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THE NOETHERITY CRITERIA OF THE RIEMANN-HILBERT PROBLEM FOR VARIABLE EXPONENT SMIRNOV CLASSES IN DOMAINS WITH PIECEWISE SMOOTH BOUNDARIES

The Riemann-Hilbert boundary value problem

$$\mathrm{Re}\left[a(t)\,\Phi^+(t)\right] = b(t) \tag{1}$$

is well-studied under different assumptions for the given and unknown elements of the problem (see [1]–[4]).

The present paper is devoted to the investigation of the problem (1) when the unknown function Φ is required to belong to Smirnov class and the condition (1) is assigned on the boundary of the domain G. In addition, it is assumed that $\Phi^+(t)$ denotes an angular boundary value of the function Φ at the point t, and equality (1) holds for almost all t on Γ.

1^0. We assume that the domain G is bounded by a simple piecewise-smooth curve Γ, the function $a(t)$ is continuous and different from zero on Γ, and $b(t)$ belongs to the Lebesgue space $L^{p(t)}(\Gamma)$ with a variable exponent $p(t)$.

As for the function $p(t)$, it is assumed that it belongs to the class $Q(\Gamma)$. This means that $p(t) \in \widetilde{\mathcal{P}}(\Gamma)$ and $\ell(\tau) = p(z(\tau)) \in \widetilde{\mathcal{P}}(\gamma)$, where $z = z(w)$ is the conformal mapping of the circle $U = \{w : |w| < 1\}$ onto G, and $\gamma = \{\tau : |\tau| = 1\}$.

$\widetilde{\mathcal{P}}(\gamma)$ denotes a class of functions $p = p(t)$ defined on Γ for which there exist positive numbers $B(p)$ and $\mathcal{E}(p)$ such that: (i) for any t_1 and t_2 from Γ, we have $|p(t_1) = p(t_2)| < B)p)|\ln(t - t_0)|^{1+\mathcal{E}}$; (ii) $\min p(t) = \underline{p} > 1$.

If in the above definition instead of a positive ε we take $\varepsilon = 0$, then we obtain a class which we denote by $\mathcal{P}(\Gamma)$.

Let $\omega(w)$ be a measurable, almost everywhere different from zero function on Γ.

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Definition 1 (See either [5], or [6], p. 72)**.** The Smirnov class $E^{p(t)}(G,\omega)$ is a set of analytic in G functions Φ for which

$$\sup_{0<r<1}\int_0^{2\pi}\big|\Phi(z(re^{i\vartheta}))\,\omega(z(re^{i\vartheta}))\big|^{p(z(e^{i\vartheta}))}\big|z'(re^{i\vartheta})\big|d\vartheta<\infty.$$

Assume $E^{p(t)}(G):E^{p(t)}(G;1)$.

By $L^{p(t)}(\Gamma)$ we denote a set of measurable on Γ functions f for which

$$\|f\|_{p(\cdot)}=\inf\left\{\lambda>0:\int_0^{\ell}\left|\frac{f(t(s))}{\lambda}\right|^{p(t(s))}ds\le 1\right\}<\infty,$$

where $t=t(s)$, $0\le s\le\ell$ is the equation of Γ with respect to the arc abscissa s.

Assume $L^{p(t)}(\Gamma;\omega)=\{f:\|f\omega\|_{p(\cdot)}<\infty\}$.

$L^{p(t)}(\Gamma;\omega)$ is the Banach space. For the conjugate space $[L^{p(t)}(\Gamma;\omega)]^*$, we have

$$\big[L^{p(t)}(\Gamma;\omega)\big]^*=L^{q(t)}\Big(\Gamma;\frac{1}{\omega}\Big),\quad p'(t)=q(t)=p(t)[p(t)-1]^{-1}$$

([7]).

Definition 2. By $K^{p(t)}(G;\omega)$ we denote a set of analytic in G functions Φ for which

$$\Phi(z)=(K_\Gamma\varphi)(z)=\frac{1}{2\pi i}\int_G\frac{\varphi(t)\,dt}{t-z}\,,\quad z\in G,\quad \varphi\in L^{p(t)}(\Gamma;\omega).$$

Assume $K^{p(t)}(G):=K^{p(t)}(G;1)$.

We denote a set of simple closed Carleson curves by R.

2^0. If the boundary Γ of the domain G belongs to R, and $p\in\mathcal{P}(\Gamma)$, then $E^{p(t)}(G)\subset K^{p(t)}(G)$, but if Γ is a piecewise-smooth curve, free from external peaks, then $E^{p(t)}(G)=K^{p(t)}(G)$ ([6], Ch. 3).

