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ON INSCRIBED AND CIRCUMSCRIBED CONVEX POLYHEDRA

This report contains several remarks about inscribed and circumscribed polyhedra. Below, we will be dealing with convex (and, more generally, simple) polyhedra lying in the n–dimensional Euclidean space $\mathbf{R}^n$, where $n \geq 2$. Recall that a polyhedron $P \subset \mathbf{R}^n$ is called simple if P is homeomorphic to the unit ball $\mathbf{B}_n$ of $\mathbf{R}^n$.

As usual, we say that two simple polyhedra P and Q in $\mathbf{R}^n$ are combinatorially isomorphic (or combinatorially equivalent, or are of the same combinatorial type) if the geometric complex canonically associated with P is isomorphic to the geometric complex canonically associated with Q.

A combinatorial type of convex polyhedra is called inscribable (respectively circumscribable) if there exists at least one representative of this type which admits an inscribed sphere (respectively, admits a circumscribed sphere).

It can easily be seen that every combinatorial type in $\mathbf{R}^2$ is simultaneously inscribable and circumscribable. On the other hand, there are many examples of combinatorial types of convex polyhedra in $\mathbf{R}^3$ which are not inscribable (respectively, are not circumscribable). For more detailed information, we refer the reader to [3], [4] and references therein. Notice that in [4] inscribable types of convex polyhedra in $\mathbf{R}^3$ are characterized in certain purely combinatorial terms. This characterization does not hold for simple polyhedra in $\mathbf{R}^3$ (cf. Example 5 below).

Example 1. According to the celebrated Steinitz theorem (see, e.g., [3]), every combinatorial type of simple polyhedra in $\mathbf{R}^3$ has a convex representative P. Moreover, there exists a convex representative Q of the type, which admits a midsphere, i.e., the sphere touching all the edges of Q. Such a sphere is also called the Koebe sphere of Q and Q itself is called a Koebe polyhedron. During many years, it was unknown whether any combinatorial type of simple polyhedra in $\mathbf{R}^3$ possesses a convex representative T such that there exists a point in the interior of T with the property that

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all perpendiculars dropped from this point to the facets of T intersect all corresponding facets. Obviously, the existence of Q solves this problem in a much stronger form.

For the sake of brevity, we shall say that a combinatorial type of convex polyhedra is singular if it is non–inscribable and non–circumscribable simultaneously.

Theorem 1. *Equip the family of all nonempty compact subsets of* $\mathbf{R}^n$ *with the standard Hausdorff metric* ρ, *and let* P *be an arbitrary convex polyhedron in* $\mathbf{R}^n$, *where* $n \geq 3$. *Then, for each strictly positive real* ε, *there exists a convex polyhedron* Q *in* $\mathbf{R}^n$ *such that:*
(1) $\rho(P, Q) < \varepsilon$;
(2) *the combinatorial type of* Q *is singular.*

For our further purposes, we need the following three simple auxiliary propositions.

Lemma 1. *Let* T *be a triangle with the side lengths* a, b, c *and let* T' *be a triangle with the side lengths* a', b', c'. *Let* γ *denote the angle of* T *between* a *and* b *and let* γ' *denote the angle of* T' *between* a' *and* b'. *Suppose also that*

$$a \leq a', \quad b \leq b', \quad \gamma < \gamma'.$$

Then if γ' *is right or obtuse, the inequality* $c < c'$ *holds true.*

The condition that γ' is right or obtuse is essential in the statement of Lemma 1.

