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ON THE CONVERGENCE AND SUMMABILITY OF FOURIER SERIES IN WEIGHTED LEBESGUE SPACES WITH A VARIABLE EXPONENT

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In the sequel, the interval (π, π) will be denoted by T. Let a 2π-periodic function p satisfy the conditions

$$1 < p_- := \operatorname*{essinf}_{x\in T} p(x) \le p(x) \le \operatorname*{esssup}_{x\in T} p(x) =: p_+ < \infty \tag{1}$$

and

$$\big|p(x) - p(y)\big| \le \frac{A}{\ln \frac{1}{|x-y|}}\,, \quad x, y \in T, \quad |x-y| \le \frac{1}{2}\,, \tag{2}$$

and let the function $\beta(x)$ satisfy the condition

$$\big|\beta(x) - \beta(x_0)\big| \le \frac{A}{\ln \frac{1}{|x-x_0|}}\,, \quad x_0 \in T, \quad |x-x_0| \le \frac{1}{2}\,. \tag{3}$$

Almost everywhere, a finite nonnegative function w will be called a weight.

The weighted Lebesgue space with a variable exponent is defined through the modular

$$I_T^p(fw) := \int\limits_T \big|f(x)w(x)\big|^{p(x)} dx$$

by means of the norm

$$\big\|f\big\|_{p(\cdot),w} = \inf\Big\{\lambda > 0 : I_T^p\Big(\frac{fw}{\lambda}\Big) \le 1\Big\}. \tag{4}$$

By $L_w^{p(\cdot)}(T)$ we denote the weighted Banach space of all 2π-periodic functions for which (4) is finite.

In what follows, we will consider weights of power-exponential $w(x) = |x - x_0|^{\beta(x)}$, where x_0 is an arbitrary point on the interval $(-\pi, \pi)$.

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{5}$$

be the trigonometric Fourier series of the function f.

In the sequel, by $S_n f$ we will denote a partial sum of the Fourier series of the function $f(x)$,

$$S_n f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

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We will also consider the Cesaro α-means and the Abelian-Poisson means of the series (5):

$$\sigma_n^\alpha f(x) = \sum_{k=0}^{n} \frac{A_{n-k}^{n-1}}{A_n^\alpha} S_k f(x), \quad A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}$$

and

$$u_f(r,x) = \frac{1}{2}\, a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) r^k, \quad 0 < r < 1.$$

In [13], we investigated the mean convergence of trigonometric series by using the method due to Cesaro in weighted Lebesgue spaces with a constant exponent p.

Our investigation will be based on the criteria of boundedness of maximal functions and singular integrals in the variable Lebesgue spaces with weight (see [3] and [4]). As for the Lebesgue spaces with a variable exponent and for integral operators, we refer the reader to [6] and [7].

The given lemma allows one to prove the theorems below.

Lemma A (see [8]). *Let $p(x)$ satisfy the conditions* (1) *and* (2), *and the function $\beta(x)$ satisfy the condition* (3). *Then for $x_0 \in T$ the norm*

$$\big\| |\cdot - x_0|^{\beta(\cdot)} \big\|_{p(\cdot)}$$

is finite if and only if

$$-\frac{1}{p(x_0)} < \beta(x_0) < \frac{1}{p'(x_0)}\,.$$

Theorem 1. *Let $p : T \to R$ satisfy the conditions* (1) *and* (2), *and the function $\beta(x)$ satisfy the condition* (3). *Then the following statements are equivalent:*

(i) $\lim\limits_{n\to\infty} \big\| S_n f(\cdot) - f \big\|_{p(\cdot),w} = 0$ *for any $f \in L_w^{p(\cdot)}(T)$;*

(ii) $-\dfrac{1}{p(x_0)} < \beta(x_0) < \dfrac{1}{p'(x_0)}\,.$

Theorem 2. *Let $p : T \to R$ satisfy the conditions* (1) *and* (2), *and the function β satisfy the condition* (3). *Assume $w(x) = |x - x_0|^{\beta(x)}$. Then the following statements are equivalent:*

(i) $\lim\limits_{n\to\infty} \big\| \sigma_n^\alpha(f,\cdot) - f \big\|_{p,w} = 0$;

(ii) $\lim\limits_{r\to 1} \big\| u(r,\cdot) - f \big\|_{p,w} = 0$;

(iii) $-\dfrac{1}{p(x_0)} < \beta(x_0) < \dfrac{1}{p'(x_0)}\,.$

The problem of convergence and summability of conjugate Fourier series is also studied.

Let

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx) \tag{6}$$

be the series, conjugate to the series (5). As above, we assume that $w(x) = |x - x_0|^{\beta(x)}$, $x_0 \in T$.

Theorem 3. *Let the function $p(x)$ satisfy the condition* (3). *Then the partial sums of the conjugate trigonometric Fourier series are mean convergent to the function $\widetilde{f}(x)$ in the space $L_w^{p(\cdot)}$ if* $-\dfrac{1}{p(x_0)} < \beta(x_0) < \dfrac{1}{p'(x_0)}\,.$

Theorem 4. *Under the conditions of Theorem* 3*, for the functions* $p(x)$ *and* $\beta(x)$ *that the trigonometric Fourier series of the function* $f \in L_w^{p(\cdot)}(T)$ *is the Fourier series of the function* $\widetilde{f}(x)$*, and*

$$\lim_{n\to\infty} \big\|\widetilde{\sigma}\,{}_n^{\alpha}(f)(x) - \widetilde{f}\big\|_{p,w} = 0.$$

An analogous theorem holds for the Abel-Poisson means.

