Proc. A. Razmadze Math. Inst. 138(2005), 111–114

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ON WEIGHTED NORM INEQUALITIES FOR FOURIER MULTIPLIERS IN LEBESGUE SPACES

(Reported on May 19, 2005)

Let $S(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing functions (see [1]). For $\varphi \in S(\mathbb{R}^n)$ the Fourier transform $\widehat{\varphi}$ is defined by

$$\widehat{\varphi}(\lambda) = \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} \varphi(x) e^{-i\lambda x}\, dx.$$

For the Fourier transform of the function f and its inverse we use the notation $F(\varphi)$ and $F^{-1}(\varphi)$, respectively.

The Fourier transform determines a topological isomorphism of the space S into itself.

Let S be the space of tempered distributions, i.e., a space of linear bounded functionals over $S(\mathbb{R}^n)$. In the sequel, the Fourier transforms will be considered in the framework of the theory of S-distributions. Here we give a definition of a weighted Triebel–Lizorkin space. (For a weightless case see [2], [3]).

Let Q_m be a subset of $\mathbb{R}^n$ given by the inequalities

$$2^{m_j} < |\lambda_j| \le 2^{m_j+1}, \quad j = 1, \dots, n; \quad m_j = 0; \pm1; \pm2; \dots.$$

We have

$$\varphi(x) = \sum_m \varphi_m(x), \quad \varphi_m(x) = \varphi_{m_1,\dots,m_n}(x) = \frac{1}{(2\pi)^{n/2}} \int\limits_{Q_m} \widehat{\varphi}(\lambda) e^{i\lambda x}\, dx.$$

Definition 1. A closure of the Schwartz space with the norm

$$\big|\varphi, L_w^{p,\theta}\big| = \bigg\{ \int\limits_{\mathbb{R}^n} \Big(\sum_m \big|\varphi_m(x)\big|^\theta \Big)^{p/\theta} dx \bigg\}^{1/p}, \quad p > 1, \quad \theta < \infty, \tag{1}$$

will be called the space $L_w^{p,\theta}(\mathbb{R}^n) = L_w^{p,\theta}$.

$L_w^{p,\theta}(\mathbb{R}^n)$ is the Banach space which coincides with the Lebesgue space $L^p(\mathbb{R}^n$ exactly to the equivalence of norms when $\theta = 2$ and $w = 1$, but for another θ-s there take place the following continuous inclusions:

$$L^{p,\theta}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n), \quad 1 < \theta \le 2,$$
$$L^p(\mathbb{R}^n) \subset L^{p,\theta}(\mathbb{R}^n), \quad \theta \ge 2.$$

The measurable function $\phi(\lambda_1, \lambda_2)$ is called a multiplier from $L_w^{p,\theta}(\mathbb{R}^2)$ to $L_v^{q,\theta}(\mathbb{R}^2)$ if the transform $T_\phi : L_w^{p,\theta} \to L_v^{q,\theta}$ defined by the equality

$$T_\phi \varphi = g(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \phi(\lambda) \widehat{\varphi} e^{i\lambda x}\, d\lambda, \tag{2}$$

2000 *Mathematics Subject Classification:* 42B05, 42C05.

Key words and phrases. Weighted norm inequalities, Fourier multipliers, Triebel–Lizorkin space.

is bounded. This means that

$$\left|T_\phi\varphi, L_v^{q,\theta}\right| \le c\left|\varphi, L_w^{p,\theta}\right|.$$

In this case we write $\phi \in M_{w,v}^{p,q,\theta}$.

In the present paper the sufficient conditions in order for $\phi \in M_{w,v}^{p,q,\theta}$ are obtained.

Definition 2. We say that the weight $\rho : \mathbb{R} \to \mathbb{R}_+$ satisfies the reverse doubling condition if there exists the constant $d > 1$ such that

$$\int\limits_{I(x,2r)} \rho(t)\,dt \ge d \int\limits_{I(x,r)} \rho(t)\,dt, \tag{3}$$

where $I(x,r)$ is an arbitrary one-dimensional interval with center at the point x and of radius r.