Lemma 2. *Let* $k > 0$ *be a natural number, and let*

$$a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k, c$$

be any strictly positive real numbers. Consider the irrational equation

$$b_1(x^2 - a_1^2)^{1/2} + b_2(x^2 - a_2^2)^{1/2} + \cdots + b_k(x^2 - a_k^2)^{1/2} = c.$$

Assuming $a_1 = \max\{a_i : 1 \leq i \leq k\}$, *this equation has a unique strictly positive solution if and only if*

$$b_2(a_1^2 - a_2^2)^{1/2} + b_3(a_1^2 - a_3^2)^{1/2} + \cdots + b_k(a_1^2 - a_k^2)^{1/2} \leq c.$$

There exists a nonzero polynomial $\chi(x^2, a_1^2, a_2^2, \dots, a_k^2, b_1^2, b_2^2, \dots, b_k^2, c^2)$ *of the variable* x^2 *whose coefficients are some polynomials of*

$$a_1^2, a_2^2, \dots, a_k^2, b_1^2, b_2^2, \dots, b_k^2, c^2,$$

such that any root of the above equation is simultaneously a root of the polynomial χ.

The proof can easily be done by induction on k.

Lemma 3. *Let $n > 0$ be a natural number. If n is odd, i.e., $n = 2m+1$, then the following formula holds true for any reals $\theta_1, \theta_2, \dots, \theta_n$:*

$$\mathrm{tg}(\theta_1 + \theta_2 + \cdots + \theta_n) = \frac{\sigma_1 - \sigma_3 + \cdots (-1)^m \sigma_n}{1 - \sigma_2 + \sigma_4 - \cdots + (-1)^m \sigma_{n-1}}.$$

If n is even, i.e., $n = 2m$, then the following formula holds true for any reals θ_1, θ_2, $\dots$, θ_n:

$$\mathrm{tg}(\theta_1 + \theta_2 + \cdots + \theta_n) = \frac{\sigma_1 - \sigma_3 + \cdots (-1)^{m-1} \sigma_{n-1}}{1 - \sigma_2 + \sigma_4 - \cdots + (-1)^m \sigma_n}.$$

Here $\sigma_1, \sigma_2, \dots, \sigma_n$ stand for the basic symmetric functions of the variables $\mathrm{tg}(\theta_1), \mathrm{tg}(\theta_2), \dots, \mathrm{tg}(\theta_n)$.

The proof of this lemma is readily obtained by induction on n.

Example 2. If P is a convex k–gon in the plane $\mathbf{R}^2$ with given successive side lengths $a_1, a_2, \dots, a_k$, then the following two assertions are equivalent:
(a) the area of P is maximal;
(b) P is inscribed in a circle.

Moreover, it can be demonstrated by using Lemma 1 that, under (b), the radius R of the circumscribed circle of P is uniquely determined. Consequently, if P' is another convex k–gon inscribed in a circle and having the same successive side lengths $a_1, a_2, \dots, a_k$, then P and P' turn out to be congruent. Recall also that, for the existence of P satisfying (b), it is necessary and sufficient that

$$2\max(a_1, a_2, \dots, a_k) < a_1 + a_2 + \cdots + a_k.$$

In other words, a convex k–gon P inscribed in a circle with the given side lengths $a_1, a_2, \dots, a_k$ exists if and only if there exists at least one simple k–gon with the same side lengths. Actually, the circumradius R of P is an algebraic function of the variables $a_1^2, a_2^2, \dots, a_k^2$. Speaking explicitly, there exists a nonzero polynomial $f(x^2, a_1^2, a_2^2, \dots, a_k^2)$ of the variable x^2, whose coefficients are some polynomials of the variables $a_1^2, a_2^2, \dots, a_k^2$, such that

$$f(R^2, a_1^2, a_2^2, \dots, a_k^2) = 0.$$

Analogously, assuming (b) and denoting the area of P by the symbol s, a nonzero polynomial $g(y^2, a_1^2, a_2^2, \dots, a_k^2)$ of the variable y^2 can be written, whose coefficients are also certain polynomials of the variables $a_1^2, a_2^2, \dots, a_k^2$, such that

$$g(s^2, a_1^2, a_2^2, \dots, a_k^2) = 0.$$

For nice proofs of these two results, see [1], [5], and references in [1]. In fact, both results can be deduced from one well–known theorem of Sabitov (see

