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V. Kokilashvili

ON THE MEAN SPHERICAL CONVERGENCE AND SUMMABILITY OF MULTIPLE FOURIER TRIGONOMETRIC SERIES

The goal of our talk is to discuss the mean spherical convergence and summability problems of multiple Fourier trigonometric series in various function spaces.

Let us consider the spherical partial sums for multiple trigonometric series

$$S_\mu(f)(x) = \sum_{|k|\le\mu} c_k e^{ikx}.$$

It is well-known that multiple trigonometric system is not a base in $L^p(\mathbb{T}^2)$ $(p \ne 2)$ in the sense of spherical convergence. It follows from the C. Fefferman's well-known result.

Theorem A ([3]). *The characteristic function of unit disk is a Fourier multiplier in $L^p(\mathbb{T}^2)$ space if and only if $p = 2$.*

Let us pose the following question: is it possible to find an integrable weight w, for which multiple trigonometric system is a base in $L^p_w(\mathbb{T}^2)$. The answer to this question is negative.

Theorem 1. *There is no integrable weight w, such that*

$$w^{-1} \in L^{1/(p-1)}, \quad w^{-1} \notin L^{1/(p-1)+\eta}$$

for all $\eta > 0$ and multiple trigonometric system is base in L^p_w $(p \ne 2)$ in the sense of spherical convergence.

Proof. On the first step of proof we use an interpolation reasoning. Let us assume the contrary: let for some $p > 1$, $p \ne 2$ and some integrable w

$$\lim_{\mu\to\infty} \|S_\mu(\cdot, f) - f\|_{p,w} = 0$$

for arbitrary $f \in L^p_w(\mathbb{T}^2)$.

This yields that the operator

$$f \longrightarrow S_\mu(\cdot, f)$$

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is uniformly bounded in $L^p_w(\mathbb{T}^2)$.

By the duality argument we conclude that the latter operator is bounded in $L^{p'}_{w^{1-p'}}(\mathbb{T}^2)$ as well. Suppose $\theta = \frac{1}{p'}$ and $\frac{1}{p_0} = \frac{1-\theta}{p} + \frac{\theta}{p'}$.

Then we can see that $p_0 \neq 2$ and $p_0 > 1$. Applying the well-known Stein-Weiss interpolation theorem we get that

$$\int\limits_{\mathbb{R}^2} |S_\mu(x,f)|^{p_0} w^{1-\theta}(x) w^{\theta(1-p')}(x)dx \leq c \int\limits_{\mathbb{R}^2} |f(x)|^{p_0} w^{1-\theta}(x) w^{\theta(1-p')}(x)dx$$

with a constant independent of μ and f.

But $(1-\theta)+\theta(1-p') = 0$. Thus we obtain that the operator $f \longrightarrow S_\mu(\cdot, f)$ is uniformly bounded in $L^{p_0}(\mathbb{T}^2)$, $p_0 \neq 2$.

According to the V. A. Ilyn's result (see [4]), the multiple trigonometric series of $\varphi \in C^\infty$ uniformly converges to φ in the sense of spherical convergence. Consequently, for that φ we have

$$\lim_{\mu\to\infty} \|\varphi - S_\mu(\cdot,\varphi)\|_{p_0} = 0.$$

On the other hand C^∞ is dense in $L^{p_0}(\mathbb{T}^2)$.

Thus by the Banach-Steinhaus theorem

$$\lim_{\mu\to\infty} \|f - S_\mu(\cdot,f)\|_{p_0} = 0$$

for every $f \in L^p(\mathbb{T}^2)$.

The latter contradicts to the fact that the multiple trigonometric system is not a base in $L^{p_0}(\mathbb{T}^2)$, $p_0 \neq 2$. $\square$

Next problem deals with the spherical summability of multiple trigonometric Fourier series in some new function spaces.

Let $s(x)$ be continuous, 2π-periodic function defined on $\mathbb{R}$. We suppose that $s(x)$ satisfies the log-Hölder continuity condition i.e. there exists a positive constant A such that for all $x, y \in \mathbb{R}$, $|x-y| < \frac{1}{2}$, the inequality

$$|s(x) - s(y)| \leq \frac{A}{-\log|x-y|}$$

holds.

