Proc. A. Razmadze Math. Inst. 167(2015), 151–155

G. Oniani

ON THE CONVERGENCE OF SPARSE MULTIPLE SERIES

Let $W \subset \mathbb{R}_+^n$, where $\mathbb{R}_+ = [0, \infty)$. For a series $\sigma = \sum_{\mathbf{m} \in \mathbb{N}^n} a_{\mathbf{m}}$ by $S_W(\sigma)$ denote its *partial sum by the set* W, i.e.,

$$S_W(\sigma) = \sum_{\mathbf{m} \in W} a_{\mathbf{m}}.$$

Note that the sum by empty set of indexes we assume to be 0.

The convergence of partial sums $S_{rW}(\sigma)$ as $r \to \infty$ will be referred as W*-convergence* of the series σ. Here rW denote the dilation of the set W by a coefficient $r > 0$, i.e., $rW = \{r\mathbf{x} : \mathbf{x} \in W\}$.

For the case $W = \{\mathbf{x} \in \mathbb{R}_+^n : x_1^2 + \cdots + x_n^2 \leq 1\}$, W-convergence is called the spherical convergence.

Recall that: a sequence $(a_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^n}$ is called *convergent* if $a_{\mathbf{m}}$ tends to the limit as $\min(m_1, \dots, m_n) \to \infty$; and a series $\sigma = \sum_{\mathbf{m} \in \mathbb{N}^n} a_{\mathbf{m}}$ is called *convergent in Pringsheim sense* if the sequence of its rectangular partial sums $S_{\mathbf{m}}(\sigma)$ is convergent.

Saying that a sequence $(a_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^n}$ *strongly converges* we mean the convergence of $(a_{\mathbf{m}})$ to the limit as $\max(m_1, \dots, m_n) \to \infty$.

Let $S \subset \mathbb{N}$ and $\lambda > 1$. A set S is said to be λ-*lacunar* if for every $m, m^* \in S$ with $m < m^*$ we have that $m^*/m > \lambda$.

A set $S \subset \mathbb{N}^n$ is called λ-lacunar if there are one-dimensional λ-lacunar sets $S_1, \dots, S_n \subset \mathbb{N}$ such that $S \subset S_1 \times \cdots \times S_n$. A set $S \subset \mathbb{N}^n$ is said to be *lacunar* if S is λ-lacunar for some $\lambda > 1$.

A sequence $(a_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^n}$ or a series $\sum_{\mathbf{m} \in \mathbb{N}^n} a_{\mathbf{m}}$ is said to be *lacunar* (λ-*lacunar*) if the set $S = \{\mathbf{m} \in \mathbb{N}^n : a_{\mathbf{m}} \neq 0\}$ is lacunar(λ-lacunar). Here note that a Haar series is lacunar at every diadic-irrational point.

By $\mathbb{I}$ we denote the unit interval $[0, 1]$.

We shall use the following notation:

$\mathbb{Z}_0$ is the set of all nonnegative integers;

Δ_k^i $(k \in \mathbb{Z}, i \in \mathbb{Z})$ is the diadic interval $[\frac{i-1}{2^k}, \frac{i}{2^k}]$;

$\widetilde{\Delta}_k^i (k \in \mathbb{Z}, i \in \mathbb{Z})$ is the open diadic interval $(\frac{i-1}{2^k}, \frac{i}{2^k})$;

$\overline{p, q}$ $(p, q \in \mathbb{N}, p \leq q)$ is the set $\{p, \dots, q\}$;

2010 *Mathematics Subject Classification:* 40A05, 42C10.

Key words and phrases. Multiple series, lacunar series, Haar series, convergence.

$\mathbb{I}_{\mathrm{d}}$ is the set of all diadic-irrational numbers of $\mathbb{I}$.

Let us recall the definition of the Haar system $(h_m)_{m\in\mathbb{N}}$: $h_1(x) = 1$ $(x \in \mathbb{I})$, and if $m = 2^k + i$ $(k \in \mathbb{Z}_0, i \in \overline{1, 2^k})$, then: $h_m(x) = 2^{k/2}$ while $x \in \widetilde{\Delta}_{k+1}^{2i-1}$; $h_m(x) = -2^{k/2}$ while $x \in \widetilde{\Delta}_{k+1}^{2i}$ and $h_m(x) = 0$ while $x \notin \Delta_k^i$. At inner points of discontinuity h_m is defined as mean value of the limits from the right and from the left, and at the ends of $\mathbb{I}$ as the limits from inside of the interval.

