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ON THE SUMMABILITY OF TRIGONOMETRIC FOURIER SERIES IN WEIGHTED LEBESGUE SPACES

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The problems on the summability of trigonometric Fourier series in weighted Lebesgue spaces in the frame of Muckenhoupt weights have been investigated in [1–5]. Using the Abel–Poisson and Cesaro methods of order α, $\alpha > 0$, we expound the results concerning the summability of trigonometric Fourier series in the weighted spaces, when weighted functions do not belong to the Muckenhoupt class A_p.

In the sequel, every nonnegative summable function will be called a weight. The 2π-periodic function $f : (-\pi, \pi) \to R'$ is said to be an element of the space $L^p_w(-\pi, \pi)$, if the norm

$$\|f\|_{p,w} = \bigg(\int\limits_{-\pi}^{\pi} |f(x)|^p w(x)\,dx\bigg)^{1/p} < \infty.$$

We introduce some classes of pairs of weight functions.

Definition 1. Let v and w be even positive increasing on $(0, \pi)$ functions. Suppose that $v(\pi-) < \infty$, $w(\pi-) < \infty$.

The pair (v, w) is said to be of the class $a_p(1 < p < \infty)$, if

$$\sup_{0<x<\pi} \bigg(\int\limits_{x}^{\pi} \frac{v(t)}{t}\,dt\bigg)\bigg(\int\limits_{0}^{x} w^{1-p'}(t)\,dt\bigg)^{p-1} < \infty.$$

Here we present an example of a pair belonging to the class a_p. Let $v(x) = |x|^{p-1}$ and $w(x) = |x|^{p-1} \ln^p \frac{2\pi}{|x|}$. Note that none of these functions belongs to the Muckenhoupt class A_p.

Definition 2. Let v and w be even positive decreasing on $(0, \pi)$ functions, where $v(\pi-) > 0$ and $w(\pi-) > 0$.

The pair (v, w) is said to be of the class $\widetilde{a}_p(1 < p < \infty)$, if

$$\sup_{0<x<\pi} \bigg(\frac{1}{x}\int\limits_{0}^{x} v(t)\,dt\bigg)\bigg(\frac{1}{x}\int\limits_{0}^{x} w^{1-p'}(t)\,dt\bigg)^{p-1} < \infty.$$

Let $u_f(x, r)$ denote the Abel–Poisson means of the trigonometric Fourier series of the 2π-periodic summable function f.

The following statements are valid.

Theorem 1. *Let $1 < p < \infty$, and let v and w be even positive increasing* (*decreasing*) *on $(0, \pi)$ functions. If $(v, w) \in a_p((v, w) \in \widetilde{a}_p)$, then:*

(i) *there exists the positive constant c such that*

$$\|u_f(\cdot, r)\|_{p,v} \le c\|f\|_{p,w}$$

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for an arbitrary $f \in L^p_w$ and r, $0 < r < 1$;

(ii) $\lim\limits_{r \to 0} \|u_f(\cdot, r) - f\|_{p,v} = 0$
for an arbitrary $f \in L^p_w$;

(iii) *the operator*

$$Nf(x) = \sup_{0<r<1} |u_f(r, x)|$$

is bounded from L^p_w to the space L^p_v.

Note that the condition (iii) on the contrary implies that $(v, w) \in a_p$ for increading weights, and $(v, w) \in \widetilde{a}_p$ for decreasaing v and w.

As it follows from the Banach-Steinhaus theorem, the conditions (i) and (ii) are equivalent.

Denote by $\sigma_n^\alpha(x, f)$ the Cesaro means of order $\alpha > 0$ of the function f.

Theorem 2. *Let $1 < p < \infty$, and let v and w be even positive increasing (decreasing) on $(0, \pi)$ functions. If $(v, w) \in a_p$ $((v, w) \in \widetilde{a}_p)$, then the following statements are valid:*

(i) *there exists the positive constant c such that*

$$\|\sigma_n(\cdot, f)\|_{p,v} \le c\|f\|_{p,w}$$

for every $f \in L^p_w(-\pi, \pi)$;

(ii) $\lim\limits_{r \to \infty} \|\sigma_n^\alpha(\cdot, f) - f\|_{p,v} = 0$
for an arbitrary $f \in L^p_w(-\pi, \pi)$;

(iii) *the operator*

$$N_1 f(x) = \sup_n |\sigma_n(f, x)|$$

is bounded from $L^p_w(-\pi, \pi)$ to the space $L^p_v(-\pi, \pi)$.

Consider now the Poisson integral in the upper half-space $R^{n+1}_+ = R^n \times (0, \infty)$:

$$u(x, y) = \int_{R^n} f(x - t)P(t, y)\, dt, \quad f \in L^1(R^n),$$

where

$$P(t, y) = \frac{c_n y}{(|t|^2 + |y|^2)^{\frac{n+1}{2}}}.$$

Theorem 3. *Let $1 < p < \infty$, and let v and w be even positive increasing (decreasing) on $(0, \infty)$ functions. Suppose that $\sigma(x) = v(|x|)$ and $\rho(x) = w(|x|)$. If $(v, w) \in a_p$, $((v, w) \in \widetilde{a}_p)$, then the following statements are valid:*

(i) *there exists the positive constant c such that*

$$\|u(\cdot, y)\|_{p,\sigma} \le c\|f\|_{p,\rho}$$

for every $f \in L^p_\rho(R^n)$ and $y > 0$;

(ii) $\lim\limits_{y \to 0} \|u(\cdot, y) - f\|_{p,\sigma} = 0$
for every $L^p_\rho(R^n)$;

(iii) *the operator*

$$Mf(x) = \sup_{y>0} |u(x, y)|$$

is bounded from $L^p_\rho(R^n)$ to the space $L^p_\sigma(R^n)$.

Finally for arbitrary weights we give the criteria governing the summability by Abel–Poisson method in $L^p_v(\mathbb{T})$ for arbitrary function $f \in L^p_w(\mathbb{T})$.

Theorem 4. *Let $1 < p < \infty$ and v and w be any weights. Then*

$$\lim_{r \to 1} \left\| u_f(\cdot, r) - f \right\|_{p,v} = 0$$

holds for arbitrary $f \in L^p_w(\mathbb{T})$ if and only if

$$\sup \frac{1}{|I|} \int_I v(x)\,dx \left(\frac{1}{|I|} \int_I w^{1-p'}(x)\,dx \right)^{p-1} < \infty$$

where the supremum is taken over all intervals with length less than 2π.

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