If $\Phi\in E^{p(t)}(G)$, then for almost all $t\in\Gamma$ there exists a non-tangential angular limit $\Phi^+(t)$, and

$$\Phi^+(t)=\frac12\big(\varphi(t)+(S\,\varphi)(t)\big),$$

where

$$(S\,\varphi)(t)=\frac{1}{\pi i}\int_\Gamma\frac{\varphi(\tau)\,d\tau}{\tau-t}\,,\quad t\in\Gamma.$$

If $p\in\mathcal{P}(\Gamma)$, then the operator $S:\varphi\to S\,\varphi$ is continuous in the space $L^{p(t)}(\Gamma)$, if and only if $\Gamma\in R$ ([9], see also [6], p. 44).

When Γ is an arbitrary piecewise-smooth curve and $p \in Q(\Gamma)$, then the problem (1) reduces equivalently to the singular integral equation $M\varphi = f$ in $L^{\ell(\tau)}(\gamma)$, where $\ell(\tau) = p(z(r))$, and $z = z(w)$ is the conformal mapping of the circle $U = \{w : |w| < 1\}$ onto G. In proving this statement, we have used both the method of Muskhelishvi reducing the problem (1) in the domain U to the Riemann problem ([1], Ch.2) and the results of item 6.7 from [6] (pp.216-8]) dealt with the fact that the function $|z'(w)|^{\frac{1}{p(e^{i\vartheta})}}$, $w = r\,e^{i\vartheta}$ in the above assumptions is equivalent to the function

$$z'(w) = \prod_{k=1}^{n} (w - a_k)^{\frac{\nu_k - 1}{p(A_k)}} \exp\left(\frac{1}{\pi i} \int\limits_{\gamma} \frac{\alpha(\tau)d\tau}{\tau - w}\right), \tag{2}$$

where A_k, $k = \overline{1,n}$ are angular points on Γ, $z(a_k) = A_k$ and $\nu_k \pi$ is the size of inner with respect to G angle at the point A_k, and $\alpha(\tau)$ is the real continuous function on γ ([12], p.146); (it is assumed here that the function f is equivalent to g if $0 < m \le \operatorname{essinf}\left|\frac{f}{g}\right|$ $\big(g \le \operatorname{esssup}\big(\left|\frac{f}{g}\right| = M < \infty\big)$.

3^0.

Definition 3. We say that the problem (1) is Noetherian in the class $E^{p(t)}(G)$, if the operator $M : \varphi \to M\varphi$ is Noetherian in the space $L^{\ell(\tau)}(\gamma)$.

When $a(t) = 1$, the problem (1) turns into the following Dirichlet problem: find the function Φ, satisfying the conditions

$$\begin{cases} \Phi \in E^{p(t)}(G), \\ \operatorname{Re}[\Phi^+(t)] = b(t). \end{cases} \tag{3}$$

It is not difficult to state that in this case $M = S_\gamma$. The conjugate to it operator M^* is equal to $(-S_\gamma)$, and this operator is considered in the space $L^{\ell'(\cdot)}(\gamma)$.

The problem, corresponding to this operator, has again the form (2) when $p(t)$ is replaced by $q(t)$. Therefore the condition of normal solvability will be

$$\int\limits_{\Gamma} g(t)\,\Psi(t)\,dt = 0. \tag{4}$$

where Ψ is the narrowing on Γ of an arbitrary solution of the homogeneous Dirichlet problem of the class $E^{q(t)}(G)$.

4^0. In deriving the basic results which will be presented in Sections 5^0 and 6^0, we have used the following

Theorem A. *Let $\Gamma \in C^1(A, \nu)$, that is, Γ is the piecewise-smooth curve with only one angular point A at which the angle size equals $\pi\nu(A)(= \pi\nu)$, $p \in Q(\Gamma)$, then:* (i) *if $0 < \nu(A) < p(A)$, the problem is uniquely solvable;*