By $H(\mathbf{x})$ $(\mathbf{x} \in \mathbb{I}^n)$ denote the *spectrum* of the multiple Haar system at a point $\mathbf{x} \in \mathbb{I}^n$, i.e., the set $\{\mathbf{m} \in \mathbb{N}^n : h_{\mathbf{m}}(\mathbf{x}) \neq 0\}$;

From the definition of the Haar system it follows easily that: if $x \in \mathbb{I}_{\mathrm{d}}$, *then* $H(x)$ is $3/2$-lacunar set, consequently, taking into account that $H(\mathbf{x}) = H(x_1) \times \cdots \times H(x_n)$, we have: $H(\mathbf{x})$ is $3/2$-lacunar at every $\mathbf{x} \in \mathbb{I}_{\mathrm{d}}^n$.

A point $\mathbf{x} \in \mathbb{I}^n$ let us call *diadic-irrational* if each coordinate of $\mathbf{x}$ is a diadic-irrational number.

In [1] for multiple Haar series it was established the following connection between the convergence in Pringsheim sense and the spherical convergence.

Theorem A. *Let $\sigma = \sum_{\boldsymbol{m}\in\mathbb{N}^n} c_{\boldsymbol{m}} h_{\boldsymbol{m}}$ be a multiple Haar series. If σ is convergent to a number s in Pringsheim sense at a diadic-irrational point $\boldsymbol{x} \in \mathbb{I}^n$ and its general term $(c_{\boldsymbol{m}} h_{\boldsymbol{m}}(\boldsymbol{x}))$ is strongly convergent to 0, then σ is spherically convergent to s at $\boldsymbol{x}$.*

Theorem A was obtained as a corollary of the following more general result (see [1]).

Theorem B. *Let $\sigma = \sum_{\boldsymbol{m}\in\mathbb{N}^n} a_{\boldsymbol{m}}$ be a lacunar numerical series. If σ converges to a number s in Pringsheim sense and its general term $(a_{\boldsymbol{m}})$ strongly converges to 0, then σ is spherically convergent to s.*

Two questions we study are:

1) Does Theorem A remain true for points $\mathbf{x} \in \mathbb{I}^n$ that are not diadic-irrational?
2) Is it possible Theorem B to be extended to a convergence of more general type than the spherical one?

Below it will be given a generalization of Theorem B in which lacunar series is changed by more general one and instead of the spherical convergence there is considered convergence of a quite general type. As a corollary of the generalization we obtain a positive answer to the first question.

Let $k \in \mathbb{N}$ and $\lambda > 1$. A set $S \subset \mathbb{N}$ let us call (k, λ)-*sparse* if there are λ-lacunar sets $S_1, \ldots, S_\nu \subset \mathbb{N}$ such that $S = S_1 \cup \cdots \cup S_k$. A set $S \subset \mathbb{N}^n$ let us call (k, λ)-*sparse* if there are (k, λ)-*sparse* one-dimensional sets $S_1, \ldots, S_n \subset \mathbb{N}$ such that $S \subset S_1 \times \cdots \times S_n$.

A set $S \subset \mathbb{N}^n$ let us call *sparse* if it is (k, λ)-*sparse* for some $k \in \mathbb{N}$ and $\lambda > 1$.

A sequence $(a_{\mathbf{m}})_{\mathbf{m}\in\mathbb{N}^n}$ or a series $\sum_{\mathbf{m}\in\mathbb{N}^n} a_{\mathbf{m}}$ we will call *sparse*(*(k,λ)-sparse*) if the set $S=\{\mathbf{m}\in\mathbb{N}^n : a_{\mathbf{m}}\neq 0\}$ is sparse ((k,λ)-sparse).

