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ON THE BASISNESS OF SYSTEMS OF EXPONENTS WITH DEGENERATE COEFFICIENTS IN WEIGHTED SUBSPACES

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Investigation of the problem on eigen values of some discontinuous differential operators leads to the study of basis properties of systems of exponents of the type

$$\left\{A^{+}(t)\cdot\omega^{+}(t)e^{int};\ A^{-}(t)\cdot\omega^{-}(t)e^{-int}\right\}_{n\geq 0,\ k\geq 1} \tag{1}$$

in spaces $L_p\equiv L_p(-\pi,\pi)$, $1<p<+\infty$ where $A^{\pm}(t)\equiv|A^{\pm}(t)|e^{i\alpha^{\pm}}(t)$ are complex-valued functions on $[-\pi,\pi]$, $\omega^{\pm}(t)$ have the representations

$$\omega^{\pm}(t)\equiv\prod_{i=1}^{l^{\pm}}\Big\{\sin\Big|\frac{t-\tau_i^{\pm}}{2}\Big|\Big\}^{\beta_i^{\pm}}, \tag{2}$$

$\{\tau_i\}_1^{l^{\pm}}\subset(-\pi,\pi)$; $\{\beta_i^{\pm}\}_1^{l^{\pm}}\subset R$ are the sets of real numbers. To show where such questions aris from, let us consider the discontinuous first order differential operators

$$L^{\pm}u\equiv u'(t)-\sum_{i=1}^{l^{\pm}}\operatorname{ctg}\left(t-\tau_i^{\pm}\right)\cdot u(t),$$

on $G^{\pm}\equiv\bigcup\limits_{i=1}^{l^{\pm}+1}(\tau_{i-1}^{\pm},\tau_i^{\pm})$, where $-\pi=\tau_0^{\pm}<\tau_1^{\pm}<\cdot<\tau_{l^{\pm}}^{\pm}<\tau_{l^{\pm}+1}^{\pm}=\pi$.

Following V. A. Il'yin [1], we start with the generalized treatment of eigen functions of the operator $L^{\pm}$; such treatment admits us to consider absolutely arbitrary boundary conditions. That is, under the eigen function of the operator $L^{\pm}$, corresponding to the eigen value λ, we mean any nonzero piecewise continuous function with points of discontinuity $\{\tau_i^{\pm}\}_l^{l^{\pm}}$ which is absolutely continuous on $G^{\pm}$ and satisfies almost everywhere on $(-\pi,\pi)$ the equation $L^{\pm}u=\lambda u$. It is not difficult to notice that the systems $\big\{\prod\limits_{i=1}^{l^{\pm}}\sin(t-\tau_i^{\pm})e^{\lambda_n t}\big\}$ themselves are the eigen functions of the operators $L^{\pm}$, respectively. Following V. A. Il'yin and E. A. Moiseev [2], we consider the system of the type (1):

$$\left\{\prod_{i=1}^{l^{+}}\sin(t-\tau_i^{+})e^{int};\ \prod_{i=1}^{l^{-}}\sin(t-\tau_i^{-})e^{-i(n+1)t}\right\}_{n\geq 0},$$

i.e., we take "halfs" of eigen functions of the operators L^{+} and L^{-} which correspond to the eigen values $\lambda_n=in$. In case $\omega^{\pm}(t)\equiv 1$, the basis properties of the system (1) under ceratin conditions imposed on the functions $A^{\pm}(t)$ have been studied by B. T. Bilalov (see, for e.g., [3]) in L_p, $1\leq p\leq+\infty$. Similar problems concerning the subject were considered by E. I. Moiseev [4]–[5] and V. F. Gaposhkina [6].

