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APPROXIMATION BY LINEAR SUMMABILITY MEANS IN WEIGHTED VARIABLE EXPONENT LEBESGUE SPACES

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Let $\mathbb{T}$ be the interval $[-\pi,\pi]$. Let $\mathcal{P}$ be the class of measurable functions $p:\boldsymbol{T}\to(1,\infty)$ such that $1<p_*:=\underset{x\in\boldsymbol{T}}{\operatorname{essinf}}\, p(x)\le p^*:=\underset{x\in\boldsymbol{T}}{\operatorname{esssup}}\, p(x)<\infty$. The conjugate exponent of $p(x)$ is defined as $p'(x):=p(x)/(p(x)-1)$. We define a class $L_{2\pi}^{p(\cdot)}$ of 2π-periodic measurable functions $f:\boldsymbol{T}\to\mathbb{R}$ satisfying the condition

$$\int_{\boldsymbol{T}}|f(x)|^{p(x)}\,dx<\infty$$

for $p\in\mathcal{P}$.

The class $L_{2\pi}^{p(\cdot)}$ is a Banach function space with the norm

$$\|f\|_{\boldsymbol{T},p(\cdot)}:=\inf\left\{\alpha>0:\int_{\boldsymbol{T}}\left|\frac{f(x)}{\alpha}\right|^{p(x)}dx\le 1\right\}.$$

A function $\omega:\boldsymbol{T}\to[0,\infty]$ will be called a weight if ω is measurable and almost everywhere positive. By $L_{\omega}^{p(\cdot)}$ we denote the class of Lebesgue measurable functions $f:\boldsymbol{T}\to\mathbb{R}$ for which $\omega f\in L_{2\pi}^{p(\cdot)}$. $L_{\omega}^{p(\cdot)}$ is called weighted Lebesgue spaces with variable exponent and is a Banach function space with the norm $\|f\|_{p(\cdot),\omega}:=\|\omega f\|_{\boldsymbol{T},p(\cdot)}$.

For given $p\in\mathcal{P}$ the class of weights ω satisfying the condition

$$\left\|\omega^{p(x)}\right\|_{A_{p(\cdot)}}:=\sup_{B\in\mathcal{B}}\frac{1}{|B|^{p_B}}\left\|\omega^{p(x)}\right\|_{L^1(B)}\left\|\frac{1}{\omega^{p(x)}}\right\|_{B,(p'(\cdot)/p(\cdot))}<\infty$$

will be denoted by $A_{p(\cdot)}$. Here $p_B:=\left(\frac{1}{|B|}\int\limits_B\frac{1}{p(x)}dx\right)^{-1}$ and $\mathcal{B}$ is the class of all intervals in $\boldsymbol{T}$.

The variable exponent $p(x)$ is said to be satisfy the *log-Hölder continuity condition* if there is a positive constant c such that

$$|p(x_1)-p(x_2)|\le\frac{c}{\log 1/|x_1-x_2|}\quad\text{for all } x_1,x_2\in\boldsymbol{T}. \tag{1}$$

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We will denote by $\mathcal{P}^{\log}$ the class of those $p \in \mathcal{P}$ satisfying (1).

We set $E_n(f)_{p(\cdot),\omega} := \inf\left\{\|f-T\|_{p(\cdot),\omega} : T \in \mathcal{T}_n\right\}$ for $f \in L_\omega^{p(\cdot)}$, where $\mathcal{T}_n$ is the class of trigonometric polynomials of degree not greater than n, $f \in L^p(\boldsymbol{T})$.

Let $\{\lambda_k^{(n)}\}$, $(k = 0,1,2,\ldots,n,n+1; k = 1,2,\ldots; \lambda_0^{(n)} = 1, \lambda_{n+1}^{(n)} = 0)$ be an arbitrary triangle matrix of numbers. For any function $f \in L_w^{L^{p(\cdot)}}(\mathcal{T})$. We consider a sequence of linear operators

$$U_n(f;x;\lambda) = \sum_{k=0}^{n} \lambda_k^{(n)} A_k(x),$$

where $A_0(x) = \frac{a_0}{2}$, $A_k(x) = a_k \cos kx + b_k \sin kx$ and a_k, b_k be the Fourier coefficients of function f.