Theorem 5. *Let the functions* p *and* β *satisfy the conditions of Theorem* 3*. Then for any* $f \in L_w^{p(\cdot)}(T)$ *the Abel-Poisson means of the series* (6) *are mean convergent in* $L_w^{p(\cdot)}(T)$ *to the function* $\widetilde{f}(x)$*.*

The above theorems are likewise valid for the weights

$$w(x) = \prod_{k=1}^{n} |x - x_k|^{\beta_k} \tag{7}$$

where $x_1, x_2, \dots, x_n$ are different points on T, and

$$-\frac{1}{p(x_k)} < \beta_k < \frac{1}{p'(x_k)}$$

for all $k = 1, 2, \dots, n$.

Let

$$u_f(x,t) = \int\limits_{R^n} \frac{t}{t^2 + (x-y)^2}\, f(y)\, dy \tag{8}$$

be the Poisson integral in the upper half-plane. As is known, $u_f(x,t)$ is the harmonic function in the upper half-plane.

Let $p : R^1 \to R^1$ be the measurable function for which the following conditions are fulfilled:

(1) $1 < p_- := \operatorname*{essinf}_{x\in R^1} p(x) \le p(x) \le \overline{p} = \operatorname*{esssup}_{x\in R^1} p(x) < \infty$;

(2) there exists the constant $A > 0$, such that

$$\big|p(x) - p(y)\big| \le \frac{A}{\ln \frac{1}{|x-y|}}\,, \quad |x - y| < \frac{1}{2}\,;$$

(3) there exists the interval $(-R, R)$, such that

$$p(x) = p_\infty = \text{const} \quad \text{for} \quad |x| > R.$$

Assume that the weight function is of the form

$$w(x) = |x|^{\beta} \prod_{k=1}^{n} |x - x_k|^{\beta_k} \tag{9}$$

under the conditions

$$-\frac{1}{p(x_k)} < \beta_k < \frac{1}{p'(x_k)}$$

and

$$-\frac{1}{p_\infty} < \beta + \sum_{k=1}^{n} \beta_k < \frac{1}{p'_\infty}\,.$$

Let us now solve the following Dirichlet problem. Let $f \in L_w^{p(\cdot)}(R^1)$. Find in the upper half-plane a harmonic function possessing in the capacity of nontangential boundary values on R^1 almost everywhere the function $f \in L_w^{p(\cdot)}(R^1)$, such that

$$\lim_{t\to 0} \big\|u(\cdot\,,t) - f\big\|_{L_w^{p(\cdot)}} = 0.$$

The solution of that problem is given by the Poisson integral in the upper half-plane given by formula (8).

References

1. I. Sharapudinov, On the uniform boundedness in L^p $(p = p(x))$ of some families of convolution operators. (Russian) *Mat. Zametki* **59**(1996), No. 2, 291–302.
2. A. P. Nachman and B. P. Osilenker, The estimates of weighted norms of some operators generated by multiple trigonometric Fourier series. (Russian) *Izv. Vyshikh Uchebn. Zavedenii* **239**(1982), No. 4, 39–50.
3. V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Proc. A. Razmadze Math. Inst.* **129**(2002), 145–149.
4. V. Kokilashvili and S. Samko, Singular integrals in weighted Lebesgue spaces with variable exponent. *Georgian Math. J.* **10**(2003), No. 1, 145–156.
5. V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Rev. Mat. Iberoamericana* **20**(2004), No. 2, 493–515.
6. L. Diening, P. Hästö and A. Nekvinda, Open problems in variable exponent Lebesgue and Sobolev spaces. Function Spaces, Diff. Operators and Nonlinear Analysis. *Proc. of the Conference held in Milovy, Bohemian-Moravian Uplands*, 2004, 38–58. *Math. Inst. of Sci. of the Czech Republic, Praha*, 2005.
7. V. Kokilashvili, On a Progress in the Theory of Integral Operators in Weighted Banach Function Spaces. Function Spaces, Diff. Operators and Nonlinear Analysis. *Proc. of the Conference held in Milovy, Bohemian-Moravian Uplands*, 2004, 152–175.
8. S. Samko, Convolution type operators in $L^{p(x)}$. *Integral Transform. Spec. Funct.* **7**(1998), No. 1–2, 123–144.
9. A. Zygmund, Trigonometric series. 2nd ed. Vols. I, II. *Cambridge University Press, New York* Vol. 1, 1959.
10. M. Khabazi, Maximal functions in $L^{p(x)}$ spaces. *Proc. A. Razmadze Math. Inst.* **135**(2004), 145–146.
11. V. Kokilashvili, On a progress in the theory of integral operators in weighted Banach function spaces. Function Spaces. Diff. Operators and Nonlinear Analysis. *Proc. of the Conference held in Milovy, Bohemian-Moravian Uplands*, 2004, 152–175. *Math. Inst. of Sci. of the Czech Republic, Praha*, 2005.
12. V. Kokilashvili and S. Samko, Weighted boundedness in Lebesgue spaces with variable exponents of classical operators on Carleson curves. *Proc. A. Razmadze Math. Inst.* **138**(2005), 106–110.
13. Ts. Tsanava, On the summability of trigonometric Fourier series in weighted Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* **138**(2005), 119–121.

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