In the sequel, the fact that ρ satisfies the reverse doubling condition will be denoted by $\rho \in RD(\mathbb{R})$.

Definition 3. We say that the function of two variables $u(x,y)$ belongs to the class A_∞ with respect to the variable x uniformly to the variable y ($u \in A_\infty^{(x)}$) if there exist positive numbers c and δ such that for any one-dimensional interval I and any measurable set $E \subset I$ the inequality

$$\frac{\int\limits_E u(x,y)\,dx}{\int\limits_I u(x,y)\,dx} \le \Big(\frac{|E|}{|I|}\Big)^\delta \tag{4}$$

uniformly for all y-s, is satisfied.

The class $A_\infty^{(y)}$ is defined analogously.

Definition 4. (see [4]) Let $1 < p < q < \infty$, $0 < \alpha < 1$, $0 < \beta < 1$. We say that a pair of weights (v,w) defined on $\mathbb{R}^2$ belongs to the class $\Gamma_{\alpha_1,\alpha_2}^{p,q}$ if the following two conditions

$$\begin{aligned} A_1 =: \sup_{\substack{a\in R,\ r>0\\ I\in D}} |I|^{-\alpha_1}\bigg(\int\limits_I \int\limits_{|y-a|<r} w^{1-p'}(x,y)\,dxdy\bigg)^{1/p'} \times \\ \times\bigg(\int\limits_I \int\limits_{|y-a|>r} \frac{v(x,y)}{|y-a|^{q\alpha_2}}\,dxdy\bigg)^{1/q} < \infty \end{aligned} \tag{5}$$

and

$$\begin{aligned} A_2 =: \sup_{a\in R,\ r>0} |I|^{-\alpha_1}\bigg(\int\limits_I \int\limits_{|y-a|>r} w^{1-p'}(x,y)|y-a|^{p'\alpha_2}\,dxdy\bigg)^{1/p'} \times \\ \times\bigg(\int\limits_I \int\limits_{|x-a|<r} v(x,y)\,dxdy\bigg)^{1/q} < \infty \end{aligned} \tag{6}$$

are fulfilled.

The main results are given in the form of the following

Theorem 1. *Let $\{\mu_m\}$, $m = (m_1, m_2)$, $m_i \in \mathbb{Z}$, $i = 1, 2$, be a family of measures such that for some positive c there takes place*

$$\int\limits_{\mathbb{R}^2} |d\mu_m| \le c$$

uniformly with respect to m.

Suppose that in every set Q_m *the measurable function* ϕ *is representable in the form*

$$\phi(\lambda_1,\lambda_2)=\int\limits_{-\infty}^{\lambda_1}\int\limits_{-\infty}^{\lambda_2}\frac{d\mu_m(t_1,t_2)}{(\lambda_1-t_1)^{1-\alpha_1}(\lambda_2-t_2)^{1-\alpha_2}}, \tag{7}$$

where $0<\alpha_1<1$, $0<\alpha_2<1$.

If $w(x,y)=w_1(x)w_2(y)$, *where* $w_1^{1-p'}\in RD(\mathbb{R})$ *and* $v\in A_\infty^{(x)}$ *uniformly to the variable* y, *and if* $(v,w)\in\Gamma^{p,q}_{\alpha_1,\alpha_2}$, $1<p<q<\infty$, *then the operator* T_ϕ *is bounded from* $L_w^{p,\theta}(\mathbb{R}^2)$ *to* $L_v^{q,\theta}(\mathbb{R})$.

Theorem 2. *Let* $1<p<q<\infty$, $\alpha_1>\frac{1}{p'}$, $\alpha_2>\frac{1}{p'}$. *Suppose that* $v\in A_\infty^{(x)}(\mathbb{R})$ $(A_\infty^{(y)}(\mathbb{R}))$ *uniformly with respect to* y *(to* x*).*

If

$$\sup_{I,J}\left\{\int\limits_I\int\limits_J v(x,y)\,dxdy\right\}|I|^{q(1/p'-\alpha_1)}|J|^{q(1/p'-\alpha_2)}<\infty \tag{8}$$

and in every set Q_m *the function* $\phi(\lambda_1,\lambda_2)$ *is representable in the form* (7), *then the operator* T_ϕ *is bounded from* $L^{p,\theta}(\mathbb{R}^2)$ *to* $L_v^{q,\theta}(\mathbb{R}^2)$.

