}). Thus for all these forms

\[
E(\tau; F_4) = E(\tau; F_2 \oplus \Phi_2) = E(\tau; \Phi_4) = 1 + \frac{120}{19^2 + 1} \sum_{n=1}^{\infty} \sigma_3^*(n)z^n, \quad (3.9)
\]

I. By Lemma 2, the difference \(\vartheta(\tau; F_4) - E(\tau; F_4)\) is a cusp form of type \((-4, 19, 1)\). Hence, by Theorem 1, there exist numbers \(c_1, c_2, c_3, c_4\) such that

\[
\vartheta(\tau; F_4) = E(\tau; F_4) + 43004 \cdot \frac{19}{9 \cdot 31 \cdot 181} \vartheta(\tau; F_2, \varphi_{11}) - 2592 \cdot \frac{19}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{11}) - 2460 \cdot \frac{19}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{22}) - 1948 \cdot \frac{19}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{33}).
\]

Equating the coefficients of \(z, z^2, z^3, z^4\) on both sides of this equality and taking into account (2.1), (3.9) and (3.6)–(3.8), by the method of successive exclusion of unknowns, we find these numbers and obtain that

\[
\vartheta(\tau; F_4) = E(\tau; F_4) + \frac{43004 \cdot 19}{9 \cdot 31 \cdot 181} \vartheta(\tau; F_2, \varphi_{11}) - \frac{2592 \cdot 19}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{11}) - \frac{2460 \cdot 19}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{22}) - \frac{1948 \cdot 19}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{33}).
\]
Equating the coefficients of $z^n$ on both sides of this equality, by (1.1), (3.9) and (3.1)–(3.4), we get the formula (I).

II. Applying the same arguments as above, except that (2.1) is replaced by (2.2) and (2.3), for the quadratic forms $F_2 \oplus \Phi_2$ and $\Phi_4$ respectively, we obtain

$$\vartheta(\tau; F_2 \oplus \Phi_2) = E(\tau; F_2 \oplus \Phi_2) + \frac{476615}{9 \cdot 31 \cdot 181} \vartheta(\tau; F_2, \varphi_{11}) - \frac{22098}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{11}) - \frac{23934}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{22}) - \frac{11672}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{33}).$$

and

$$\vartheta(\tau; \Phi_4) = E(\tau; \Phi_4) + \frac{185915}{2 \cdot 3 \cdot 31 \cdot 181} \vartheta(\tau; F_2, \varphi_{11}) - \frac{11600}{31 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{11}) - \frac{25967}{2 \cdot 3 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{22}) - \frac{10853}{2 \cdot 3 \cdot 181} \vartheta(\tau; \Phi_2, \varphi_{33}).$$

From these identities, as above, we get the formulas (II) and (III).

Remark. The formula for $\vartheta(n; \Phi_4)$ in a few another view was obtained by Shavgulidze ([3], p. 214, Theorem 5).

References


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