V. Kokilashvili

BOUNDEDNESS CRITERION FOR THE CAUCHY
SINGULAR INTEGRAL OPERATOR IN WEIGHTED
GRAND LEBESGUE SPACES AND APPLICATION TO THE
RIEMANN PROBLEM

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We present the necessary and sufficient condition for the Cauchy singular
integral and Hardy-Littlewood maximal function defined on the Carleson
curves to be bounded in weighted grand Lebesgue spaces. Then we solve
the Riemann boundary value problem for analytic functions in the class of
the Cauchy-type integrals with the density from the grand Lebesgue \( L^p(\Gamma) \)
spaces. We consider the case when a coefficient in boundary condition is
everywhere nonvanishing continuous function and the right side function
belongs to the same \( L^p \) space. The solvability conditions are established
and the explicit formulas for solutions are given.

Let \( \Gamma = \{ t \in \mathbb{C} : t = t(s), \ 0 \leq s \leq l > \infty \} \) be a simple rectifiable curve
with a arc-length measure \( \nu \). In the sequel we use the notation:
\[
D(t, r) := \Gamma \cap B(t, r), \quad r > 0
\]
where \( B(t, r) = \{ z \in \mathbb{C} : |z - t| < r \} \).

We recall that a rectifiable curve \( \Gamma \) is called Carleson curve (regular curve)
if there exists a constant \( c_0 > 0 \) not depending on \( t \) and \( r \) such that
\[
\nu D(t, r) \leq c_0 r.
\]

The weighted grand Lebesgue space \( L^p_w(\Gamma) \) \( 1 < p < \infty \) is a Banach
function space defined by the norm
\[
\| f \|_{L^p_w(\Gamma)} = \sup_{\epsilon > 0, \epsilon < p - 1} \left( \frac{\epsilon}{\nu t} \int_{\Gamma} |f(t)|^{p-\epsilon} w(t) d\nu \right)^{\frac{1}{p-\epsilon}} \quad (1)
\]

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integral.
where \( w \) is an almost everywhere positive integrable function on \( \Gamma \) (i.e. weight).

It is worth mentioning that the following continuous embeddings hold:

\[
L^p_w(\Gamma) \subset L^p(\Gamma) \subset L^{p- \varepsilon}_w(\Gamma).
\]

The grand Lebesgue space \( L^p(\Gamma) \) was introduced by T. Iwaniec and C. Sbordone \[6\].

Our goal is to give a complete characterization of that weight functions \( w \) which govern one-weighted norm inequalities in grand Lebesgue spaces for the following two operators: the Cauchy singular integral

\[
(S_T f)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{f(\tau)}{\tau - t} \, d\tau
\]

and the Hardy-Littlewood maximal function

\[
(M_T f)(t) = \sup_{r>0} \frac{1}{r} \int_{D(t, r)} |f(\tau)| \, d\nu
\]

defined on the Carleson curves.

This problem in classical Lebesgue spaces for the Hardy-Littlewood function and Hilbert transform defined on the real line was solved in papers \[12\] and \[5\] respectively. Recently, for the same integral transforms defined on a finite interval, the solution of one-weighted problem in the grand Lebesgue spaces was done in \[3\] and \[10\]. G. David’s well-known theorem states that for the boundedness of \( S_T \) in \( L^p(\Gamma) \) it is necessary and sufficient that \( \Gamma \) should be the Carleson curve \[2\].

**Theorem 1.** Let \( 1 < p < \infty \) and let \( \Gamma \) be the Carleson curve of finite length. Then the following conditions are equivalent:

\begin{align*}
\text{(i)} & \quad S_T \text{ is bounded in } L^p_w(\Gamma); \\
\text{(ii)} & \quad M_T \text{ is bounded in } L^p_w(\Gamma); \\
\text{(iii)} & \quad \sup_{D(z, r)} \frac{1}{r} \int w(\tau) \, d\nu \left( \frac{1}{r} \int_{D(z, r)} w^{1 - p'}(\tau) \, d\nu \right)^{p-1} < \infty
\end{align*}

where the supremum is taken over all \( z \in \Gamma \) and \( r, 0 < r < \text{diam} \Gamma \).

In the case of classical Lebesgue spaces for the equivalence of the boundedness of the operator \( S_T \) in \( L^p_w(\Gamma) \) and (4) we refer to \[1\] and \[7\].

For the real line condition (4) coincides with the well-known B. Muckenhoupt’s \( A_p \) condition.

In the sequel both of operators \( S_T \) and \( M_T \) we denote by \( T_T \).

From Theorem 1 we deduce the following
Corollary. Let $1 < p < \infty$. Operator $T_{\Gamma}$ is bounded in $L^p(\Gamma)$ if and only if condition (4) is satisfied i.e. $\Gamma$ is the Carleson curve.

Note that the following vector-valued analogy of Theorem 1 holds.

