Proc. A. Razmadze Math. Inst. 143(2007), 140–142

Ts. Tsanava

THE TWO-WEIGHTED INEQUALITIES FOR GENERALIZED MARCINKIEWICZ INTEGRALS IN WEIGHTED LEBESQUE SPACES WITH VARIABLE EXPONENT

(Reported on 07.03.2007)

Let P be a closed set of the space R^n and $\delta(y)$ be a distance from the point y to the set P. J. Marcinkiewicz was the first who studied the following integral transforms [1]:

$$Jf(x) = \int\limits_P \frac{(\delta(y))^\lambda}{|x-y|^{n+\lambda}} f(y)\,dy, \quad \lambda > 0 \tag{1.1}$$

and

$$Jf(x) = \int\limits_{\{y:\delta(y)\le\delta_0<1\}} \frac{(\log\frac{1}{\delta(y)})^{-1}}{|x-y|^n} f(y)\,dy. \tag{1.2}$$

These integrals are of importance in the theory of Fourier series. Modification of the above integrals have been considered by L. Carleson [2] and A. Zygmund [3]. For example, A. Zygmund studied the following integral transformations:

$$(J^*f)(x) = \int\limits_{R^n} \frac{[\delta(y)]^\lambda}{(|x-y|+\delta(y))^{n+\lambda}} f(y)\,dy, \tag{1.3}$$

$$(J^*f)(x) = \int\limits_{\{\delta(y)\le\delta_0<1\}} \frac{[\lg\frac{1}{\delta(y)}]^{-1}}{(|x-y|+\delta(y))^n} f(y)\,dy. \tag{1.4}$$

It is evident that if $x \in P$, then $(J^*f)(x)$ and $(J^*f)(x) = (Jf)(x)$. In the theory of singular and hypersingular integrals the most important turned out to be the integrals written in the form (1.3) and (1.4). In this section we present the two-weight inequality for generalized Marcinkiewicz integrals in weighted Lebesque spaces with variable exponent. This generalization has been considered by Calderon [4] who proved the one-weighted inequality for the Muckenhoupt A_p classes in Lebesque spaces with constant exponent.

Let on the set $(0,\infty)\times[0,\infty)$ be defined the nonnegative function $\varphi(\rho,t)$ satisfying the following conditions:

(1) for every fixed t, $t\in[0,\infty)$ the function $(\rho+t)^{-n}\varphi(\rho,t)$ is nonincreasing with respect to ρ, and

$$\lim_{\rho\to\infty}(\rho+t)^{-n}\varphi(\rho,t) = 0;$$

(2) there exists the positive constant c such that

$$\int\limits_0^\infty \rho^{n-1}(\rho+t)^{-n}\varphi(\rho,t)\,d\rho \le c,$$

for every nonnegative t.

2000 *Mathematics Subject Classification:* 42B205, 47B38.

Key words and phrases. Marcinkiewicz integrals, two-weighted inequalities.

Let $\psi(y) \geq 0$ be a measurable function such that the function $\varphi(|x–y|, \psi(y))$ is measurable. Consider the integral

$$Kf(x) = \int\limits_{R^n} \frac{\varphi(|x-y|, \psi(y))}{(|x-y| + \psi(y))^n} \varphi\big(|x-y|, \psi(y)\big) f(y)\, dy. \tag{1.5}$$

Obviously, when

$$\varphi(\rho, t) = \frac{t^2}{(\rho + t)^2}, \quad \psi(y) = \delta(y),$$

we obtain $Kf(x) = J^* f(x)$.

let q be a measurable function on $\mathbb{R}^n$ such that $1 \leq q(x) \leq \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} q(x) < \infty$. Suppose that ρ is a weight function on $\mathbb{R}^n$, i.e. ρ is a non-negative, almost everywhere positive function on $\mathbb{R}^n$. By $L^{q(\cdot)}(\mathbb{R}^n)$ we denote the space of all measurable functions f on $\mathbb{R}^n$ such that

$$S_{q(\cdot)}(f) := \int\limits_{\mathbb{R}^n} |f(x)|^{q(x)} dx < \infty.$$

This is a Banach space with respect to the norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} := \inf \left\{ \lambda > 0 : S_{q(\cdot)}\left(f/\lambda\right) \leq 1 \right\}.$$

The weighted variable Lebesgue space $L^{q(\cdot)_\rho(\mathbb{R}^n)}$ is defined by the norm

$$\|f\|_{L^{q(\cdot)}_\rho} = \|f\rho\|_{L^{q(\cdot)}}.$$

For the variable Lebesgue spaces we refer to [6], [7].