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1. The Basisness of the System $\{e^{int}\}_{-\infty}^{+\infty}$ in Weighted Spaces

To investigate the subsequent questions, we have first to establish the basisness of the classical system of exponents $\{e^{int}\}_{-\infty}^{+\infty}$ in the weight space $L_{p,\nu}$ on the interval $(-\pi,\pi)$:

$$L_{p,\nu} \overset{def}{\equiv} \big\{f : \|f\|_{p,\nu} < +\infty\big\},$$

where

$$\|f\|_{p,\nu} \equiv \bigg(\int\limits_{-\pi}^{\pi} \big|f(x)\big|^p \nu(x)\,dx\bigg)^{1/p}, \quad \nu(x) > 0,$$

almost everywhere.

So, let $\varphi(x)$ be some function satisfying the condition

$$\varphi(x) \in L_1(-\pi,\pi), \ \ \varphi(x) > 0 \quad (-\pi,\pi). \tag{3}$$

Consider a harmonic function

$$\varphi(x) \equiv \varphi(\tau,x) = \frac{1}{2\pi} \int\limits_{-\pi,\pi}^{\pi} \varphi(t) \frac{1-r^2}{1+r^2-2r\cdot\cos(t-x)}\,dt, \tag{4}$$

where $0 \le r < 1$, $z = r\cdot e^{ix}$. Its conjugate function we denote by $\psi(z) \equiv \psi(r,x)$. Let $\exists C > 0$ for which

$$\varphi(r,x) \ge C\big|\psi(r,x)\big|, \tag{5}$$

where

$$\left.\begin{array}{rl} C > 0 & \text{for } \ p \ge 2 \\ C > \Big|\operatorname{tg}\dfrac{p\pi}{2}\Big| & \text{for } \ 1 < p \le 2 \end{array}\right\} \tag{6}$$

We assume that the weight $\nu(x) \ge 0$ satisfies almost everywhere the condition

$$\nu(x); \ \ \nu^{1-q}(x) \in L_1(-\pi,\pi), \tag{7}$$

where $q : \frac{1}{p} + \frac{1}{q} = 1$ is the conjugate number.

The following theorem is valid.

Theorem 1. *Let the weight $\nu(x)$ satisfy the condition* (6) *and, moreover, for the function $\varphi(x) \equiv \nu(x)$ the expressions* (4) *and* (5) *hold. Then the system of exponents $\{e^{int}\}_{-\infty}^{\infty}$ forms a basis in $L_{p,\nu}$, $1 < p < +\infty$.*

Using one result of K. I. Babenko [7], from the above theorem we obtain the following

Corollary 1. *Let $\nu \equiv \prod\limits_{i=0}^{n} |x-x_i|^{\beta_i}$, where $-\pi \le x_0 < x_1 \cdots < x_n < \pi$, $-1 < \beta_i < p-1$. Then the system $\{e^{int}\}_{-\infty}^{+\infty}$ forms a basis in $L_{p,\nu}$, $1 < p < +\infty$.*

2. The Basisness of the System of Exponents in Weighted Subspaces

Let H_p^+; H_p^- be the standard Hardy classes of analytic functions respectively inside and outside of the unit circle; m is the order of the principal part of the Loran-series expansion at infinity of the function from H_p^-. Denote by L_p^+ and ${}_mL_p^-$ narrowings of the functions respectively from ${}_mH_p^+$ and H_p^- on the unit circle. It is easy to see that L_p^+ and ${}_mL_p^-$ are the subspaces of the space $L_p(-\pi,\pi)$. Since any part of the basis in the Banach space is the basis of its own closed linear span, it is clear that the systems $\{e^{int}\}_{n\ge 0}$ and $\{e^{-int}\}_{n\ge m}$ are the bases of the spaces L_p^+ and ${}_mL_p^-$, respectively. Moreover, we have the expansion

$$L_p = L_p^+ + {}_1L_p,$$

i.e., $\forall f \in L_p$ is uniquely representable in the form $f = f^+ + f^-$, where $f^= \in L_p^+$, $f^- \in {}_1L_p^-$. Let now $\nu^{\pm}(x)$ be the function almost everywhere measurable on $-\pi, \pi$. We introduce into consideration the following weight spaces:

