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ON THE NORM ESTIMATE OF DEVIATION BY LINEAR SUMMABILITY MEANS AND AN EXTENSION OF THE BERNSTEIN INEQUALITY

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We begin with some definitions. Let $\mathbb{T}$ be the interval $[-\pi, \pi]$. Let $\mathcal{P}$ be the class of Lebesgue measurable functions $p : \boldsymbol{T} \to (1, \infty)$ such that $1 < p_* := \operatorname*{essinf}_{x \in \boldsymbol{T}} p(x) \leq p^* := \operatorname*{esssup}_{x \in \boldsymbol{T}} p(x) < \infty$. The conjugate exponent of $p(x)$ is defined as $p'(x) := p(x) / (p(x) - 1)$. We define a class $L_{2\pi}^{p(\cdot)}$ of 2π periodic measurable functions $f : \boldsymbol{T} \to \mathbb{R}$ satisfying the condition

$$\int_{\boldsymbol{T}} |f(x)|^{p(x)} \, dx < \infty$$

for $p \in \mathcal{P}$.

The class $L_{2\pi}^{p(\cdot)}$ is a Banach space with the norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \alpha > 0 : \int_{\boldsymbol{T}} \left| \frac{f(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}.$$

A function $\omega : \boldsymbol{T} \to [0, \infty]$ will be called a weight if ω is measurable and almost everywhere positive. We will denote by $L_{\omega}^{p(\cdot)}$, the class of Lebesgue measurable functions $f : \boldsymbol{T} \to \mathbb{R}$ satisfying $\omega f \in L_{2\pi}^{p(\cdot)}$. $L_{\omega}^{p(\cdot)}$ is called weighted variable exponent Lebesgue space and is a Banach space with the norm $\|f\|_{p(\cdot),\omega} := \|\omega f\|_{p(\cdot)}$.

For given $p \in \mathcal{P}$ the class of weights ω satisfying the condition

$$\left\| \omega^{p(x)} \right\|_{A_{p(\cdot)}} := \sup_{B \in \mathcal{B}} \frac{1}{|B|^{p_B}} \left\| \omega^{p(x)} \right\|_{L^1(B)} \left\| \frac{1}{\omega^{p(x)}} \right\|_{B,(p'(\cdot)/p(\cdot))} < \infty$$

will be denoted by $A_{p(\cdot)}$. Here $p_B := \left(\frac{1}{|B|} \int\limits_B \frac{1}{p(x)} dx \right)^{-1}$ and $\mathcal{B}$ is the class of all intervals in $\boldsymbol{T}$.

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The variable exponent $p(x)$ is said to be satisfy local *log-Hölder continuity condition* if there is a positive constant c such that

$$|p(x_1) - p(x_2)| \leq \frac{c}{\log 1/|x_1 - x_2|} \quad \text{for all } x_1, x_2 \in \boldsymbol{T}. \tag{1}$$

We will denote by $\mathcal{P}^{\log}$ the class of those $p \in \mathcal{P}$ satisfying (1). For $f \in L_w^{p(\cdot)}(\mathbb{T})$ now we can define the generalized moduli of smoothness for $p \in \mathcal{P}^{\log}$, $\omega \in A_{p(\cdot)}$ and $f \in L_\omega^{p(\cdot)}$ as

$$\Omega(f, \delta)_{p(\cdot),\omega} := \sup_{0<h\leq\delta} \|(I - \mathcal{A}_h) f\|_{p(\cdot),\omega}, \quad \delta \geq 0.$$

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

be the Fourier series of the function $f \in L^1(\mathbb{T})$. Let $\sigma_n^\alpha(\cdot, f)$ $(\alpha > 0)$ be the Cesaro means of the series, that is

$$\sigma_n^\alpha(x, f) = \frac{1}{\pi} \int\limits_{\mathbb{T}} f(x+t) K_n^\alpha(t) dt,$$

where

$$K_n^\alpha(t) = \sum_{k=0}^{n} \frac{A_{n-k}^{\alpha-1} D_k(t)}{A_n^\alpha}$$

is the Fejér kernel and

$$D_k(t) = \frac{\sin\left(k + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}}$$

is the Dirichlet kernel, with

$$A_n^\alpha = \begin{pmatrix} n+\alpha \\ \alpha \end{pmatrix} \approx \frac{n^\alpha}{\Gamma(\alpha+1)}.$$

Let also $U_r(\cdot, f)$ $(0 \leq r < 1)$ be the Abel-Poisson means of the function f, that is

$$U_r(x, f) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} P_r(x-t) f(t) dt,$$

where

$$P_r(t) = \frac{1 - r^2}{1 - 2r \cos t + r^2}$$

is the Poisson kernel.

Theorem 1. *Let us suppose that* $p \in \mathcal{P}^{\log}$, $\omega^{-p_0} \in A_{\left(\frac{p(\cdot)}{p_0}\right)'}$ for some $p_0 \in (1, p_*)$. *Then the following estimates hold:*

$$\|\sigma_n^\alpha(\cdot, f) - f\|_{p(\cdot),\omega} \leq cn\Omega\left(\frac{1}{n}, f\right)_{p(\cdot),\omega}$$

and

$$\|U_r(\cdot,f)-f\| \le \frac{c}{1-r}\Omega\left(f,1-r\right)_{p(\cdot),\omega},$$

where a constant c does not depend on n, r and f.

Theorem 2. *Let us suppose that $p \in \mathcal{P}^{\log}$, $\omega \in A_{(p(\cdot))'}$ then for arbitrary trigonometric polynomial $t_n(x)$ the Bernstein type inequality holds*

$$\|t_n'\|_{p(\cdot),\omega} \le cn\|t_n\|_{p(\cdot),\omega},$$

where a constants c does not depend on n and t_n.

When $p(x)$ is a constant and the weight w belongs to the Muckenhoupt A_p class, for the estimates presented above we refer the readers to [1].

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References

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