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ON SHARP WEIGHTED BOUNDS FOR ONE–SIDED OPERATORS NORMS

Introduction

One of the important problems of Harmonic Analysis is to characterize a weight w guaranteeing the boundedness of a given integral operator T in the weighted L^p_w space. An important class of such weights is A_p. It is well-known that it guarantees one-weight characterization for many operators, e.g., for the Hardy–Littlewood maximal operator, singular integrals, etc (see, for example, the papers by B. Muchenhoupt [16], R. Hunt, B. Muckenhoupt, and R. Wheeden [8], R. Coifman and R. Fefferman [4], etc.). However, the sharp dependence of the operator norm $\|T\|_{L^p_w}$ in terms of A_p characteristic of w is known only for some operators. The interest in the sharp weighted norm for singular operators is motivated by application in partial differential equations (see e.g the papers by K. Astala, T. Iwaniec and E. Saksman [2], S. Petermichl and A. Volberg, [19], etc).

Let us denote by M the Hardy–Littlewood maximal operator defined on $\mathbb{R}^n$

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int\limits_Q |f(y)|dy,$$

where the supremum is taken over all cubes $Q \ni x$ with sides parallel to the coordinate axes.

According to the Muckenhoupt [16] theorem, if $1 < p < \infty$, then M is bounded in $L^p_w(\mathbb{R}^n)$ if and only if $w \in A_p(\mathbb{R}^n)$.

In 1993 S. Buckley [3] investigated the sharp A_p bound for M. In particular, he proved that

$$\|M\|_{L^p_w(\mathbb{R}^n)} \le c_{p,n} \|w\|^{\frac{1}{p-1}}_{A_p(\mathbb{R}^n)}, \quad 1 < p < \infty,$$

Moreover, he showed that the power $\frac{1}{p-1}$ is best possible in the sense that we can not replace $\|w\|^{\frac{1}{p-1}}_{A_p}$ by $\psi(\|w\|_{A_p})$ for any positive non-decreasing

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function ψ growing slowly than $x^{\frac{1}{p-1}}$. From here, in particular, it follows that for any $\lambda > 0$,

$$\sup_{w \in A_p} \frac{\|M\|_{L^p_w}}{\|w\|_{A_p}^{\frac{1}{p-1}-\lambda}} = \infty.$$

Sharp weighted bounds for fractional maximal functions and Riesz potentials on Euclidean spaces were established in the paper [12]. We refer also to [5] for mixed type weighted bounds for these operators. Similar problems for maximal functions and singular integrals were studied in [17], [19], [6], [10], [14], [11] etc.

Our aim in this note is to give sharp weighted estimates for one–sided maximal and potential operators.

Finally we mention that we also proved Buckley type theorems for strong maximal functions, multiple potentials and singular integrals. The latter results will be announced later.

1. Preliminaries

Let w be a weight function on a subset Ω of $\mathbb{R}^n$. We denote by $L^p_w(\Omega)$, $1 < p < \infty$, the set of all measurable functions $f : \Omega \to \mathbb{R}$ for which the norm

$$\|f\|_{L^p_w(\Omega)} = \Big(\int\limits_{\Omega} |f(x)|^p w(x) dx \Big)^{\frac{1}{p}}$$

is finite. If $w \equiv \text{const}$, then we denote $L^p_w(\Omega) = L^p(\Omega)$.

We denote by $L^{p,\infty}_w(\Omega)$ the weak weighted Lebesgue space on Ω.

Let us introduce one–sided A_p characteristics:

Definition 1.1. Let $1 < p < \infty$. We say that a weight function w defined on $\mathbb{R}$ satisfies $A^+_p(\mathbb{R})$ condition ($w \in A^+_p(\mathbb{R})$), if

$$\|w\|_{A^+_p(\mathbb{R})} := \sup_{\substack{x \in \mathbb{R} \\ h > 0}} A^+(p, x, h) < \infty,$$

where

$$A^+(p, x, h) := \Big(\frac{1}{h} \int\limits_{x-h}^{x} w(t) dt \Big) \Big(\frac{1}{h} \int\limits_{x}^{x+h} w^{1-p'}(t) dt \Big)^{p-1} < \infty.$$