(ii) *if* $\nu(A) > p(A)$*, then the problem is solvable ambiguously, and a general solution is given by the equality*

$$\Phi(z) = \Phi_b(z) \pm C\,\frac{w(z) + w(A)}{w(z) - w(A)}\,,$$

where Φ_b *is a particular solution of the inhomogeneous problem,* C *is an arbitrary constant, and* $w(z)$ *is the inverse function to* $z = z(w)$*;* (iii) *if* $\nu_k = p(A_k)$*, or* $\nu_k = 0$*, then for the problem to be solvable, it is necessary and sufficient that the condition*

$$w^+(t) \int\limits_{\Gamma} \frac{b(z(\tau))\,d\tau}{\omega^+(t)(\tau - t)} \in L^{p(t)}(\Gamma) \tag{5}$$

is fulfilled, where $\omega(w) = (w - w(A))^{-\frac{1}{p(A)}}\omega_0(w)$*, and* $\omega_0(w) = \exp \int\limits_{\gamma} \frac{\alpha(\tau)d\tau}{\tau - w}$ *is the function from the representation* (2).

$$z'(w) = \prod_{k=1}^{n}(w - a_k)^{-\frac{1}{\ell(a_k)}} \exp \int\limits_{\gamma} \frac{\alpha(\tau)d\tau}{\tau - w}\,.$$

Theorem A is a consequence of the results from [10]–[11] (see also [6], p. 221).

5^0. By $C^1(A_1, \dots, A_n; \nu_1, \dots, \nu_n)$ we denote a set of piecewise-smooth curves with angular points $A_1, \dots, A_n$ at which the angle sizes are equal to $\pi\,\nu(A_k)$, $0 \le \nu(A_k) \le 2$. The set of the same piecewise-Lyapunov curves we denote by $C^{1,2}(A_1, \dots, A_n; \nu_1, \dots, \nu_n)$.

Lemma 1. *If*

$$\Gamma \in C^1(A_1, \dots, A_n; \nu_1, \dots, \nu_n), \quad p(t) \in Q(\Gamma), \tag{6}$$

and either

$$\nu(A_k) = p(A_k) \quad or \quad \nu(A_k) = 0, \tag{7}$$

then the Dirichlet problem (3) *is not normally solvable and, consequently, it is not Noetherian one.*

The proof runs as follows: for any curve with the conditions (6)–(7), we construct the function $b_k(t)$ for which equalities (4) are fulfilled, but the condition (5) is violated.

Lemma 2. *If the condition* (5) *holds and either*

$$T = \big\{A_k : \nu(A_k) = p(A_k) \quad or \quad \nu(A_k) = 0\big\} = \varnothing, \tag{8}$$

then the problem (2) *is Noetherian in* $E^{p(\cdot)}(G)$*.*

From Lemmas 1 and 2 we arrive at

Theorem 1. *The Dirichlet problem* (3) *in the conditions* (5) *is Noetherian in* $E^{p(\cdot)}(G)$, *if and only if* $T = \varnothing$, *and its index* $\varkappa$ *is defined by the equality*

$$\varkappa = \operatorname{card} T = \operatorname{card}\left\{A_k : \nu(A_k) > p(A_k)\right\} -$$
$$- \operatorname{card}\left\{A_k : \max\left(0, \frac{p(A_k)-2}{p(A_k)-1}\right) < \nu(A_k) < 2\right\}. \qquad (9)$$

Lemma 3. *If the operator* M *corresponds to the Riemann-Hilbert problem* (1) *in the class* $E^{p(t)}(G)$, *where* $a(t) \in C(\Gamma)$, $a(t) \neq 0$, $b \in L^{p(t)}(\Gamma)$, *then*

$$M = D + V, \qquad (10)$$

where D *is the operator corresponding to the Dirichlet problem* (3), *and* V *is the compact operator in* $L^{p(t)}(\Gamma)$.

Theorem 2. *For the Riemann-Hilbert problem to be Noetherian in the class* $E^{p(t)}(G)$, *it is necessary and sufficient that the condition* (8) *is fulfilled; if it is fulfilled, its index is calculated by the equality* (9).

The proof of this theorem follows from Theorem 1 and Lemma 3 with the use of Atkinson's theorem according to which by adding to the Noetherian operator the compact one we get the Noetherian operator with the same index ([13]).

As for the condition $a(t) \in C(\Gamma)$, $a(t) \neq 0$ from (10), the following theorem is valid.

Theorem 3. *If* $\Gamma \in R$, $p \in \mathcal{P}(\Gamma)$, *then for the Riemann-Hilbert problem to be Noetherian in* $E^{p(t)}(G)$, *it is necessary that the condition*

$$\operatorname{essinf} |a(t)| > 0 \qquad (11)$$

is fulfilled.

This theorem (and the more general one) for $p = \text{const}$ has been proved in [14] (pp.256-8). Following this proof and using the properties of functions from $E^{p(t)}(G)$, we state that Theorem 3 is valid.

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