Theorem 3. *Let the function* ϕ *be continuous outside the coordinate axes and have continuous partial derivatives*

$$\frac{\partial^k}{\partial\lambda_1^{k_1}\cdot\partial\lambda_2^{k_2}},\quad 0\le k\le 2,\quad k=k_1+k_2.$$

Moreover, assume that

$$\left|\lambda_1^{k_1+\alpha_1-1}\cdot\lambda_2^{k_2+\alpha_2-1}\frac{\partial\phi(\lambda_1,\lambda_2)}{\partial\lambda_1^{k_1}\cdot\partial\lambda_2^{k_2}}\right|\le M,\quad 0<\alpha_1<1,\quad 0<\alpha_2<1, \tag{9}$$

where M *does not depend on* λ_1 *and* λ_2.

Suppose that the pair of weights (v,w) *satisfies the conditions of Theorem* 1, *then the operator* T_ϕ *is bounded from* $L_w^{p,\theta}$ *to* $L_v^{q,\theta}$.

Theorem 4. *Let* $1<p<q<\infty$. *If for the function* $\phi(\lambda_1,\lambda_2)$ *the conditions* (9) *hold, and the weight* v *satisfies the condition of Theorem* 2, *then the operator* T_ϕ *is bounded from* $L^{p,\theta}(\mathbb{R}^2)$ *to* $L_v^{q,\theta}(\mathbb{R}^2)$.

In the sequel, we will give a more convenient (from the practical point of view) sufficient condition such that the function ϕ is a multiplier of the class $M_{w,v}^{p,q,\theta}$. It is known that the characteristic function of the parallelepiped J with sides parallel to the coordinate axes, belongs to the class $M_{w,v}^{p,q,\theta}$. The generalization of the result is given by the following

Theorem 5. *Let the function* $\phi(\lambda_1,\lambda_2)$ *over the parallelepiped* $\overline{J}((\lambda_1,\lambda_2):a_j\le\lambda_j\le b_j;\ j=1,2)$ *have the continuous derivatives*

$$\frac{\partial^m\phi}{\partial\lambda_1^{m_1}\cdot\partial\lambda_2^{m_2}},\quad 0\le m\le 2,\quad m=m_1+m_2,$$

where m_j *takes the values* 0 *and* 1, *and suppose that these partial derivatives satisfy the condition*

$$\left|\frac{\partial^m\phi}{\partial\lambda_1^{m_1}\cdot\partial\lambda_2^{m_2}}\right|\le\frac{M_0}{(b_1-a_1)^{m_1+\alpha_1-1}(b_2-a_2)^{m_2+\alpha_2-1}}. \tag{10}$$

Then for the pair of weights (v,w) *under the conditions of Theorem* 1 (*Theorem* 2), *the function* ψ *coinciding with the function* ϕ *over the parallelepiped* $\overline{J}$ *and has zero outside it, belongs to the class* $M_{w,v}^{p,q,\theta}$. *Moreover, for the operator* T_ψ *there takes place the inequality*

$$\left|T_\psi f,L_v^{q,\theta}\right|\le cM_0\left|f,L_w^{p,\theta}\right|,$$

where the constant c does not depend on f and φ *(in particular, on the location of the parallelepiped* $\overline{J}$*).*

The proofs of all the above-formulated theorems are carried out by the same methods, i.e., by the representation of the operator under consideration as a composition of certain elementary transformations.

The one-weighted estimates for the Fourier multipliers in the framework of Muckenhoupt A_p class have been established in [5]. As for the two-weighted theory for multipliers of Fourier transforms on the line, it has been developed in [6]–[9].

Note that Theorems 1–5 extend the results of papers [3] and [10], [11].

Acknowledgement

The author would like to express his thanks to Prof. V. Kokilashvili for helpful discussions about the subject.

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