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ROTATION OF COORDINATE AXES AND DIFFERENTIATION OF INTEGRALS WITH RESPECT TO TRANSLATION INVARIANT BASES

A mapping B defined on $\mathbb{R}^n$ is said to be a *differentiation basis* if for every $x \in \mathbb{R}^n$, $B(x)$ is a family of bounded measurable sets with positive measure and containing x, such that there exists a sequence $R_k \in B(x)$ $(k \in \mathbb{N})$ with $\lim\limits_{k\to\infty} \operatorname{diam} R_k = 0$.

For $f \in L(\mathbb{R}^n)$, the upper and the lower limit of integral means $\frac{1}{|R|}\int_R f$ as $R \in B(x), \operatorname{diam} R \to 0$, are called *the upper and the lower derivative,* respectively, *of the integral of f at a point x.* If the upper and the lower derivative coincide, then their common value is called the *derivative of $\int f$ at a point* x and denoted by $D_B(\int f, x)$. We say that the *basis B differentiates* $\int f$ (or $\int f$ is differentiable with respect to B) if $\overline{D}_B(\int f, x) = \underline{D}_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$. If this is true for each f in the class of functions X we say that B differentiates X.

Denote by $\mathbf{I} = \mathbf{I}(\mathbb{R}^n)$ the basis of intervals, i.e., the basis for which $\mathbf{I}(x)$ $(x \in \mathbb{R}^n)$ consists of all n-dimensional intervals containing x. Note that differentiation with respect to $\mathbf{I}$ is called *strong differentiation.*

For a basis B, we denote by $\overline{B}$ the union of families $B(x)$ $(x \in \mathbb{R}^n)$.

A basis B is called:

translation invariant (briefly, TI-basis) if $B(x) = \{x + I : I \in B(0)\}$ for every $x \in \mathbb{R}^n$;

homothecy invariant (briefly, HI-basis) if for every $x \in \mathbb{R}^n, R \in B(x)$ and a homothethy H with the centre at x we have that $H(R) \in B(x)$;

sub-basis of a basis B'(denoted as $B \subset B'$) if $B(x) \subset B'(x)$ for every $x \in \mathbb{R}^n$;

formed of sets from the class Δ if $\overline{B} \subset \Delta$;

Busemann–Feller basis if $(x \in \mathbb{R}^n, R \in B(x), y \in R) \Rightarrow R \in B(y)$.

Let us introduce the following notation:

$\mathfrak{B}_{\mathrm{TI}}$ is the class of all translation invariant bases;

$\mathfrak{B}_{\mathrm{HI}}$ is the class of all homothecy invariant bases;

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$\mathfrak{B}_{\mathrm{BF}}$ is the class of all Busemann–Feller bases;

$\mathfrak{B}_B$ is the class of all subbases of a basis B;

$\mathfrak{B}_{\mathrm{NL}}$ is the class of all bases which does not differentiate $L(\mathbb{R}^n)$.

Note that if $B \in \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}}$, then $B \in \mathfrak{B}_{\mathrm{TI}}$.

For a basis B by F_B ($F_B(x)$) denote the class of all functions $f \in L(\mathbb{R}^n)$ the integrals of which are differentiable with respect to B (are differentiable with respect to B at a point x).

We say that a function f is *reduced in the class F by a transformation of a variable* γ if $f \circ \gamma \in F$.

A class of functions F is called *invariant with respect to a class of transformations of a variable* Γ if $(f \in F, \gamma \in \Gamma) \Rightarrow f \circ \gamma \in F$.

In what follows the dimension of the space $\mathbb{R}^n$ is assumed to be greater than 1.

Denote by Γ_n the family of all rotations in the space $\mathbb{R}^n$. Clearly, when $F = L(\mathbb{R}^n)$, the question of invariance of the class F with respect to rotations is trivial.

The dependence of the properties of functions of several variables on a choice of coordinate axes (i.e. on a rotation of the standard orthogonal coordinate system) were studied by different authors.

A. Zygmund posed the following problem (see [3, Ch. IV, §2]): *Can an arbitrary function $f \in L(\mathbb{R}^2)$ be reduced in the class $F_{\mathbf{I}}$ by means of rotation of coordinate axes*? J. Marstrand [7] gave the negative answer to this question by constructing a function $f \in L(\mathbb{R}^2)$, such that $f \circ \gamma \notin F_{\mathbf{I}}$ for any rotation $\gamma \in \Gamma_2$. Various generalizations of this result are established in the papers [6], [8] and [10].

In the works [5] by G. Lepsveridze, [9] by G. G. Oniani and [11] by A. Stokolos it was proved that the class $F_{\mathbf{I}}$ is not invariant with respect to linear changes of a variable, in particular with respect to rotations. An analogous result was established by O. Dragoshanski [1] for the class of continuous functions of two variables, having an a.e. converging Fourier series (Fourier integral) in Pringsheim sense.