Further, a weight function w defined on $\mathbb{R}$ satisfies $A^-_p(\mathbb{R})$ condition ($w \in A^-_p(\mathbb{R})$) if

$$\|w\|_{A^-_p(\mathbb{R})} := \sup_{\substack{x \in \mathbb{R} \\ h > 0}} A^-(p, x, h) < \infty,$$

where

$$A^{-}(p,x,h) := \left(\frac{1}{h}\int\limits_{x}^{x+h} w(t)dt\right)\left(\frac{1}{h}\int\limits_{x-h}^{x} w^{1-p'}(t)dt\right)^{p-1}.$$

Remark 1.1. It is easy to see that any increasing weight satisfies $A_p^+(\mathbb{R})$ condition, while any decreasing weight w belongs to $A_p^-(\mathbb{R})$. In fact, $A_p^+(\mathbb{R}) \subset A_p(\mathbb{R})$, $A_p^-(\mathbb{R}) \subset A_p(\mathbb{R})$. More precisely, $A_p(\mathbb{R}) = A_p^+(\mathbb{R}) \cap A_p^-(\mathbb{R})$. Further, the Lebesgue differentiation theorem yields that

2. One–sided Maximal Functions

Let $0 \le \alpha < 1$. We define one-sided (fractional) maximal operators:

$$M_\alpha^+ f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int\limits_{x}^{x+h} |f|dx \qquad x \in \mathbb{R},$$

$$M_\alpha^- f(x) = \sup_{h>0} \frac{1}{h^{1-\alpha}} \int\limits_{x-h}^{x} |f|dx \qquad x \in \mathbb{R}.$$

If $\alpha = 0$, then we have one-sided Hardy-Littlewood maximal operators denoted by M^+ and M^-, respectively.

E. Sawyer [20] proved that if $1 < p < \infty$, then M^+ is bounded in $L_w^p(\mathbb{R})$ if and only if $w \in A_p^+(\mathbb{R})$; M^- is bounded in $L_w^p(\mathbb{R})$ if and only if $w \in A_p^-(\mathbb{R})$.

Our result regarding M^+ and M^- reeds as follows:

Theorem 2.1. *Let $1 < p < \infty$. Then*

(a) *there is a positive constant depending only on p such that*

$$\|M^+\|_{L_w^p(\mathbb{R})} \le c\|w\|_{A_p^+(\mathbb{R})}^{\frac{1}{p-1}}$$

and the exponent $\frac{1}{p-1}$ is best possible.

(b) *there is a positive constant c depending only on p such that the inequality*

$$\|M^-\|_{L_w^p(\mathbb{R})} \le c\|w\|_{A_p^-(\mathbb{R})}^{\frac{1}{p-1}}$$

holds and the exponent $\frac{1}{p-1}$ is best possible.

Let $1 < p, q < \infty$. We say that a weight function w defined on $\mathbb{R}$ satisfies the $A_{p,q}^+(\mathbb{R})$ condition ($w \in A_{p,q}^+(\mathbb{R})$), if

$$\|w\|_{A_{p,q}^+(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ h>0}} \left(\frac{1}{h}\int\limits_{x-h}^{x} w^q(t)dt\right)\left(\frac{1}{h}\int\limits_{x}^{x+h} w^{-p'}(t)dt\right)^{q/p'} < \infty.$$

Further, a weight function w satisfies the $A^-_{p,q}(\mathbb{R})$ condition ($w \in A^-_{p,q}(\mathbb{R})$), if

$$\|w\|_{A^-_{p,q}(\mathbb{R})} := \sup_{\substack{x\in\mathbb{R}\\ h>0}} \left(\frac{1}{h}\int\limits_x^{x+h} w^q(t)dt\right)\left(\frac{1}{h}\int\limits_{x-h}^{x} w^{-p'}(t)dt\right)^{q/p'} < \infty.$$

K. Andersen and E. Sawyer [1] showed that if $1 < p < \infty$, $0 < \alpha < 1/p$, $q = \frac{p}{1-\alpha p}$, then:

(a) M^+_α is bounded from $L^p_{w^p}(\mathbb{R})$ to $L^q_{w^q}(\mathbb{R})$ if and only if $w \in A^+_{p,q}(\mathbb{R})$;

(b) M^-_α is bounded from $L^p_{w^p}(\mathbb{R})$ to $L^q_{w^q}(\mathbb{R})$ if and only if $w \in A^-_{p,q}(\mathbb{R})$.

