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N. Danelia, V. Kokilashvili and Ts. Tsanava

TWO WEIGHT UNIFORM BOUNDEDNESS CRITERIA FOR THE CESÁRO MEANS WITH VARIABLE ORDER

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In this paper we present the necessary and sufficient conditions ensuring two-weight uniform boundedness for Cesáro summability means with variable order for univariate and multiple Fourier trigonometric series. We give two-weight uniform estimates criteria for above-mentioned Fourier operators in general case when we have different weights and exponents of classical Lebesgue spaces on both sides of inequalities.

Let $\mathbb{T}$ be the interval $[-\pi, \pi]$. Let w be a 2π-periodic almost everywhere non-negative integrable function. We denote by $L^p_w(\mathbb{T})$, $1 \leq p < \infty$ the Banach function space of all measurable 2π-periodic functions f, for which the norm

$$\|f\|_{p,w} = \Bigg(\int\limits_{\mathbb{T}} |f(x)|^p w(x)dx \Bigg)^{1/p} < \infty.$$

The uniform boundedness problem of Cesáro and Abel-Poisson means of functions from weighted Lebesgue space was studied by M. Rosenbloom [1] and B. Muckenhoupt [2]. In the paper [1] a characterization of the weights w for which the Cesáro $(C, 1)$ and Abel-Poisson means are uniformly bounded in weighted Lebesgue space L^p_w $(1 < p < \infty)$ has been done. Later on B. Muckenhoupt [2] established that the condition referred in [2] is equivalent to the condition A_p, that is

$$\sup_I \frac{1}{|I|} \int\limits_{\mathbb{I}} w(x)dx \Bigg(\frac{1}{|I|} \int\limits_{\mathbb{I}} w^{1-p'}(x)dx \Bigg)^{p-1} < \infty,$$

where $p' = p/(p-1)$ and supremum is taken over all intervals whose lengths are not greater than 2π. In two weight setting by B. Muckenhoupt [3] has been proved that the necessary and sufficient condition for the uniform

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boundedness of the Abel-Poisson means as operators from L_w^p to L_v^p is

$$\sup_I \frac{1}{|I|}\int\limits_I v(x)dx\left(\frac{1}{|I|}\int\limits_I w^{1-p'}(x)dx\right)^{p-1} < \infty. \tag{1}$$

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \tag{2}$$

be the Fourier series of the function $f \in L^1(\mathbb{T})$. Let $\sigma_n^\alpha(\cdot, f)$ $(\alpha > 0)$ be the Cesáro means of the series (2), that is

$$\sigma_n^\alpha(x, f) = \frac{1}{\pi}\int\limits_{\mathbb{T}} f(x+t)\, K_n^\alpha(t)\, dt,$$

where

$$K_n^\alpha(t) = \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha-1} D_k(t)}{A_n^\alpha}, \quad \alpha > 0$$

is the Fejer kernel and

$$D_k(t) = \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

is the Dirichlet kernel, with

$$A_n^\alpha = \left(\begin{array}{c} n+\alpha \\ \alpha \end{array}\right) \approx \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

In the paper [4] by the second author and A. Guven has been proved

Theorem A. *Let $1 < p \le q < \infty$. Then the inequality*

$$\|\sigma_n^\alpha(\cdot, f)\|_{q,v} \le c\; n^{\frac{1}{p}-\frac{1}{q}}\,\|f\|_{p,w}, \quad \alpha > 0 \tag{3}$$

holds for arbitrary $f \in L_w^p(\mathbb{T})$, where the constant c does not depend on n and f, if and only if $(v, w) \in \mathcal{A}_{p,q}(\mathbb{T})$ i. e.

$$\sup_I \left(\frac{1}{|I|}\int\limits_I v(x)dx\right)^{\frac{1}{q}}\left(\frac{1}{|I|}\int\limits_I w^{1-p'}(x)dx\right)^{\frac{1}{p'}} < \infty.$$

The latter condition is appeared in the papers [3], [4] by B. Muckenhoupt.

In present paper we study the Cesáro means of variable order

$$\sigma_n^{\alpha_n}(x, f) = \frac{1}{\pi}\int\limits_{\mathbb{T}} f(x+t)\, K_n^{\alpha_n}(t)\, dt,$$

where

$$K_n^{\alpha_n}(t) = \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha_n-1} D_k(t)}{A_n^{\alpha_n}}, \quad \alpha_n > 0$$

with the condition $\lim_{n\to\infty} \alpha_n = \alpha$, $\alpha > 0$.

It is evident that when $\alpha_n \equiv 0$ we have $\sigma_n^0(x,f) = S_n(x,f)$-partial sums of Fourier trigonometric series.

It should be stressed that an idea to introduce and study of linear methods of summability of variable orders comes from D. E. Menshov (see e. g. [6]). For the considerable results on divergence problems of summability means of variable order in the spaces C and L^1, we refer to the papers by Sh. Tetunashvili [7], [8].