[6], [7]). According to it, if U is a convex polyhedron in $\mathbf{R}^3$, which is of a fixed combinatorial type and all facets of which are triangles, then

$$h(v^2, l_1^2, l_2^2, \dots, l_m^2) = 0,$$

where $h(z^2, l_1^2, l_2^2, \dots, l_m^2)$ is a nonzero polynomial of the variable z^2 (depending only on the combinatorial type of U), whose coefficients are some polynomials of the variables $l_1^2, l_2^2, \dots, l_m^2$ (here $l_1, l_2, \dots, l_m$ denote the lengths of all edges of U). Applying Sabitov's this result to a bipyramid whose base is a convex k–gon P inscribed in a circle and whose height passes through the center of the circle and varies ranging over $]0, +\infty[$, one can obtain the existence of the above–mentioned polynomials f and g for P. Furthermore, one can associate to the length d of an arbitrary diagonal of P the corresponding equality

$$\phi(d^2, a_1^2, a_2^2, \dots, a_k^2) = 0,$$

where $\phi(t^2, a_1^2, a_2^2, \dots, a_k^2)$ is a nonzero polynomial of the variable t^2 (depending only on P and the place of this diagonal in P with respect to the sides of P), whose coefficients are some polynomials of the side lengths $a_1^2, a_2^2, \dots, a_k^2$.

Theorem 2. *Let Q be a convex polyhedron in $\mathbf{R}^n$ which is of a given combinatorial type and is inscribed in a sphere. Then the family*

$$(l_1, l_2, \dots, l_m)$$

of the lengths of all edges of Q determines Q up to the congruence of polyhedra.

Moreover, the radius R of the circumscribed sphere of Q is uniquely determined by $l_1, l_2, \dots, l_m$ and is an algebraic function of $l_1^2, l_2^2, \dots, l_m^2$.

The first assertion in Theorem 2 may be interpreted as a strong form of rigidity of inscribed convex polyhedra, because the combinatorial and metric structures of the 1–skeleton of Q completely determine Q as a solid body in $\mathbf{R}^n$. The proof of Theorem 2 can be carried out by induction on n, taking into account Lemma 2 and Gaifullin's recent far–going extension of Sabitov's theorem to the case of all n–dimensional convex polyhedra with simplicial facets (see [2]). Notice that, according to the statement of Theorem 2, for the circumradius R of Q we have the equality

$$\Phi(R^2, l_1^2, l_2^2, \dots, l_m^2) = 0,$$

where $\Phi(z^2, l_1^2, l_2^2, \dots, l_m^2)$ is a nonzero polynomial of the variable z^2 (depending only on the combinatorial type of Q), whose coefficients are some polynomials of the variables $l_1^2, l_2^2, \dots, l_m^2$. The latter equality is deducible by induction on n, simultaneously with parallel establishing analogous facts for the lengths of all diagonals of Q (cf. Example 2).

Let P be a convex k–gon in $\mathbf{R}^2$ which admits an inscribed circle and let, as earlier, $a_1, a_2, \dots, a_k$ denote the successive side lengths of P. Here we have a situation radically different from the case of a k–gon with a circumscribed circle. First of all, if k is even, i.e., $k = 2m$, then the relation

$$a_1 + a_3 + a_5 + \cdots = a_2 + a_4 + a_6 + \cdots$$

should be true and the radius r of an inscribed circle is not uniquely determined in this case.

If k is odd, i.e., $k = 2m + 1$, then we must have the relations

$$\begin{aligned} 2\tau_1 &= a_1 - a_2 + a_3 - \cdots > 0, \\ 2\tau_2 &= a_2 - a_3 + a_4 - \cdots > 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ 2\tau_{k-1} &= a_{k-1} - a_k + a_1 - \cdots > 0, \\ 2\tau_k &= a_k - a_1 + a_2 - \cdots > 0. \end{aligned}$$

