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ON BELLMAN'S TRANSFORM OF THE TRIGONOMETRIC FOURIER SERIES IN THE WEIGHTED LEBESGUE SPACES WITH A VARIABLE EXPONENT

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In the present article we present the results concerning the weighted estimate for Bellman's transform of trigonometric Fourier series in the Lebesgue spaces with a variable exponent.

For the classical Lebesgue spaces, an analogous problem has been considered in [1], [2], [3], [4] and [5].

Let $T = (-\pi, \pi)$. Below we will consider a measurable function $p : T \to R^1$, such that

$$0 < p_- := \operatorname*{ess\,inf}_T p(x) \le p(x) \le p_+ := \operatorname*{ess\,sup}_T p(x) < \infty. \tag{1}$$

By $L^{p(\cdot)}(T)$ we denote a set of all measurable functions on T for which

$$I_p(f) = \int\limits_T |f(x)|^{p(x)} dx < \infty.$$

When $p(x)$ is a measurable bounded function with values in $[1, \infty)$ then $L^{p(\cdot)}(T)$ is the Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf \Big\{ \lambda > 0 : I_p\Big(\frac{f}{\lambda}\Big) \le 1 \Big\}.$$

Definition. By $\mathcal{P}_0(T)$ is denoted a set of functions satisfying the condition (1) for which are fulfilled the following conditions:

(i) $p(0) = \lim\limits_{x \to 0} p(x)$:

(ii) $|p(x) - p(0)| \le \frac{c}{\ln |x|}$ for $|x| \le \frac{1}{2}$.

Let

$$b_n = \int\limits_{-\pi}^{\pi} f(x) \sin nx dx \tag{2}$$

be the Fourier coefficients of 2π-periodic summable function. Using the sequence $\{b_n\}$, we construct a new sequence

$$B_k = \sum_{k=n+1}^{\infty} \frac{b_k}{k} + \frac{b_n}{2n}, \quad n \ge 1.$$

The following theorem is valid.

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Theorem 1. *Let the functions* $p : T \to R^1$ *and* $\beta : T \to R^1$ *belong to the set* $\mathcal{P}_0(t)$, $p_- > 1$ *and*

$$0 \le \beta(0) < \frac{1}{p(0)}.$$

Further, we assume that the function $q : T \to R^1$ belongs to the class $\mathcal{P}_0(t)$ with the condition

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \beta(0).$$

Let (2) be the Fourier coefficients of the function $f \in L^{p(\cdot)}$. Then the trigonometric series

$$\sum_{k=1}^{\infty} B_k \sin kx$$

is the Fourier series of the function F for which the inequality

$$\|t^{\beta(t)} F(t)\|_{L^{q(\cdot)}(T)} \le c\|f\|_{L^{p(\cdot)}(T)}$$

holds.

Theorem 2. *Let the functions* $p : T \to R^1$ *and* $\beta : T \to R^1$ *belong to the set* $\mathcal{P}_0(t)$, $p_- > 1$ *and*

$$0 \le \beta(0) < \frac{1}{p(0)}.$$

Further, we assume that the function $q : T \to R^1$ belongs to the class $\mathcal{P}_0(t)$, $q_- > 1$ with the condition

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \beta(0).$$

Let (2) be the Fourier coefficients of the function $f \in L^{p(\cdot)}$. Then the trigonometric series

$$\sum_{k=1}^{\infty} A_k \sin kx,$$

where

$$A_k = \frac{1}{k}\left(\sum_{j=1}^{k-1} b_j + \frac{b_k}{2}\right),$$

is the Fourier series of the function Φ for which the inequality

$$\|t^{\beta(t)} \Phi(t)\|_{L^{q(\cdot)}(T)} \le c\|f\|_{L^{p(\cdot)}(T)}$$

holds.

The analogous statements are true for cosine Fourier series.

The proofs of presented theorems are is based on the result obtained in [6].

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