In the sequel we denote the class of 2π-periodic functions satisfying the log-Hölder continuity condition by $\mathcal{P}^{\log}$. Further, we say that $s \in \mathcal{P}$ if

$$1 < s_- \leq s_+ < \infty,$$

where

$$s_- = \inf_{\mathbb{T}^n} |s(x)|, \quad s_+ = \sup_{\mathbb{T}^n} |s(x)|.$$

Definition 1. Let $p \in \mathcal{P}$ and $\theta > 0$. Denote by $L^{p(\cdot),\theta}(\mathbb{T}^{\ltimes})$ the class of those 2π-periodic measurable functions for which

$$\|f\|_{p(\cdot),\theta} = \sup_{0<\varepsilon<p_- -1} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|f\|_{p(\cdot)-\varepsilon,\theta}$$

where

$$\|f\|_{p(\cdot)-\varepsilon} = \inf_{\lambda>0}\left\{\lambda : \int_{\mathbb{T}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)-\varepsilon} dx \le 1\right\}.$$

These spaces were introduced and studied in view of boundedness of harmonic analysis operators by V. Kokilashvili and A. Meskhi [6]. The spaces $L^{p(\cdot),\theta}(\mathbb{T}^n)$ unify two nonstandard Banach function spaces: variable exponent Lebesgue spaces as a special case of Musielak-Orlicz spaces and grand Lebesgue spaces. For the variable exponent Lebesgue spaces we refer the readers to [2], [1]. The grand Lebesgue spaces were introduced by T. Iwaniec and C. Sbordone [5]. It should be noted that the grand variable exponent Lebesgue spaces $L^{p(\cdot),\theta}$ are non-reflexive, non-separable and non-rearrangement invariant.

The closure of the space $L^{p(\cdot)}(\mathbb{T})$ by the norm of $L^{p(\cdot),\theta}(\mathbb{T})$, $\theta > 0$, does not coincide with the latter space. Let us denote this closure by $\widetilde{L}^{p(\cdot),\theta}(\mathbb{T})$. This subspace of $L^{p(\cdot),\theta}$ is a set of functions for which

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p_- -\varepsilon}} \|f\|_{p(\cdot)-\varepsilon} = 0.$$

Now let us consider the Riesz summability means for multiple trigonometric series

$$R_\mu^s(f)(x) = \sum_{|m|\le\mu} \left(1 - \frac{|m|^2}{\mu^2}\right) c_m e^{imx}, \quad m = (m_1, \dots, m_n).$$

Theorem B ([7]). *Let $1 < p < \infty$, $s > (n-1)\left|\frac{1}{p} - \frac{1}{2}\right|$, then*

$$\lim_{\mu\to\infty} \|R_\mu^s(f) - f\|_p = 0.$$

Moreover, when $s > \frac{n-1}{2}$, then $\lim\limits_{\mu\to\infty} R_\mu^s(f) = f(x)$ almost everywhere for arbitrary $f \in L^p(\mathbb{T}^n)$.

The following statements are true:

Theorem 2. *Let $p \in \mathcal{P} \bigcap \mathcal{P}^{\log}$, $\theta > 0$ and let $s > \frac{n-1}{2}$. Then the operator*

$$R_*^s(x) = \sup_{\mu>0} |R_\mu^s(f)(x)|$$

is bounded in $L^{p(\cdot),\theta}(\mathbb{T}^n)$.

From this theorem we can conclude

Theorem 3. *Let* $p \in \mathcal{P} \bigcap \mathcal{P}^{\log}$, $\theta > 0$ *and let* $s > \frac{n-1}{2}$. *Then*

$$\lim_{\mu \to \infty} \|R_{\mu}^{s}(f) - f\|_{p(\cdot),\theta} = 0$$

for arbitrary $f \in \widetilde{L}^{p(\cdot),\theta}$.

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Author's address:

A. Razmadze Mathematical Institute
Iv. Javakhishvili Tbilisi State University
6, Tamarashvili St., Tbilisi 0177
Georgia
E-mail: kokil@rmi.ge