It is easy to see that if $x\in\mathbb{I}\setminus\mathbb{I}_{\mathrm{d}}$, then $H(x)$ is $(2,3/2)$-sparse set. Consequently, taking into account relation $H(\mathbf{x})=H(x_1)\times\cdots\times H(x_n)$, we have: $H(\mathbf{x})$ is $(2,3/2)$-sparse at every $\mathbf{x}\in\mathbb{I}^n\setminus\mathbb{I}_{\mathrm{d}}^n$.

Let $\alpha\geq 1$. By $\mathcal{W}_n(\alpha)$ we denote the class of all sets $W\subset\mathbb{R}_+^n$ for which there is a number $t>0$ such that $[0,t]^n\subset W\subset[0,\alpha t]^n$; and by $\overline{\mathcal{W}}_n(\alpha)$ we denote the class of all sets $W\subset\mathbb{R}_+^n$ for which there is a number $t>0$ such that $W\cap[0,t]^n=\emptyset$ and $W\subset[0,\alpha t]^n$. The union of the classes $\mathcal{W}_n(\alpha)$ and $\overline{\mathcal{W}}_n(\alpha)$ will be denoted by $\mathcal{W}_n^*(\alpha)$.

Let us introduce the following notation:

$$\mathcal{W}_n=\bigcup_{\alpha\geq 1}\mathcal{W}_n(\alpha),\ \overline{\mathcal{W}}_n=\bigcup_{\alpha\geq 1}\overline{\mathcal{W}}_n(\alpha),\ \mathcal{W}_n^*=\bigcup_{\alpha\geq 1}\mathcal{W}_n^*(\alpha);$$
$$t_W=\sup\{t>0:[0,t]^n\subset W\}\quad(W\in\mathcal{W}_n);$$
$$t_W=\sup\{t>0:[0,t]^n\cap W=\emptyset\}\quad(W\in\overline{\mathcal{W}}_n);$$
$$\pi_i(\mathbf{x})=(x_1,\dots,x_{i-1},x_{i+1},x_n)\ \ (i\in\overline{1,n},\ \mathbf{x}\in\mathbb{R}^n);$$
$$W(i,t)=\pi_i(\{\mathbf{x}\in W: x_i=t\})\ \ (W\subset\mathbb{R}^n,\ i\in\overline{1,n},\ t\in\mathbb{R}).$$

Let $\alpha\geq 1$. By $\mathcal{V}_2(\alpha)$ ($\overline{\mathcal{V}}_2(\alpha)$) we denote the class of all sets $W\in\mathcal{W}_2(\alpha)$ ($W\in\overline{\mathcal{W}}_2(\alpha)$) such that $W(i,t)$ consists of not more then α one-dimensional segments for every $i\in\overline{1,n}$ and $t\geq t_W$. The class $\mathcal{V}_2^*(\alpha)$ is defined as the union of the classes $\mathcal{V}_2(\alpha)$ and $\overline{\mathcal{V}}_2(\alpha)$. For arbitrary dimension $n>2$, $\mathcal{V}_n(\alpha)$ ($\overline{\mathcal{V}}_n(\alpha)$) we define as the class of all sets $W\in\mathcal{W}_n(\alpha)$ ($W\in\overline{\mathcal{W}}_n(\alpha)$) such that for every $i\in\overline{1,n}$ and $t\geq t_W$, $W(i,t)$ is either empty set or belongs to the class $\mathcal{V}_{n-1}^*(\alpha)$. The class $\mathcal{V}_n^*(\alpha)$ will be defined as the union of the classes $\mathcal{V}_n(\alpha)$ and $\overline{\mathcal{V}}_n(\alpha)$.

The union of the classes $\mathcal{V}_n(\alpha)$ ($\alpha\geq 1$) let us denote by $\mathcal{V}_n$.

We will say that a set $W\subset\mathbb{R}_+^n$ has a *bounded variation* if $W\in\mathcal{V}_n$.

It is easy to see that every ball $B_n=\{\mathbf{x}\in\mathbb{R}_+^n : x_1^2+\cdots+x_n^2\leq 1\}$ ($n\geq 2$) and every set of the type $C\cap\mathbb{R}_+^2$, where C is a two-dimensional convex set having the origin as interior point, has a bounded variation.