One of main result of this paper is

Theorem 1. *Let $1 < p \le q < \infty$. The the inequality*

$$\left\|\sigma_n^{\alpha_n}(\cdot,f)\right\|_{q,v} \le c\; n^{\frac{1}{p}-\frac{1}{q}} \left\|f\right\|_{p,w}, \quad \alpha > 0 \tag{4}$$

holds with a constant c independent of n, α_n and f, if and only if $(v,w) \in \mathcal{A}_{p,q}(\mathbb{T})$.

Let us now discuss the two-dimensional case.

Let $\mathbb{T}^2 = \mathbb{T}\times\mathbb{T}$ and w be a weight function on $\mathbb{T}^2$. We denote by $L_w^p\left(\mathbb{T}^2\right)$, $1 \le p < \infty$, the space of functions $f(x,y)$ which are $2\pi-$periodic with respect to each variable, such that

$$\left\|f\right\|_{p,w} = \left(\int\limits_{\mathbb{T}^2} \left|f(x,y)\right|^p w(x,y)\,dxdy\right)^{1/p} < \infty.$$

Let the function $f \in L^1\left(\mathbb{T}^2\right)$ has the Fourier series

$$\begin{aligned} f(x,y) \quad \sim \quad \sum_{m,n=0}^{\infty} \lambda_{mn} & \left(a_{mn}\cos mx\cos ny + b_{mn}\sin mx\cos ny + \right. \\ & \left. + c_{mn}\cos mx\sin ny + d_{mn}\sin mx\sin ny\right), \end{aligned} \tag{5}$$

where

$$\lambda_{mn} = \begin{cases} \dfrac{1}{4}, & m=n=0, \\ \dfrac{1}{2}, & m=0, \quad n>0 \quad \text{or} \quad m>0, \quad n=0, \\ 1, & m>0, \quad n>0. \end{cases}$$

Let also

$$\sigma_{mn}^{(\alpha_m,\beta_n)}(x,y,f) = \frac{\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n} A_{m-i}^{\alpha_m-1} A_{n-j}^{\beta_n-1} S_{ij}(x,y,f)}{A_m^{\alpha_m} A_n^{\beta_n}}, \quad (\alpha_m > 0, \beta_n > 0)$$

and $\lim_{m\to\infty} \alpha_m = \alpha$, $\lim_{n\to\infty} \beta_n = \beta$, $\alpha > 0$, $\beta > 0$.

Definition 1. The pair (v, w) is said to belong to the class $\mathcal{A}_{p,q}\left(\mathbb{T}^2\right)$ if the condition

$$\sup_J \left(\frac{1}{|J|}\int_J v\,(x,y)\,dxdy\right)^{1/q}\left(\frac{1}{|J|}\int_J w^{1-p'}\,(x,y)\,dxdy\right)^{1/p'} < \infty \qquad (6)$$

holds, where the supremum is taken over all rectangles J with sides parallel to the coordinate axes and with lengths not greater than 2π.

Theorem 2. *Let $1 < p \le q < \infty$. Then the condition $(v, w) \in \mathcal{A}_{p,q}\left(\mathbb{T}^2\right)$ is necessary and sufficient for the validity of the inequality*

$$\left\|\sigma_{mn}^{(\alpha_m,\beta_n)}\,(\cdot,\cdot,f)\right\|_{q,v} \le c\ \ (mn)^{\frac{1}{p}-\frac{1}{q}}\,\|f\|_{p,w}\,, \quad \alpha_m > 0,\ \beta_n > 0 \qquad (7)$$

for every $f \in L^p_w\left(\mathbb{T}^2\right)$, where the constant c is independent of m, n, α_m, β_n and f.

On the base of Theorems 1 and 2 we derive the following norm summability theorems in two-weighted setting

Theorem 3. *Let $1 < p < \infty$ and $(v, w) \in \mathcal{A}_{p,p}(\mathbb{T})$. Then we have*

$$\lim_{n\to\infty} \|f - \sigma_n^{\alpha_n}(\cdot, f)\|_{p,v} = 0$$

for arbitrary $f \in L^p_w(\mathbb{T})$.

Theorem 4. *Let $1 < p < \infty$. $(v, w) \in \mathcal{A}_{p,p}(\mathbb{T}^2)$. Then*

$$\lim_{m,n\to\infty} \|f - \sigma_{mn}^{(\alpha_m,\beta_n)}(\cdot, f)\|_{p,v} = 0$$

for all $f \in L^p_w(\mathbb{T}^2)$.

Example. Let $w(x) = |x|^{p-1}\ln^p\frac{2\pi}{|x|}$ and $v(x) = |x|^{p-1}$. Then for the pair (v, w) Theorem 1 is valid with $p = q$. The same pair governs also validity of Theorem 3.

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Authors' addresses:

N. Danelia

Department of Exact and Natural Sciences,
I. Javakhishvili Tbilisi State University
2, University st., Tbilisi 0186, Georgia

V. Kokilashvili

A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
6, Tamarashvili St., Tbilisi 0177, Georgia

Ts. Tsanava

Georgian Technical University,
77, M. Kostava, Tbilisi 0175, Georgia