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BOUNDARY VALUE PROBLEM FOR A WAVE EQUATION WITH POWER NONLINEARITY IN THE ANGULAR DOMAINS

In a plane of independent variables x and t we consider the wave equation with power nonlinearity of the type

$$Lu := \square u + \lambda |u|^{\alpha} u = F(x,t), \tag{1}$$

where λ and α are the given real numbers, $\alpha > 0$; $F = F(x,t)$ is the given and $u = u(x,t)$ is an unknown real functions; $\square := \partial^2/\partial t^2 - \partial^2/\partial x^2$.

By $D : \gamma_2(t) < x < 0$, $t > 0$ we denote an angular domain lying inside of the characteristic angle $\Lambda_0 : t > |x|$ and bounded by the ray $\gamma_1 : x = 0$, $t \geq 0$ and by the smooth noncharacteristic curve $\gamma_2 : x = \gamma_2(t)$, $t \geq 0$, i.e., $|\gamma_2'(t)| \neq 1$, $t \geq 0$, emanating from the origin $O(0,0)$. Under these assumptions, obviously, $-t < \gamma_2(t) < \gamma_1(t) = 0$, $t > 0$; $-1 < \gamma_2'(t)| \leq 0$, $t \geq 0$, $\gamma_i(0) = 0$, $i = 1,2$.

Suppose $D_T := D \cap \{t < T\}$ and $\gamma_{i,T} := \gamma_i \cap \{t \leq T\}$, $T > 0$, $i = 1,2$. Obviously, for $T = \infty$, we have $D_\infty = D$, $\gamma_{i,\infty} := \gamma_i$, $i = 1,2$.

For equation (1), we consider the boundary value problem when the oblique derivative of a solution is given on $\gamma_{1,T}$ and a solution itself of equation (1) is given on $\gamma_{2,T}$. The problem is formulated as follows: find in the domain D_T a solution $u = u(x,t)$ of equation (1) under the boundary conditions

$$(l_1 u_x + l_2 u_t)|_{\gamma_{1,T}} = 0, \tag{2}$$

$$u|_{\gamma_{2,T}} = 0, \tag{3}$$

where l_1, and l_2 are the given continuous functions, and $(|l_1| + |l_2|)|_{\gamma_1} \neq 0$.

Note that in the linear case, i.e., when $\lambda = 0$ in (1), and instead of the boundary conditions (2), (3) are considered the conditions

$$(\alpha_i u_x + \beta_i u_t)|_{\gamma_{i,T}} = 0, \quad i = 1,2; \quad u(0,0) = 0, \tag{4}$$

the problem (1), (4) in the domain D_T has been studied in [1–6]. It should also be noted that the problem (1)–(3) is equivalent to the problem (1),

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(4) when the direction of (α_2, β_2) coincides with that of the tangent to the curve $\gamma_{2,T}$ at any of its points.

In the case of nonlinear equation (1), when homogeneous Dirichlet conditions $u|_{\gamma_{i,T}} = 0$, $i = 1,2$ are taken on γ_1 and γ_2, and one of those curves γ_1 or γ_2 is the characteristic, this problem has been studied in [7–9], while when $u_x|_{\gamma_{1,T}} = 0$, $u|_{\gamma_{2,T}} = 0$, where $\gamma_{1,T} : x = 0$, $0 \le t \le T$, and $\gamma_{2,T} : x = -t$, $0 \le t \le T$ is the characteristic of equation (1), the problem is studied in [10,11]. As is pointed out in [1,6], such type of problems arise in mathematical modeling of small harmonic wedge oscillations in a supersonic flow and string oscillations in a cylinder filled with a viscous liquid.

Suppose

$$\overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T) := \big\{ v \in C^2(\overline{D}_T) : (l_1 v_x + l_2 v_t)|_{\gamma_{1,T}} = 0,\\ v|_{\gamma_{2,T}} = 0 \big\}, \quad \gamma_T := \gamma_{1,T} \cup \gamma_{2,T}.$$

Definition 1. Let $F \in C(\overline{D}_T)$; $l_1, l_2 \in C(\gamma_{1,T})$. The function u is said to be a strong generalized solution of the problem (1)–(3) of the class C in the domain D_T, if $u \in C(\overline{D}_T)$ and there exists a sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T)$ such that $u_n \to u$ and $Lu_n \to F$ in the space $C(\overline{D}_T)$, as $n \to \infty$.

Remark 1. Obviously, a classical solution of the problem (1)–(3) from the space $\overset{\circ}{C}{}^2(\overline{D}_T, \gamma_T)$ is a strong generalized solution of that problem of the class C in the domain D_T in a sense of Definition 1.

Definition 2. Let $F \in C(\overline{D}_\infty)$; $l_1, l_2 \in C(\gamma_{1,\infty})$. We say that the problem (1)–(3) is globally solvable in the class C, if this problem for any finite $T > 0$ has at least one strong generalized solution of the class C in the domain D_T in a sense of Definition 1.

Definition 3. Let $F \in C(\overline{D}_\infty)$; $l_1, l_2 \in C(\gamma_{1,\infty})$. The function $u \in C(\overline{D}_\infty)$ is said to be a global strong generalized solution of the problem (1)–(3) of the class C in the domain D_∞, if for any finite $T > 0$ the function $u|_{D_T}$ is a strong generalized solution of that problem of the class C in the domain D_T in a sense of Definition 1.