We derived the following statement:

Theorem 2.2. *Suppose that* $0 < \alpha < 1$ *and* $1 < p < 1/\alpha$. *We set* $q = \frac{p}{1-\alpha p}$. *Then*

(a) *there is a positive constant* c *depending only on* p *and* α *such that the inequality*

$$\|wM^+_\alpha\|_{L^q} \le c\|w\|^{\frac{p'}{q}(1-\alpha)}_{A^+_{p,q}(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}$$

holds and the exponent $\frac{p'}{q}(1-\alpha)$ *is best possible.*

(b) *there is a positive constant* c *depending only on* p *and* α *such that the inequality*

$$\|wM^-_\alpha\|_{L^q} \le c\|w\|^{\frac{p'}{q}(1-\alpha)}_{A^-_{p,q}(\mathbb{R})}\|f\|_{L^p(\mathbb{R})}$$

holds and the exponent $\frac{p'}{q}(1-\alpha)$ *is best possible.*

3. One–sided Potentials

Now we give the sharp weighted bounds for one–sided fractional integrals. Let $\mathcal{R}_\alpha$ and $\mathcal{W}_\alpha$ be operators defined as follows:

$$\mathcal{R}_\alpha f(x) = \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}}dt, \qquad x > 0,$$

$$\mathcal{W}_\alpha f(x) = \int\limits_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}}dt, \qquad x > 0.$$

K. Andersen and E. Sawyer [1] proved that $\mathcal{R}_\alpha$ is bounded from $L^p_{w^p}(\mathbb{R})$ to $L^q_{w^q}(\mathbb{R})$ if and only if $w \in A^-_{p,q}(\mathbb{R})$; further, $\mathcal{W}_\alpha$ is bounded from $L^p_{w^p}(\mathbb{R})$ to $L^q_{w^q}(\mathbb{R})$ if and only if $w \in A^+_{p,q}(\mathbb{R})$;

We proved the following statements:

Theorem 3.1. *Let* $0 < \alpha < 1$, $1 \le p < 1/\alpha$ *and let* q *satisfy* $q = \frac{p}{1-\alpha p}$. *Then*

(a)

$$\|\mathcal{R}_\alpha\|_{L^p_{w^p} \to L^q_{w^q}} \le c\|w\|^{(1-\alpha)\max\{1,p'/q\}}_{A^-_{p,q}(\mathbb{R})}.$$

Furthermore, this estimate is sharp in the sense that the exponent $(1-\alpha)\max\{1,p'/q\}$ *is best possible;*

(b)

$$\|\mathcal{W}_\alpha\|_{L^p_{w^p} \to L^q_{w^q}} \le c\|w\|^{(1-\alpha)\max\{1,p'/q\}}_{A^+_{p,q}(\mathbb{R})}.$$

Moreover, the exponent $(1-\alpha)\max\{1,p'/q\}$ *is best possible.*

The proofs of these results are based on two-weight criteria established under the conditions of Gabidzashvili–Kokilashvili type (see the monograph [7], Ch. 2, for these criteria).

Let

$$(I_\alpha f)(x) = \int\limits_{\mathbb{R}} \frac{f(t)}{|x-t|^{1-\alpha}}dt, \quad x \in \mathbb{R},$$

be the Riesz potential operator on $\mathbb{R}$, where $0 < \alpha < 1$. The appropriate fractional maximal function on $\mathbb{R}$ is given by

$$(M_\alpha f)(x) = \frac{1}{h^{1-\alpha}} \int\limits_{x-h}^{x+h} |f(t)|dt, \quad x \in \mathbb{R},$$

where $0 \le \alpha < 1$.

The following pointwise inequalities can be checked immediately:

$$(M_\alpha f)(x) \le (M^+_\alpha f)(x) + (M^-_\alpha f)(x);$$
$$\max\big\{(M^+_\alpha f)(x), (M^-_\alpha f)(x)\big\} \le (M_\alpha f)(x),$$
$$(I_\alpha f)(x) = (\mathcal{R}_\alpha f)(x) + (\mathcal{W}_\alpha f)(x);$$
$$\max\big\{(\mathcal{R}_\alpha f)(x), (\mathcal{W}_\alpha f)(x)\big\} \le (I_\alpha f)(x), \quad f \ge 0.$$

Taking these estimates into account and the results formulated above we derive the following corollaries on the real line:

