Proc. A. Razmadze Math. Inst. 164(2014), 98–99

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NOTE ON SINGULAR LEBESGUE–STIELTJES MEASURES

1. Definitions and Notation. A mapping B defined on $\mathbb{R}^n$ is said to be a differentiation basis (briefly: basis) if for every $x \in \mathbb{R}^n$, $B(x)$ is a family of open sets containing x such that there exists a sequence $\{R_k\} \subset B(x)$ with $\operatorname{diam} R_k \to 0$ $(k \to \infty)$.

By $|E|$ we will denote the Lebesgue measure of a Lebesgue measurable set $E \subset \mathbb{R}^n$.

For a Lebesgue-Stieltjes measure μ and a basis B, the numbers

$$\overline{D}_B(\mu, x) = \varlimsup_{R \in B(x), \operatorname{diam} R \to 0} \frac{\mu(R)}{|R|}, \quad \underline{D}_B(\mu, x) = \varliminf_{R \in B(x), \operatorname{diam} R \to 0} \frac{\mu(R)}{|R|}$$

are called the upper and the lower derivative with respect to B, respectively, of μ at a point x. If the upper and the lower derivative coincide, then their common value is called the derivative with respect to B of μ at a point x and is denoted by $D_B(\mu, x)$.

A basis B is said to differentiate a Lebesgue–Stieltjes measure μ if $D_B(\mu, x)$ exists for almost all $x \in \mathbb{R}^n$.

For $f \in L(\mathbb{R}^n)$ a basis B is said to differentiate $\int f$ if $D_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^n$.

A basis B is said to differentiate a class $X \subset L(\mathbb{R}^n)$ if B differentiates $\int f$ for every $f \in X$.

A basis B is called translation invariant if $B(x) = \{x + R : R \in B(0)\}$ for every $x \in \mathbb{R}^n$.

Let B_1 and B_2 be bases in $\mathbb{R}^n$. B_1 is said to be a sub-basis of B_2 (entry: $B_1 \subset B_2$) if $B_1(x) \subset B_2(x)$ for every $x \in \mathbb{R}^n$.

Below everywhere we will assume that dimension n is greater then 1.

Denote by $\mathbf{Q}$ and $\mathbf{I}$, the bases defined as follows:

$\mathbf{Q}(x)$ $(x \in \mathbb{R}^n)$ consists of all n-dimensional cubic intervals containing x;
$\mathbf{I}(x)$ $(x \in \mathbb{R}^n)$ consists of all n-dimensional intervals containing x.

Note that according to the classical theorem of Lebesgue (see e.g. [1, Ch. V, §5]) the basis $\mathbf{Q}$ differentiates every Lebesgue–Stieltjes measure and

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2010 *Mathematics Subject Classification:* 28A15.
Key words and phrases. Lebesgue–Stieltjes measure, singular measure, differentiation basis.

furthermore $\mathbf{Q}$ differentiates $L(\mathbb{R}^n)$. On the other hand by virtue of results of Saks and of Busemann and Feller (see e.g. [2, Ch. IV, § 2]) the basis $\mathbf{I}$ does not differentiate $L(\mathbb{R}^n)$. Here we note also that by Stokolos [3, 4] it was given a geometrical characterization of translation invariant sub-bases of $\mathbf{I}$ that does not differentiate $L(\mathbb{R}^n)$.

A Lebesgue–Stieltjes measure μ is called singular if there is a Borel set E such that: $|E| = 0$ and $\mu(A) = \mu(A \cap E)$ for every Borel set A.

2. Result. It is known that (see e.g. [1,Ch.V,§7]) if μ is a singular Lebesgue–Stieltjes measure, then

$$D_{\mathbf{Q}}(\mu, x) = 0 \text{ almost everywhere.}$$

The similar essertion is not valid for the basis $\mathbf{I}$, moreover it is true the following result.

Theorem. *Let B be a translation invariant sub-bases of $\boldsymbol{I}$ that does not differentiate $L(\mathbb{R}^n)$. Then there exists a singular Lebesgue-Stieltjes measure μ such that*

$$\overline{D}_B(\mu, x) = \infty \textit{ almost everywhere.}$$

Corollary. *There exists a singular Lebesgue–Stieltjes measure μ such that*

$$\overline{D}_{\boldsymbol{I}}(\mu, x) = \infty \textit{ almost everywhere.}$$

Acknowledgement

The second author was supported by Shota Rustaveli National Science Foundation Grant No: 31/48.

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