G. Karagulyan [4] gave, in the two-dimensional case, a complete characteristic of singularities from the standpoint of differentiability with respect to a basis $\mathbf{I}$ which may have the integral of a fixed function for various choices of a coordinate system. The multi-dimensional aspect of this question was studied in [10].

M. Dyachenko [2] considered a problem of invariance with respect to Γ_2 of two-dimensional classes of functions with bounded variation in various senses.

For a basis B denote by S_B the class of all non-negative functions $f \in L(\mathbb{R}^n)$ such that $\overline{D}_{B(\gamma)}\left(\int f, x\right) = \infty$ almost everywhere for every $\gamma \in \Gamma_n$.

The theorem below extends the result of J. Marstrand to quite wide class of bases.

Theorem 1. *If $B \in \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}} \cap \mathfrak{B}_{\mathrm{NL}}$, then the class S_B is non-empty.*

The result on the non-invariance of the class $F_{\mathbf{I}}$ with respect to rotations can be extended to bases from the class $\mathfrak{B}_{\mathbf{I}} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. In particular, the following theorem is true.

Theorem 2. *If $B \in \mathfrak{B}_{\mathbf{I}} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$, then the class F_B is not invariant with respect to rotations, moreover, there exists a non-negative function $f \in F_{\mathbf{I}}$ such that $f \circ \gamma \notin F_B$ for some $\gamma \in \Gamma_n$.*

Let us consider the problem: *What kind of singularities from the standpoint of differentiability with respect to a given basis B may have the integral of a fixed function for various choices of coordinate axes?*

Let B be a basis in $\mathbb{R}^n$ and $\gamma \in \Gamma_n$. The *γ-rotated basis B* is defined as follows

$$B(\gamma)(x) = \{x + \gamma(I - x) : I \in B(x)\} \quad (x \in \mathbb{R}^n).$$

Suppose B is translation invariant. Then it is easy to verify that the differentiation of the integral of a "rotated" function $f \circ \gamma$ with respect to B at a point x is equivalent to the differentiation of the integral of f with respect to the "rotated" basis $B(\gamma^{-1})$ at a point $\gamma^{-1}(x)$. Consequently, we can reduce the study of the behavior of functions $f \circ \gamma$ $(\gamma \in \Gamma_n)$ with respect to the basis B to the study of the behavior of f with respect to rotated bases $B(\gamma)$ $(\gamma \in \Gamma_n)$. This approach will be used in the sequel.

In connection to the posed problem let us introduce the following definitions:

Let B and H are bases in $\mathbb{R}^n$ and $E \subset \Gamma_n$. Let us call E a *$W_{B,H}$-set* (*$W^+_{B,H}$-set*), if there exists a function $f \in L(\mathbb{R}^n)$ $(f \in L(\mathbb{R}^n), f \geq 0)$ such that: 1) $f \notin F_{B(\gamma)}$ for every $\gamma \in E$; and 2) $f \in F_{H(\gamma)}$ for every $\gamma \notin E$;

Let B and H are bases in $\mathbb{R}^n$ and $E \subset \Gamma_n$. Let us call E an *$R_{B,H}$-set* (*$R^+_{B,H}$-set*), if there exists a function $f \in L(\mathbb{R}^n)$ $(f \in L(\mathbb{R}^n), f \geq 0)$ such that: 1) $f \notin F_{B(\gamma)}(x)$ almost everywhere for every $\gamma \in E$; and 2) $f \in F_{H(\gamma)}$ for every $\gamma \notin E$.

When $B = H$ we will use terms $W_B(W^+_B, R_B, R^+_B)$-set, and when $B = H = \mathbf{I}$ - terms $W(W^+, R, R^+)$-set.

The definitions of R, R^+ and W-sets were introduced in [9], [8] and [4], respectively.

Now the problem can be formulated as follows: *For a given basis B what kind of sets $E \subset \Gamma_n$ are $W_B(W^+_B, R_B, R^+_B)$-sets?*

The set of two-dimensional rotations Γ_2 can be identified with the circumference $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, if to a rotation γ we put into correspondence the complex number $z(\gamma)$ from $\mathbb{T}$, the argument of which is equal to the value of the angle by which the rotation about the origin takes place in the positive direction under the action of γ.

The distance $d(\gamma, \sigma)$ between points $\gamma, \sigma \in \Gamma_2$ is assumed to be equal to the length of the smallest arch of the circumference $\mathbb{T}$ connecting points $z(\gamma)$ and $z(\sigma)$.