Corollary 3.1 ([3]). *Let* $1 < p < \infty$, *then*

$$\|M\|_{L^p_w(\mathbb{R})} \le c\|w\|^{\frac{1}{p-1}}_{A_p(\mathbb{R})}$$

and the exponent $\frac{1}{p-1}$ *is best possible.*

Corollary 3.2 ([12]). *Suppose that* $0 < \alpha < 1$, $1 < p < 1/\alpha$ *and that* q *is such that* $1/q - 1/p - \alpha = 0$. *Then*

$$\|wM_\alpha\|_{L^q} \le c\|w\|^{\frac{p'}{q}(1-\alpha)}_{A_{p,q}(\mathbb{R})} \|f\|_{L^p(\mathbb{R})}$$

and the exponent $\frac{p'}{q}(1-\alpha)$ *is best possible.*

Corollary 3.3 ([12])**.** *Let* $0<\alpha<1$, $1\le p<1/\alpha$ *and let* q *satisfy* $q=\frac{p}{1-\alpha p}$. *Then*

$$\|I_\alpha\|_{L^p_{w^p}\to L^q_{w^q}} \le c\|w\|_{A_{p,q}(\mathbb{R})}^{(1-\alpha)\max\{1,p'/q\}}.$$

and the exponent $(1-\alpha)\max\{1,p'/q\}$ *is best possible.*

4. Mixed Type Estimates

Observe that taking the limit as $p\to\infty$, the following equalities hold:

$$\lim_{p\to\infty} A^+_{p,x,h} = \left(\frac{1}{r}\int\limits_{x-r}^{x} w(x)dx\right)\exp\left(\frac{1}{r}\int\limits_{x}^{x+r}\log\frac{1}{w(t)}dt\right)$$

and

$$\lim_{p\to\infty} A^-_{p,x,h} = \left(\frac{1}{r}\int\limits_{x}^{x+r} w(x)dx\right)\exp\left(\frac{1}{r}\int\limits_{x-r}^{x}\log\frac{1}{w(t)}dt\right)$$

There naturally arises a question regarding the mixed type bounds involving so called one–sided Hruščev [9] $A_{\infty,\exp}$ characteristics for norms of one-sided operators.

Let us denote

$$\|\sigma\|_{A^+_{\infty,\exp}} := \sup_{a,r}\left(\frac{1}{r}\int\limits_{a-r}^{a}\sigma(t)dt\right)\exp\left(\frac{1}{r}\int\limits_{a}^{a+r}\log\frac{1}{\sigma(t)}dt\right);$$

$$\|\sigma\|_{A^-_{\infty,\exp}} := \sup_{a,r}\left(\frac{1}{r}\int\limits_{a-r}^{a}\sigma(t)dt\right)\exp\left(\frac{1}{r}\int\limits_{a}^{a+r}\log\frac{1}{\sigma(t)}dt\right).$$

We say that $w\in A^\pm_\infty$ if $w\in A^\pm_r$, for some r. For the definition and properties of one–sided A_∞ weights we refer e.g., to [15].

By Jensen's inequality it follows that

$$\|w\|_{A^\pm_{\infty,\exp}} \le \|w\|_{A^\pm_r},\quad r>1.$$

The following statements give the mixed type estimates for one–sided operators:

Theorem 4.1. *Suppose that* $1<p<\infty$ *and* $0<\alpha<1/p$. *We set* $q=\frac{p}{1-\alpha p}$. *Then the following statements hold:*

(i) *there is a positive constant* c *depending only on* p *and* α *such that*

$$\|M^+_\alpha\|_{L^p_u\to L^q_v} \le c\|w\|^{1/q}_{A^+_{p,q}}\|w^{-p'}\|^{1/q}_{A^-_{\infty,\exp}}.$$

(ii) *there is a positive constant* c *depending only on* p *and* α *such that*

$$\|M_\alpha^-\|_{L_u^p\to L_v^q} \le c\|w\|_{A_{p,q}^-}^{1/q}\|w^{-p'}\|_{A_{\infty,\exp}^+}^{1/q}.$$

Theorem 4.2. *Let $1<p<\infty$ and $0<\alpha<1/p$. We set $q=\frac{p}{1-\alpha p}$. Then*
(a)

$$\|\mathcal{R}_\alpha\|_{L_u^p\to L_v^{q,\infty}} \le c\|w\|_{A_{p,q}^-}^{1/q}\|w^q\|_{A_{\infty,\exp}^-}^{1/p'}$$

where the positive constant c depends only on p and α.

(b) *if $v\in A_\infty^+$, then*

$$\|\mathcal{W}_\alpha\|_{L_u^p\to L_v^{q,\infty}} \le c\|w\|_{A_{p,q}^+}^{1/q}\|w^q\|_{A_{\infty,\exp}^+}^{1/p'}$$

where the positive constant c depends only on p and α.