In the sequel we will use the following notation:

$$p_-(E) := \operatorname*{ess\,inf}_{x \in E} p(x); \quad p_+(E) := \operatorname*{ess\,sup}_{x \in E} p(x),$$

where $p(\cdot)$ is a measurable function on $\mathbb{R}^n$ and E is a measurable set in $\mathbb{R}^n$.

Further, we denote:

$$p_0(x) := p_-(\{y : |y| < |x|\}), \ \ \widetilde{p}_0(x) := \begin{cases} p_0(x) & \text{if } |x| \leq 1 \\ p_c = \text{const} & \text{if } |x| > 1, \end{cases}$$

and $\widetilde{p}_0(x) = \dfrac{\widetilde{p}_0(x)}{\widetilde{p}_0(x) - 1}$

In the sequel we assume that K and M are the Calderón-Zygmund and the Hardy-Littlewood maximal operator respectively defined on $\mathbb{R}^n$.

Definition. We say that a function $p(\cdot)$ satisfies the Dini-Lipschitz condition on $\mathbb{R}^n$ ($p(\cdot) \in DL(\mathbb{R}^n)$) if

$$|p(x) - p(y)| \leq \frac{A}{\ln \frac{1}{|x-y|}}; \ \ 0 < |x-y| \leq 1/2; \ \ x, y \in \mathbb{R}^n.$$

Based on two-weighted estimate for maximal function, obtained in [5] we come to the following

Theorem. *Let* $1 < p_- \leq p(x) \leq p_+ < \infty$. *Suppose that* $p(\cdot) \in DL(\mathbb{R}^n)$ *and* $p(x) \equiv p_c \equiv$ const *for* $|x| > 1$. *Suppose that* v *and* w *are functions increasing on* $\mathbb{R}_+$ *and satisfying the condition*

$$B := \sup_{t>0} \Bigg(\int\limits_{|x|>t} (v(|x|)/|x|^n)^{p(x)} \Bigg(\int\limits_{|y|<t} w^{-(\widetilde{p}_0)'(x)}(|y|) dy \Bigg)^{\frac{p(x)}{(\widetilde{p}_0)'(x)}} dx \Bigg) < \infty.$$

Then there exists a positive constant c such that

$$\|(Kf)(\cdot)v(|\cdot|)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c\|f(\cdot)w(|\cdot|)\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

holds.

References

1. J. Marcinkiewicz, Sur quelques integrales du type de Dini. *Ann. Soc. Polon. Math.* **17**(1938), 42–50.
2. L. Carleson, On convergence and growth of partial sums of Fourier series. *Acta Math.* **116**(1966), 135–157.
3. A. Zygmund, On certain lemmas of Marcinkiewicz and Carleson. *J. Approximation Theory* **2**(1969), 249–257.
4. A.-P. Calderón, On an integral of Marcinkiewicz. *Studia Math.* **57**(1976), No. 3, 279–284.
5. D. E. Edmunds, V. Kokilashvili and A. Meskhi, Two-weight estimates in $L^{p(x)}$ spaces for classical integral operators and applications to the norm summability of Fourier trigonometric series. *Proc. A. Razmadze Math. Inst.* **142**(2006), 123–128.
6. S. Samko, On a progress in the theory of Lebesque spaces with variable exponent: maximal and singular operators. *Integral Transforms Spec. Funct.*, **16**(2005), No. 5-6, 461–482.
7. V. Kokilashvili, On a progress in the theory of integral operators in weighted Banach function spaces. In *"Function Spaces, Differential Operators and Nonlinear Analysis", Proceedings of the Conference held in Milovy, Bohemial-Moravian Uplands* , May 28-June 2, 2004. *Math. Inst. Acad. Sci. Czech Republic, Prague.*

Acknowledgement

This work was supported by the Grant GNSF/ST 06/3-010 and INTAS Grant No. 05-1000008-8157.

Author's Address:

Georgian Technical University
Department of Mathematics
77, M. Kostava St. Tbilisi 0175
Georgia