Our aim is to estimate the norm deviation

$$R_n(f;\lambda)_{L_w^{p(\cdot)}} = \|f(x) - U_n(f;x;\lambda)\|_{L_w^{p(\cdot)}}$$

by the best approximation of function $f \in L_w^{p(\cdot)}$.

Theorem 1. *Let $\{\lambda_k^{(n)}\}$ be a nondecreasing sequence of numbers. Let us suppose, that $p \in \mathcal{P}^{\log}$, $\omega^{-p_0} \in A_{\left(\frac{p(\cdot)}{p_0}\right)'}$ for some $p_0 \in (1, p_*)$.*

Then the following estimate holds

$$R_n(f;\lambda)_{L_w^{p(\cdot)}} \leq c_{p(\cdot),w} \left\{ \sum_{\nu=0}^{m} \mu_{2^{\nu+1}}^{(n)\gamma} E_{2^\nu-1}^{\gamma}(f)_{L_w^{p(\cdot)}} \right\}^{1/\gamma},$$

where $\gamma := \min\{2, p_*\}$ and

$$\mu_\nu^{(n)} = 1 - \lambda_\nu^{(n)}, \quad (\nu = 0,1,2,\ldots,n,n+1).$$

From Theorem 1 we can deduce the following corollary's:

Corollary 1. *Let*

$$\lambda_k^{(n)} = 1 - \left(\frac{k}{n+1}\right)^r, \quad (k = 0,1,2,\ldots,n; r \geq 1)$$

be the Zygmund's means of summability. Then we have the following estimate

$$R_n(f;\lambda)_{L_w^{p(\cdot)}} \leq \frac{c_{p(\cdot),w}}{n^r} \left\{ \sum_{\nu=0}^{n} \nu^{\gamma r-1} E_{\nu-1}^{\gamma}(f)_{L_w^{p(\cdot)}} \right\}^{1/\gamma}$$

when $\gamma := \min\{2, p_*\}$.

Corollary 2. *For the Bernstein-Rogozinsky means*

$$\lambda_k^{(n)} = \cos\frac{k\pi}{2n+1}, \quad (k = 0,\ldots n)$$

we obtain the following inequality

$$R_n(f;\lambda)_{L_w^{p(\cdot)}} \le \frac{c_{p(\cdot),w}}{n^2}\left\{\sum_{\nu=0}^{n}\nu^{2\gamma-1}E_{\nu-1}^{\gamma}(f)_{L_w^{p(\cdot)}}\right\}^{1/\gamma},$$

where $\gamma := \min\{2, p_*\}$.

In particular, for the Fejer means when

$$E_n(f) \le \frac{c_1}{n}$$

we obtain the estimate

$$R_n(f;\lambda)_{L_w^{p(\cdot)}} \le \frac{1}{n}(\ln)^{1/\gamma}$$

where $\gamma := \min\{2, p_*\}$.

Moreover, we are able to estimate from below the norm of deviation by linear summability means in weighted variable exponent Lebesgue spaces. Namely, the following assertions are valid:

Theorem 2. *Let*

$$\lambda_k^{(n)} = 1 - \left(\frac{k}{n+1}\right)^r, \quad (k = 0, 1, 2, \ldots, n; r \ge 1)$$

be the Zygmund's means of summability. Then we have the following estimate

$$R_n(f;\lambda)_{L_w^{p(\cdot)}} \ge \frac{c'_{p(\cdot),w}}{n^r}\left\{\sum_{\nu=0}^{n}\nu^{\beta r-1}E_{\nu-1}^{\beta}(f)_{L_w^{p(\cdot)}}\right\}^{1/\beta}$$

when $\beta := \max\{2, p^*\}$.

Theorem 3. *For the Bernstein-Rogozinsky means*

$$\lambda_k^{(n)} = \cos\frac{k\pi}{2n+1}, \quad (k = 0, \ldots n)$$

we obtain the following inequality

$$R_n(f;\lambda)_{L_w^{p(\cdot)}} \ge \frac{c''_{p(\cdot),w}}{n^2}\left\{\sum_{\nu=0}^{n}\nu^{2\beta-1}E_{\nu-1}^{\beta}(f)_{L_w^{p(\cdot)}}\right\}^{1/\beta},$$

where $\beta := \max\{2, p^*\}$.

When $p(x)$ is a constant, $1 < p < \infty$ and weight $w(x) = 1$, for these estimates we refer the readers to the paper [3].

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