The set of the rotations $\gamma_k (k \in \overline{0,3})$, where $z(\gamma_k) = e^{i\pi k/2}$ is denoted by Π.

For a non-empty set $E \subset \Gamma_n$, denote by $B(E)$ the basis, for which $B(E)(x)$ $(x \in \mathbb{R}^n)$ is the union of all families $B(\gamma)(x)$ where $\gamma \in E$.

The following theorems give necessary conditions for singularity sets.

Theorem 3. *For arbitrary basis B in $\mathbb{R}^2$ each W_B-set has $G_{\delta\sigma}$ type.*

Theorem 4. *For arbitrary basis B in $\mathbb{R}^2$ each R_B-set has G_δ type.*

For non-empty sets $E_1 \subset \Gamma_2$ and $E_2 \subset \Gamma_2$ denote $E_1E_2 = \{\gamma_1 \circ \gamma_2 : \gamma_1 \in E_1, \gamma_2 \in E\}$. A set $E \subset \Gamma_2$ let us call symmetric if $E = \Pi E$.

A basis $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}}$ let us call *symmetric*, if $R \in B(0) \Rightarrow s(R) \in B(0)$, where s is a symmetry of $\mathbb{R}^2$ with respect to the line $\{x \in \mathbb{R}^2 : x_1 = x_2\}$.

G. Karagulyan [4] established the following characterization of two-dimensional W and R-sets: *$E \subset \Gamma_2$ is W-set (R-set) if and only if E is symmetric and of $G_{\delta\sigma}$ type (is symmetric and of G_δ type).*

The results given below characterize W_B and R_B-sets for a quite wide class of bases.

Theorem 5. *If $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}} \cap \mathfrak{B}_{\mathrm{NL}}$, then:*
1) *every symmetric set $E \subset \Gamma_2$ of $G_{\delta\sigma}$ type is $W_{B,\mathbf{I}}$-set;*
2) *every symmetric set $E \subset \Gamma_2$* of *G_δ type is $R_{B,\mathbf{I}}$-set.*

Corollary 1. *If B is symmetric and $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{BF}} \cap \mathfrak{B}_{\mathrm{HI}} \cap \mathfrak{B}_{\mathrm{NL}}$, then:*
1) *a set $E \subset \Gamma_2$ is $W_{B,\mathbf{I}}(W_B)$-set if and only if E is symmetric and of $G_{\delta\sigma}$ type;*
2) *a set $E \subset \Gamma_2$ is $R_{B,\mathbf{I}}(R_B)$-set if and only if E is symmetric and of G_δ type.*

For bases from the class $\mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$ there are valid the following results.

Theorem 6. *Let $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. Then for every not more than countable set $E \subset \Gamma_2$ and for every sequence of its neighbourhoods (V_k) there is a non-negative function $f \in L(\mathbb{R}^2)$ such that:*
1) *for every $\gamma \in E$, $\overline{D}_{B(\gamma)}(\int f, x) = \infty$ almost everywhere;*
2) *for every $k \in \mathbb{N}$, $f \in F_{\mathbf{I}(\Gamma_2 \setminus \Pi V_k)}$. Consequently, for every $\gamma \notin \bigcap\limits_{k=1}^{\infty} \Pi V_k$ we have that $f \in F_{\mathbf{I}(\gamma)}$;*

3) *If* $f \notin F_{B(\gamma)}$ *for some* $\gamma \in \Gamma_2$, *then* $f \notin F_{B(\gamma)}(x)$ *almost everywhere.*

Corollary 2. *Let* $B \in \mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. *Then:*

1) *every not more than countable set* $E \subset \Gamma_2$ *is* $W^+_{B,\mathbf{I}}$*-set;*

2) *every not more than countable symmetric set of* G_δ *type is* $R^+_{B,\mathbf{I}}$*-set;*

3) *there exists an* $R^+_{B,\mathbf{I}}$*-set of the second category and consequently, of the continuum cardinality.*

Corollary 3. *Let* B *is a symmetric basis from the class* $\mathfrak{B}_{\mathbf{I}(\mathbb{R}^2)} \cap \mathfrak{B}_{\mathrm{TI}} \cap \mathfrak{B}_{\mathrm{NL}}$. *Then:*

1) *not more than countable set* $E \subset \Gamma_2$ *is a* $W^+_{B,\mathbf{I}}(W_{B,\mathbf{I}},\, W^+_B, W_B)$*-set if and only if* E *is symmetric;*

2) *not more than countable set* $E \subset \Gamma_2$ *is an* $R^+_{B,\mathbf{I}}(R_{B,\mathbf{I}},\, R^+_B, R_B)$*-set if and only if* E *is symmetric and of* G_δ *type.*

Theorem 6 and it's corollaries for the case $B = \mathbf{I}$ were proved in [9].

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