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INTEGRAL OPERATORS IN GRAND VARIABLE
LEBESGUE SPACES

1. Introduction

In this note new Banach function spaces are introduced. These spaces unify two non-standard function spaces: variable exponent Lebesgue spaces and grand Lebesgue space. Comprehensive study and some aspects of applications of one these spaces were delivered in the recently published books [1], [6], [23]. The variable exponent Lebesgue space represents the special case of that introduced by W. Orlicz in the 30-th of the last century and then generalized by I. Musielak and W. Orlicz. H. Nakano [28] then specified it.

The grand Lebesgue spaces were introduced in the 90-th of the last century by T. Iwaniec and C. Sbordone [12]. Lately number of problems of Harmonic analysis and the theory of non-linear differential equations were studied in these spaces (see e.g. the papers [9], [16], [17], [18], [15], [29], [20], [21], etc.).

The spaces introduced in this paper are non-reflexive, non-separable and non-rearrangement invariant. The boundedness results of the Hardy-Littlewood maximal and Calderón-Zygmund operators defined on spaces of homogeneous type are given. From the above-mentioned solutions quite a number of interesting results are obtained.

2. Preliminaries

Throughout the paper we assume that \((X, d, \mu)\) is a space of homogeneous type (SHT) with finite measure, i.e. \(X\) is a set, \(d\) is a quasi-metric on \(X\) and \(\mu\) is a finite measure on \(X\) satisfying the well-known doubling condition. We will assume that \(X\) does not contain any atoms. Let \(p\) be a measurable function on \(X\) satisfying the condition

\[
1 < p_- \leq p_+ < \infty,
\]

(1)

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where
\[ p_- := \inf_X p; \quad p_+ := \sup_X p. \]

We denote the class of all exponent satisfying condition (1) by \( \mathcal{P}(X) \).

Let us denote by \( \mathcal{D}(X) \) the class of bounded functions on \( X \) with compact support, \( d_X \) be the diameter of \( X \).

Let \( p(\cdot) \in \mathcal{P}(X) \). By the symbol \( L^{p(\cdot)} \) we denote the variable exponent Lebesgue spaces (see e.g. [26], [6] for the definition). Further, let \( \theta > 0 \).

We denote by \( L^{p(\cdot),\theta}(X) \) the class of all measurable functions \( f : X \to \mathbb{R} \) for which the norm
\[
\| f \|_{L^{p(\cdot),\theta}(X)} := \sup_{0 < \varepsilon < p_+ - 1} \frac{\theta}{\varepsilon^{\theta-\varepsilon}} \| f \|_{L^{p(\cdot) - \varepsilon}(X)}
\]
is finite.

Together with the space \( L^{p(\cdot),\theta}(X) \) it is interesting to consider the space \( \mathcal{L}^{p(\cdot),\theta}(X) \) which is defined with respect to the norm
\[
\| f \|_{\mathcal{L}^{p(\cdot),\theta}(X)} := \sup_{0 < \varepsilon < p_+ - 1} \| \varepsilon^{\frac{\theta}{\varepsilon^{\theta-\varepsilon}}} f \|_{L^{p(\cdot) - \varepsilon}(X)}.
\]

It is obvious that
\[ \mathcal{L}^{p(\cdot),\theta}(X) \hookrightarrow L^{p(\cdot),\theta}(X). \]

Further, there exists a function \( f \) such that \( f \in L^{p(\cdot),\theta}(X) \) but \( f \notin \mathcal{L}^{p(\cdot),\theta}(X) \).

It can be checked that \( L^{p(\cdot),\theta}(X) \) and \( \mathcal{L}^{p(\cdot),\theta}(X) \) are Banach spaces.

Remark. Let \( X \) be a bounded domain in \( \mathbb{R}^n \), \( d \) be an Euclidean metric, and let \( \mu \) be the Lebesgue measure. If \( p = p_\text{c} = \text{const} \), then \( L^{p(\cdot),\theta}(X) = \mathcal{L}^{p(\cdot),\theta}(X) \) is the grand Lebesgue space \( L^{p(\cdot),\theta}(X) \) introduced in [10]. In the case \( p = p_\text{c} = \text{const} \) and \( \theta = 1 \), then we have Iwaniec-Sbordone [12] space \( L^{p(\cdot)} \). The space \( L^{p(\cdot)} \) naturally arises, for example, to study integrability problems of the Jacobian under minimal hypothesis (see [12]), while \( L^{p(\cdot),\theta} \) is related to the investigation of the nonhomogeneous \( n \)-harmonic equation \( \text{div} A(x, \nabla u) = \mu \) (see [3]).