Let $f = (f_1, f_2, \ldots, f_n, \ldots)$ be a vector-valued function when $f_k (k = 1, 2, \ldots)$ are measurable functions defined on $\Gamma$.

Theorem 2. Let $1 < p, \theta < \infty$ and let $\Gamma$ be the Carleson curve of finite length. Then the inequality

$$\left\| \left( \sum_{j=1}^{\infty} |T_{\Gamma} f_j(t)|^\theta \right)^{1/\theta} \right\|_{L^p(\Gamma)} \leq c \left\| \left( \sum_{j=1}^{\infty} |f_j(t)|^\theta \right)^{1/\theta} \right\|_{L^p(\Gamma)}$$

holds with a constant $c$ independent of $f$ if and only if condition (4) is fulfilled.

Definition. Let $D$ be a simply connected bounded domain bounded by a rectifiable curve $\Gamma$. By $E^p(D)$ $1 \leq p < \infty$ we denote a set of all analytic functions $\Phi$ in $D$ for which there exists a sequence of closed curves $\Gamma_{r} \subset D$ converging to $\Gamma$ such that

$$\sup_n \|\Phi\|_{L^p(\Gamma_{r})} < \infty.$$  

Theorem 3. Let $1 < p < \infty$ and $\Gamma$ be a Carleson curve. For arbitrary $\Phi \in E^p(D)$ the representation by the Cauchy type integral

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - z} d\tau$$

holds with $\varphi \in L^p(\Gamma)$.

Note that the class $E^p$ is an analogy of the Smirnov class of analytic functions.

Let $\Gamma$ be an oriented rectifiable simple closed curve in the complex plane. We denote by $D^+$ and $D^-$ the bounded and unbounded component of $\mathbb{C} \setminus \Gamma$, respectively.

One of the goal of this talk is to investigate the Riemann problem: find an analytic function $\Phi$ on the complex plane cut along $\Gamma$ whose boundary values satisfy the conjugate condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in \Gamma,$$  

when $G$ and $g$ are given functions on $\Gamma$, and $\Phi^+$ and $\Phi^-$ are boundary values of $\Phi$ on $\Gamma$ from inside and outside $\Gamma$, respectively. This problem is also known as the problem of linear conjugation.

Problem (5) comes from Riemann [14]. Important results on which the posterior solution of problem (4) was based, were obtained by Yu. Sokhotski, D. Hilbert, I. Plemely and T. Carleman. A complete solutions of the
Riemann problem in the frame of Hölder continuous functions was given in the papers of Gakhov [4] and N. Muskhelishvili [13]. We refer also to the works [7–9], [11], [15] for investigation of the Riemann problem in classical $L^p$ spaces.

Let

$$K^p(\Gamma) = \left\{ \Phi(z) : \Phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\tau)d\tau}{\tau - z}, \ z \notin \Gamma \text{ with } \varphi \in L^p(\Gamma) \right\}. \quad (6)$$

We proceed the solution of the Riemann problem in the following setting: let $\Gamma$ be Carleson curve. Let $G$ be a continuous function on $\Gamma$ with the condition $G(t) \neq 0, \ t \in \Gamma$. Let $\omega = \frac{1}{2\pi} \text{arg}G(t)|_{\Gamma}$. Find an analytic function $\Phi \in K^p(\Gamma), \ 1 < p < \infty$, satisfying the condition (5), where $g \in L^p(\Gamma)$.

**Theorem.** The following statements hold:

(i) for $\omega \geq 0$, problem (5) is unconditionally solvable in the class $K^p(\Gamma)$ and all its solutions are given by

$$\Phi(z) = \frac{X(z)}{2\pi i} \int_\Gamma \frac{g(\tau)}{X^+(\tau)(\tau - z)}d\tau + X(z)Q_{\omega-1}(z) \quad (7)$$

with

$$X(z) = \begin{cases} \exp h(z), & z \in D^+ \\ (z - z_0)^{-\omega} \exp h(z), & z \in D^-, \ z_0 \in D^+, \end{cases} \quad (8)$$

$$h(z) = \frac{1}{2\pi i} \int_\Gamma \frac{\ln|\{t - z_0\}^{-\omega}G(t)|}{t - z}dt, \quad (9)$$

where $Q_{\omega-1}$ is an arbitrary polynomial of degree $\omega - 1$ ($Q_{-1}(z) = 0$);

(ii) for $\omega < 0$, problem (5) is solvable in the class $K^p(\Gamma)$ if and only if

$$\int_\Gamma \frac{g(t)t^k}{X^+(t)}dt = 0, \ k = 0, 1, \ldots, |\omega| - 1; \quad (10)$$

and under these conditions problem (5) has the unique solution given by (7) with $Q_{\omega-1} = 0$.

In forthcoming talks we plan to discuss the Riemann problem in the case of oscillating coefficients $G$ and approximation problems in grand Lebesgue spaces with weight.

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Author’s address:
A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia
E-mail: kokil@rmi.acnet.ge