Definition 4. Let $F \in C(\overline{D}_\infty)$; $l_1, l_2 \in C(\gamma_{1,\infty})$. We say that the problem (1)–(3) is locally solvable in the class C, if there exists a positive number $T_0 = T_0(F)$ such that for $T \le T_0$ this problem has at least one strong generalized solution of the class C in the domain D_T in a sense of Definition 1.

Theorem 1. *Let $\lambda > 0$, $F \in C(\overline{D}_T)$; $\gamma_{2,T} \in C^2([0,T])$, $l_i \in C^1(\gamma_{1,T})$, $i = 1,2$, $(l_1 l_2)|_{\gamma_{1,T}} \ge 0$, and in case $(l_1 l_2)(O) = 0$, the curves $\gamma_{1,T}$ and $\gamma_{2,T}$*

do not have a common tangent line at the point O. Then the problem (1)–(3) *has a unique strong generalized solution of the class C in the domain D_T in a sense of Definition* 1.

Theorem 2. *Let $F \in C(\overline{D}_\infty)$; $\gamma_{2,\infty} \in C^2$, $l_i \in C^1(\gamma_{1,\infty})$, $i = 1,2$, $(l_1 l_2)|_{\gamma_{1,\infty}} \geq 0$, and in case $(l_1 l_2)(O) = 0$, the curves $\gamma_{1,\infty}$ and $\gamma_{2,\infty}$ do not have a common tangent line at the point O. Then for $\lambda < 0$, the problem* (1)–(3) *is locally solvable in the class C in a sense of Definition* 4.

Remark 2. Note that if the conditions of Theorem 1 are fulfilled, then a strong generalized solution u of the problem (1)–(3) of the class C in the domain D_T belongs to the space $C^1(\overline{D}_T)$, and under the additional requirement that $F \in C^1(\overline{D}_T)$, this solution belongs to the space $C^2(\overline{D}_T)$, that is, it will be classical. In both cases the boundary conditions (2) and (3) are fulfilled pointwise. If the conditions of Theorem 1 are fulfilled for $T = +\infty$, then the problem (1)–(3) is globally solvable in the class C in a sense of Definition 2, and under the additional requirement that $F \in C^1(\overline{D}_\infty)$, the problem (1)–(3) has a unique global classical solution $u \in C^2(\overline{D}_\infty)$.

When $(l_1 l_2)(O) = 0$, and the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ have a common tangent line at the point O, the solvability of the problem (1)–(3) will depend on the tangency order of the curves $\gamma_{1,T}$ and $\gamma_{2,T}$ at the point O and on the direction of (l_1, l_2) in the vicinity of that point.

By virtue of the requirements imposed on the curve $\gamma_2 \in C^2$, it can be easily verified that if $\eta(t) = t - \gamma_2(t)$, then $\eta'(t) = 1 - \gamma_2'(t) > 0$ and, consequently, there exists an inverse function $t = \zeta(\eta)$, where the function $\tau(\eta) = \zeta(\eta) + \gamma_2(\zeta(\eta))$ satisfies the conditions

$$\tau \in C^2, \quad \tau(0) = 0, \quad \tau'(\eta) > 0, \quad 0 < \tau(\eta) < \eta, \quad \eta > 0, \tag{5}$$

and if the curves γ_1 and γ_2 have a common tangent line at the point O, the equality

$$\tau'(0) = 1 \tag{6}$$

holds.

It will be assumed below that the curves γ_1 and γ_2 have at the point O the first order tangency. Then owing to (5) and (6), we have

$$\tau(\eta) = \eta - \tau_0 \eta^2 + \sigma(\eta)\eta^2, \quad \tau_0 = -\frac{\tau''(0)}{2} > 0, \quad \eta \geq 0, \tag{7}$$

where $\sigma(\eta) = o(\eta)$, as $\eta \to 0+$, i.e., $\lim\limits_{\eta \to 0+} \sigma(\eta) = 0$.

Assuming $a(t) := \dfrac{l_2 - l_1}{l_2 + l_1}(t)$, $t \geq 0$, for $(l_1 l_2)(O) = 0$ we have $|a(0)| = 1$. According to what has been said above, we assume below that

$$a(t) \operatorname{sign} a(0) = 1 + a_0 t^m + \mu(t) t^m, \quad 0 \leq t \leq \varepsilon, \tag{8}$$

where $a_0 = \text{const} \neq 0$, $m = \text{const} \geq 1$, $\mu(t) = o(t)$, as $t \to 0+$, i.e., $\lim\limits_{t\to 0+} \mu(t) = 0$, and ε is a sufficiently small positive number.

Theorem 3. *Let $\lambda > 0$, $F \in C(\overline{D}_T)$; $\gamma_{2,T} \in C^2([0,T])$, $l_i \in C^1(\gamma_{1,T})$, $i = 1,2$, $(l_1 l_2)|_{\gamma_{1,T}} \geq 0$, $(l_1 l_2)(O) = 0$, and the curves γ_1 and γ_2 have at the point O the first order tangency. Then if the conditions* (7) *and* (8) *are fulfilled, the problem* (1)–(3) *has a unique strong generalized solution u of the class C in the domain D_T in a sense of Definition* 1, *if at least one of the following three conditions:* i) $m > 1$; ii) $m = 1$, $a(0)a_0 < 0$; iii) $m = 1$, $a(0)a_0 > 0$, $2|a_0| < \tau_0$ *is fulfilled.*

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