**Proposition A.** The spaces \( L^{p(\cdot),\theta}(X) \) and \( \mathcal{L}^{p(\cdot),\theta}(X) \) are complete. The closure of \( L^{p(\cdot)}(X) \) in \( L^{p(\cdot),\theta}(X) \) (resp. in \( \mathcal{L}^{p(\cdot),\theta}(X) \)) consists of those \( f \in L^{p(\cdot),\theta}(X) \) (resp. \( f \in \mathcal{L}^{p(\cdot),\theta}(X) \)) for which
\[
\lim_{\varepsilon \to 0} \frac{\varepsilon^{\frac{\theta}{\varepsilon^{\theta-\varepsilon}}} f(\cdot)}{\| f(\cdot) \|_{L^{p(\cdot) - \varepsilon}(X)}} = 0.
\]

**Proposition B.** Let \( p \in \mathcal{P}(X) \). Then the following embeddings hold:
\[
\begin{align*}
L^{p(\cdot)}(X) & \hookrightarrow L^{p(\cdot),\theta}(X) \hookrightarrow L^{p(\cdot) - \varepsilon}(X), \quad 0 < \varepsilon < p_- - 1; \\
L^{p(\cdot)}(X) & \hookrightarrow \mathcal{L}^{p(\cdot),\theta}(X) \hookrightarrow L^{p(\cdot) - \varepsilon}(X), \quad 0 < \varepsilon < p_- - 1.
\end{align*}
\]
We define the Hardy–Littlewood maximal operator on $X$ by

$$(M_X f)(x) = \sup_{0 < r < d(x)} \frac{1}{\mu B(x, r)} \int_{B(x, r)} |f(y)| d\mu(y), \quad x \in X,$$

where $B(x, r)$ is the ball in $X$ with center $x$ and radius $r$.

**Definition 1.** Suppose that $\tilde{\mathcal{P}}_{loc}^{\log}(X)$ is the class of those exponents $p$ satisfying the local log-Hölder continuity condition: there is a positive constant $c_0$ such that for all $x, y \in X$ with $d(x, y) < 1/2$,

$$|p(x) - p(y)| \leq \frac{c_0}{-\ln(d(x, y))}.$$

Further, let $\mathcal{P}_{loc}^{\log}(X)$ be the class of those exponents satisfying the condition: there exists a positive constants $a$ and $b$ such that if $d(x, y) < b$, then

$$|p(x) - p(y)| \leq \frac{a}{-\ln(\mu B(x, d(x, y)))}.$$

It is easy to check that $\mathcal{P}_{loc}^{\log}(X) \subset \tilde{\mathcal{P}}_{loc}^{\log}(X)$.

The boundedness of $M_X$ in $L^{p(\cdot)}(X)$ spaces was established by L. Diening [5] for Euclidean spaces and by M. Khabazi [13] for an SHT.

3. The Main Results

Now we formulate the main results of this paper:

**Theorem 1** (General-type theorem). Let $p \in \mathcal{P}(X)$ and let $\theta > 0$.

(a) Suppose that $\mathcal{F}$ is a family of pairs $(f, g)$ such that

$$\|f\|_{L^{p(\cdot)} - \varepsilon} \leq c_{p, \varepsilon} \|g\|_{L^{p(\cdot)} - \varepsilon}.$$

If

$$\sup_{0 < \varepsilon < \sigma} c_{p, \varepsilon} < \infty$$

for some positive constant $\sigma$, then for all $(f, g) \in \mathcal{F},$

$$\|f\|_{L^{p(\cdot)} - \varepsilon (X)} \leq c \|g\|_{L^{p(\cdot)} - \varepsilon (X)}.$$

(b) Suppose that $\mathcal{F}$ is a family of pairs $(f, g)$ such that

$$\|e^{\frac{\theta}{\varepsilon}} f\|_{L^{p(\cdot)} - \varepsilon (X)} \leq b_{p, \varepsilon} \|e^{\frac{\theta}{\varepsilon}} g\|_{L^{p(\cdot)} - \varepsilon (X)}$$

for some positive constant $b_{p, \varepsilon}$. If

$$\sup_{0 < \varepsilon < \sigma} b_{p, \varepsilon} < \infty$$

for some positive constant $\sigma$, then there exists a positive constant $c$ such that for all $(f, g) \in \mathcal{F},$

$$\|f\|_{L^{p(\cdot)} - \varepsilon (X)} \leq c \|g\|_{L^{p(\cdot)} - \varepsilon (X)}.$$
Theorem 2. Let \( p \in \mathcal{P}(X) \cap \mathcal{P}_{\log}^{\text{loc}}(X) \) and let \( \theta > 0 \). Then the Hardy–Littlewood maximal operator \( M_X \) is bounded in \( L^{p(\cdot),\theta}(X) \).