Theorem 1. *Let $\sigma=\sum_{\boldsymbol{m}\in\mathbb{N}^n} a_{\boldsymbol{m}}$ be a sparse numerical series. If σ converges to a number s in Pringsheim sense and its general term $(a_{\boldsymbol{m}})$ strongly converges to 0, then σ is W-convergent to s for any set W with bounded variation.*

Since every Haar series is a sparse numerical series at any point $\mathbf{x}\in\mathbb{I}^n$ then from Theorem 1 we obtain the following corollary.

Corollary 1. *Let $\sigma=\sum_{\boldsymbol{m}\in\mathbb{N}^n} c_{\boldsymbol{m}}h_{\boldsymbol{m}}$ be a multiple Haar series. If σ is convergent to a number s in Pringsheim sense at a point $\boldsymbol{x}\in\mathbb{I}^n$ and its* *general term $(c_{\mathbf{m}}h_{\mathbf{m}}(\boldsymbol{x}))$ is strongly convergent to* 0*, then σ is W-convergent to s for any set W with bounded variation.*

Remark 1. Theorem 1 for the case of lacunary series and spherical convergence(instead of W-convergence) was proved in [1]; and for the case of two-dimensional and lacunary series was proved in [2].

In [1] (see Corollary 3) it was proved that if $f \in L(\ln^+ L)^{n-2}(\mathbb{I}^n)$, then the general term $(c_{\mathbf{m}}h_{\mathbf{m}}(\mathbf{x}))$ of Fourier-Haar series of f strongly converges to zero almost everywhere. Consequently, from Corollary 1 we obtain the next result.

Corollary 2. *Let $f \in L(\ln^+ L)^{n-2}(\mathbb{I}^n)$ and $\sigma(f)$ be Fourier-Haar series of f. If $\sigma(f)$ converges to $f(\mathbf{x})$ in Pringsheim sense at every point $\mathbf{x}$ from a set E, then there is a set $E^* \subset E$ with $|E^*| = |E|$ such that for every point $\mathbf{x}$ from E^*, $\sigma(f)$ is W-convergent to $f(\mathbf{x})$ for any set W with bounded variation.*

Taking into account that(see [3] or [4]) Fourier-Haar series of every function f from $L(\ln^+ L)^{n-1}(\mathbb{I}^n)$ converges almost everywhere, from Corollary 2 we deduce the following result.

Corollary 3. *If $f \in L(\ln^+ L)^{n-1}(\mathbb{I}^n)$ then there is a set $E \subset \mathbb{I}^n$ of full measure such that for every point $\mathbf{x}$ from E, Fourier-Haar series of f is W-convergent to $f(\mathbf{x})$ for any set W with bounded variation.*

Remark 2. Corollaries 2 and 3 for the case of spherical convergence (instead of W-convergence) was proved in [1]; and for two-dimensional case was proved in [2]. Note also that an analogue of Corollary 3 for the case of triangular convergence of two-dimensional Fourier-Haar series was established in [5].

Acknowledgement

The author was supported by Shota Rustaveli National Science Foundation (project no. 31/48).

References

1. G. G. Oniani, On the convergence of multiple Haar series. (Russian) *Izv. Ross. Akad. Nauk Ser. Mat.* **78** (2014), No. 1, 99–116; translation in *Izv. Math.* **78** (2014), No. 1, 90–105.
2. G. G. Oniani, The convergence of double Fourier-Haar series over homothethic copies of sets. (Russian) *Mat. Sb.* **205** (2014), No. 7, 73–94; translation in *Sb. Math.* **205** (2014), No. 7, 983–1003.
3. O. P. Dzagnidze, Representation of measurable functions of two variables by double series. (Russian) *Soobšč. Akad. Nauk Gruzin. SSR* **34** (1964), 277–282.
4. T. Sh. Zerekidze, Convergence of multiple Fourier-Haar series and strong differentiability of integrals. (Russian) *Trudy Tbiliss. Mat. Inst. Razmadze* **76** (1985), 80–99.

5. K. R. Muradyan, On the divergence of triangular and sectorial sums of double Fourier series. *Ph.D. Thesis, Yerevan State University,* **2014**.

Author's address:

Akaki Tsereteli State University
59, Tamar Mepe St., Kutaisi 4600, Georgia
E-mail: oniani@atsu.edu.ge