Denoting the radius of the inscribed circle of P by r, we can write

$$\tau_1/r = \mathrm{tg}(\alpha_1),\ \tau_2/r = \mathrm{tg}(\alpha_2), \dots,\ \tau_k/r = \mathrm{tg}(\alpha_k),$$

where for each natural index $i \in [1, k]$, we have $\alpha_i = \pi/2 - \beta_i/2$ and β_i stands for the measure of the interior angle of P at its i–th vertex. Since

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = \pi, \quad \mathrm{tg}(\alpha_1 + \alpha_2 + \cdots + \alpha_k) = 0,$$

we obtain by virtue of Lemma 3 that

$$\sigma_1 r^{2m} - \sigma_3 r^{2m-2} + \cdots + (-1)^m \sigma_k = 0,$$

where $\sigma_1, \sigma_3, \dots, \sigma_k$ are basic symmetric functions of $\tau_1, \tau_2, \cdots, \tau_k$. It is not difficult to see that the equation

$$\sigma_1 x^m - \sigma_3 x^{m-1} + (-1)^m \sigma_k = 0$$

always has m strictly positive roots and the largest root of it is equal to r^2. Moreover, in this case r is uniquely determined by $a_1, a_2, \dots a_k$.

Example 3. If the side lengths of a convex pentagon in the plane $\mathbf{R}^2$ are

$$a_1 = a_2 = 1, \quad a_3 = a_4 = 2, \quad a_5, = 5,$$

then such a pentagon does not admit an inscribed circle, because in this case $\tau_4 < 0$. Similarly, if the side lengths of a convex hexagon in $\mathbf{R}^2$ are

$$a_1 = 1,\ a_2 = 2,\ a_3 = 6,\ a_4 = 4,\ a_5 = 5,\ a_6 = 6,$$

then such a hexagon also does not admit an inscribed circle. On the other hand, if all values τ_1, τ_2, τ_3, τ_4, τ_5 corresponding to a convex pentagon P with the side lengths a_1, a_2, a_3, a_4, a_5 are strongly positive, then, as was stated above, there exists an inscribed circle for P. Its radius r coincides with the largest root of the equation

$$\sigma_1 x^4 - \sigma_3 x^2 + \sigma_5 = 0,$$

where σ_1, σ_3, and σ_5 are basic symmetric functions of τ_1, τ_2, τ_3, τ_4, τ_5. We thus conclude that in this case the inradius r of P can be constructed with the aid of compass and ruler. Notice also that the second (smaller) strictly positive root of this equation coincides with the inradius of a pentagonal star having the same side lengths.

Example 4. Let P be a cube in $\mathbf{R}^3$ whose all edges have length $a > 0$ and let Q be a parallelepiped in $\mathbf{R}^3$ any facet of which is a rhombus with the side length also equal to a and with an acute angle equal to $\alpha > 0$. Then:

(1) P and Q are combinatorially equivalent;

(2) both P and Q admit inscribed spheres (whose radii differ from each other);

(3) the 1–skeletons of P and Q are metrically isomorphic.

Thus, in contrast to Theorem 2, the combinatorial and metrical structure of the 1–skeleton of a convex polyhedron in $\mathbf{R}^3$ does not determine uniquely the radius of its inscribed sphere and, in general, this radius cannot be an algebraic function of the edge lengths.

Example 5. Take in $\mathbf{R}^3$ a triangular prism P and at all vertices of P cut off six sufficiently small pyramids so that all of them would be pairwise disjoint. The obtained convex polyhedron P' is of a non–circumscribable type, so its dual convex polyhedron Q' is of a non–inscribable type. However, there are simple polyhedra of the same type as Q' which are inscribed in a sphere. More generally, it can be demonstrated that if a simple polyhedron $V \subset \mathbf{R}^3$ with k facets is such that $k-1$ of its facets are triangles, then there always exists a simple polyhedron $V' \subset \mathbf{R}^3$ of the same combinatorial type as V, which is inscribed in a sphere. We thus see that the duality between inscribable and circumscribable types of convex polyhedra in the space $\mathbf{R}^3$ fails to be true when one deals with inscribed simple polyhedra and circumscribed convex polyhedra in $\mathbf{R}^3$.

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