Let \( k : X \times X \setminus \{(x, x) : x \in X\} \to \mathbb{R} \) be a measurable function satisfying the conditions:

\[
|k(x, y)| \leq \frac{c}{\mu B(x, d(x, y))}, \quad x, y \in X, \quad x \neq y;
\]

\[
|k(x_1, y) - k(x_2, y)| + |k(y, x_1) - k(y, x_2)| \leq c\omega \left( \frac{d(x_2, x_1)}{d(x_2, y)} \right) \frac{1}{\mu B(x_2, d(x_2, y))}
\]

for all \( x_1, x_2, y \) with \( d(x_2, y) > d(x, x_2) \), where \( \omega \) is a positive, non-decreasing function on \((0, \infty)\) satisfying \( \Delta_2 \) condition \( (\omega(2t) \leq c\omega(t), t > 0) \) and the Dini condition \( \int_0^1 \omega(t)/tdt < \infty \).

We also assume that for some \( p_0, 1 < p_0 < \infty \), and all \( f \in L^{p_0}(X) \) the limit

\[
(Kf)(x) = \text{p.v.} \int_X k(x, y)f(y)d\mu(y)
\]

exists almost everywhere on \( X \) and that \( K \) is bounded in \( L^{p_0}(X) \).

The following statement is known (see [24], [25]) (for Euclidean spaces see [7], [3]).

Theorem A. Let \( p \in \mathcal{P}(X) \cap \mathcal{P}_{\log}^{\text{loc}}(X) \). Then \( K \) is bounded in \( L^{p(\cdot)}(X) \).

Theorem 3. Let \( p \in \mathcal{P}(X) \cap \mathcal{P}_{\log}^{\text{loc}}(X) \) and let \( \theta > 0 \). Then there is a positive constant \( c \) depending only on \( p \) such that the following inequality

\[
\|Kf\|_{L^{p(\cdot),\theta}(X)} \leq c\|f\|_{L^{p(\cdot),p(X)}}, \quad f \in \mathcal{D}(X),
\]

holds, where the positive constant \( c \) does not depend on \( f \).

Regarding the space \( L^{p(\cdot),\theta}(X) \) we have the following statement:

Theorem 4. Let \( p \) satisfy the conditions of Theorem 2. Then the operator \( M_X \) is bounded in \( L^{p(\cdot),\theta}(X) \).

4. Some Applications

Let \( \Gamma \subset \mathbb{C} \) be a connected rectifiable curve and let \( \nu \) be arc-length measure on \( \Gamma \). By definition, \( \Gamma \) is regular if there is a positive constant \( c \) such that

\[
\nu(D(z, r) \cap \Gamma) \leq cr
\]

for every \( z \in \Gamma \) and all \( r > 0 \), where \( D(z, r) \) is a disc in \( \mathbb{C} \) with center \( z \) and radius \( r \). The reverse inequality

\[
\nu(D(z, r) \cap \Gamma) \geq r
\]
holds for all $z \in \Gamma$ and $r < L/2$, where $L$ is a diameter of $\Gamma$. If we equip $\Gamma$ with the measure $\nu$ and the Euclidean metric, the regular curve becomes an SHT.

The associate kernel in which we are interested is

$$k(z, w) = \frac{1}{z - w}.$$ 

The Cauchy integral

$$S_{\Gamma} f(t) = \int_{\Gamma} \frac{f(\tau)}{t - \tau} d\nu(\tau)$$

is the corresponding singular operator.

The above-mentioned kernel in the case of regular curves is a Calderón-Zygmund kernel. As was proved by G. David [4], a necessary and sufficient condition for continuity of the operator $S_{\Gamma}$ in $L^{r}(\Gamma)$, where $r$ is a constant ($1 < r < \infty$), is that $\Gamma$ is regular.

We denote by $M_{\Gamma}$ the Hardy–Littlewood maximal operator defined on $\Gamma$.

The above-formulated results yield the next statement:

**Proposition 1.** Let $\Gamma$ be a regular curve. Suppose that $p \in P(\Gamma) \cap P_{\text{log}}^{\text{loc}}(\Gamma)$. Assume that $L < \infty$. Then

(i) $M_{\Gamma}$ is bounded in $L^{p(\cdot), \theta}(\Gamma)$;

(ii) $M_{\Gamma}$ is bounded in $L^{p(\cdot), \theta}(\Gamma)$;

(iii) the operator $S_{\Gamma}$ is a bounded operator in $L^{p(\cdot), \theta}(\Gamma)$.

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**References**


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