$$L_{p,\nu}^+ \stackrel{def}{\equiv} \{f \in L_1^+ : \|f\|_{p,\nu^+} < +\infty\},$$
$${}_mL_{p,\nu}^- \stackrel{def}{\equiv} \{f \in {}_mL_1^- : \|f\|_{p,\nu^-} < +\infty\},$$

where

$$\|f\|_{p,\nu^\pm} \equiv \left(\int_{-\pi}^{\pi} |f(t)|^p \cdot \nu^{\pm}(t)\, dt \right)^{1/p}.$$

Assume that the weight $\nu^{\pm}(x) \ge 0$ satisfies almost everywhere the condition

$$\nu^{\pm}(x);\, [\nu^{\pm}(x)]^{1-q} \in L_1(-\pi, \pi). \tag{8}$$

Theorem 2. *Let the weight $\nu^+(x)$ $(\nu^-(x))$ satisfy the condition* (8) *and, moreover, for the function $\varphi(x) \equiv \nu^-(x))$ the conditions* (5) *and* (6) *be fulfilled. Then the system $\{e^{int}\}_{n\ge 0}$ $(\{e^{-int}\}_{n\ge m})$ forms a basis in the space L^+_{p,ν^+} $({}_mL^-_{p,\nu^-})$, $1 < p < +\infty$.*

Using again the results of [7], we have

Corollary 2. *Let $\nu(x) \equiv \prod\limits_{i=0}^{l} |x - x_i|^{\beta_i}$, where $-\pi \le x_0 < x_1 < \cdots < x_l < \pi$, $-1 < \beta_i < p - 1$, $\forall i = \overline{1, l}$. Then the system $\{e^{int}\}_{n\ge m}$ $(\{e^{-int}\}_{n\ge m})$ forms a basis in the space $L^+_{p,\nu}$ $({}_mL^-_{p,\nu})$, $1 < p < +\infty$.*

3. The Basisness in L_p

Using the above-stated results, we can establish the basisness of the system (1) in L_p. Thus, let the functions $\omega^{\pm}(t)$ be defined by formulas (2), where $\{\tau_i^{\pm}\} : -\pi \le \tau_l^{\pm} < \cdots < \tau^{\pm}_{l^{\pm}}$ are some points, and

$$\{\tau_i^+\} \cap \{\tau_i^-\} = \{\varnothing\}. \tag{9}$$

Moreover, the following condition regarding the function $A^{\pm}(t)$ holds:

(a) $\alpha^{\pm}(t)$ are the piecewise-Hölder functions in $[-\pi, \pi]$; $\{s_i\}_1^r \subset [-\pi, \pi)$ is the set of points of discontinuity of the function $\theta(t) \equiv \alpha^+(t) - \alpha^-(t)$. Note that $\{\tau_i^{\pm}\} \cap \{s_i\}_1^r = \{\varnothing\}$ and the condition $0 < \|A^{\pm}\|_\infty < +\infty$, where $\|\cdot\|_\infty$ is the norm in L_∞, is fulfilled. Denote by $\|h_i\|_1^r$ oscillations of the function $\theta(t)$ at the points $s_i : h_i = \theta(s_i + 0) - \theta(s_i - 0)$, $i = \overline{1, r}$.

Theorem 3. *Let complex-valued functions $A^{\pm}(t)$ defined by the representations* (2) *satisfy the condition* (*a*) *with respect to the functions $\omega^{\pm}(t)$, and let the condition* (8) *hold. Then if the conditions*

$$-\frac{1}{p} < \beta_i^{\pm} < \frac{1}{q}, \quad i = \overline{1, l^{\pm}};$$
$$-\frac{2\pi}{q} < h_k < \frac{2\pi}{p}, \quad k = \overline{1, r};$$

are fulfilled, then the system (1) *forms a basis in L_p.*

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