Theorem 4.3. *Let $1<p<\infty$ and $0<\alpha<1/p$. We set $q=\frac{p}{1-\alpha p}$. Then*
(a) *there is a positive constant c depending only on p and α such that*

$$\|\mathcal{R}_\alpha\|_{L_u^p\to L_v^q} \le c\|w\|_{A_{p,q}^-}^{1/q}\Big(\|w^q\|_{A_{\infty,\exp}^-}^{1/p'}+\|w^{-p'}\|_{A_{\infty,\exp}^+}^{1/q}\Big);$$

(b) *there is a positive constant c depending only on p and α such that*

$$\|\mathcal{W}_\alpha\|_{L_u^p\to L_v^q} \le c\|w\|_{A_{p,q}^+}^{1/q}\Big(\|w^q\|_{A_{\infty,\exp}^+}^{1/p'}+\|w^{-p'}\|_{A_{\infty,\exp}^-}^{1/q}\Big);$$

Remark 4.4. One–weighted sharp estimates (see Theorems 2.1, 3.1 and 3.1) now follow from Theorems 4.1 and 4.3.

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References

1. K. F. Andersen and E. Sawyer, Weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators. *Trans. Amer. Math. Soc.* **308** (1988), No. 2, 547–558.
2. K. Astala, T. Iwaniec and E. Saksman, Beltrami operators in the plan. *Duke Math. J.* **107** (2001), No. 1, 27–56.
3. S. M. Buckley, Estimates for operator norms on weighted space and reverse Jensen inequalities, *Trans. Amer. Math. Soc.* **340** (1993), No. 1, 253–272.
4. R. Coifman and R. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, *Studia. Math.* **51** (1974), 241–250.
5. D. Cruz-Uribe, SFO and K. Moen, A fractional Muckenhoupt-Wheeden theorem and its consequences, *Intgr. Equ. Oper. Theory* **76**(2013), 421-446.
6. O. Dragičević, L. Grafakos, C. Pereyra and S. Petermichl, Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces. *Publ. Math.* **49** (2005), No. 1, 73–91.

7. D. E. Edmunds, V. Kokilashvili and A. Meskhi, On Fourier multipliers in weighted Triebel-Lizorkin spaces. *J. Inequal. Appl.* **7** (2002), No. 4, 555–591.
8. R. Hunt, B. Muckenhoupt and R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.* **176** (1973), 227–251.
9. S. Hruščev, A description of weights satisfying the A_∞ condition of Muckenhoupt, *Proc. Amer. Math. Soc.* **90**(1984), No.2, 253–257.
10. T. Hytönen, The sharp weighted bound for general Calderón-Zygmund operators. *Ann. of Math.* **175** (2012), No. 3, 1473–1506.
11. T. Hytönen and C. Perez, Sharp weighted bounds involving A_∞, *Anal. PDE* **6** (2013), No. 4, 777–818.
12. M. T. Lacey, K. Moen, C. Perez and R. H. Torres, Sharp weighted bounds for fractional integral operators. *J. Funct. Anal.* **259** (2010), No. 5, 1073–1097.
13. A. Lerner, An elementary approach to several results on the Hardy–Littlewood maximal operator, *Proc. Amer. Math. Soc.* **136** (2008), No. 8, 2829–2833.
14. A. K. Lerner and K. Moen. Mixed $A_p - A_{infty}$ estimates with one supremum. *Preprint*, arXiv:1212.0571, 2013.
15. F. Martin–Reyes, L. Pick and A. De la Torre, A_∞^+ condition, *Canad. Math. J.* **45**(1993), No. 6, 1231–1244.
16. B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207–226.
17. S. Petermichl, The sharp bound for the Hilbert transform in weighted Lebesgue spaces in terms of the classical A_p characteristic, *Amer. J. Math.* **129** (2007), No. 5, 1355–1375.
18. S. Petermichl, The sharp weighted bound for the Riesz transforms, *Proc. Amer. Math. Soc.* **136** (2008), No. 4, 1237–1249.
19. S. Petermichl and A. Volberg, Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular. *Duke Math. J.* **112** (2002), No. 2, 281–305.
20. E. Sawyer, Weighted inequalities for the one-sided Hardy-Littlewood maximal functions. *Trans. Amer. Math. Soc.* **297** (1986), No. 1, 53–61.

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