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ON MAPPING PROPERTIES OF BELLMAN TRANSFORM IN WEIGHTED LEBESGUE SPACES

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In the present report we present the results on the boundedness of the Fourier operator generated by the Bellman transform [1] from one weighted Lebesgue space into the other. Moreover, we consider weighted Lebesgue spaces both with constant and with variable exponent.

Let $f(x)$ be a 2π-periodic function. Almost everywhere positive function ρ is called a weighted function.

The weighted Lebesgue space $L^p_\rho(-\pi,\pi)$ is called the Banach space of all those measurable 2π-periodic functions for which

$$\|f\|_{L^p_\rho(-\pi,\pi)} = \Bigg(\int\limits_{-\pi}^{\pi} |f(x)|^p \rho(x)\, dx \Bigg)^{1/p} < \infty.$$

Theorem 1. *Let $1 < p \le q < \infty$. The weighted functions v and w are assumed to satisfy the condition*

$$\sup_{0<x<\pi} \Bigg(\int\limits_x^\pi \frac{v(t)}{t}\, dt \Bigg)^{\frac{1}{q}} \Bigg(\int\limits_0^x w^{1-p'}(x)\, dx \Bigg)^{\frac{1}{p}} < \infty. \tag{1}$$

Next, let the even function $f \in L^p_w(-\pi,\pi)$ and let

$$f(x) \sim \sum_{n=1}^\infty a_n \cos nx \tag{2}$$

be its Fourier series. Then the trigonometric series

$$\sum_{n=1}^\infty A_n \cos nx, \tag{3}$$

where $A_n = \sum\limits_{k=n+1}^\infty \frac{a_k}{k} + \frac{1}{2^n} a_n$, is the Fourier series of some function $F \in L^q_v(-\pi,\pi)$, and there exists an independent of f constant $c > 0$, such that

$$\|F\|_{L^q_v} \le |f|_{L^p_w}. \tag{4}$$

The following theorems are valid.

Theorem 2. *Let $p : (0,\pi) \to [1,\infty)$ be a bounded measurable function, $p(0) > 1$ and*

$$\lim_{x\to 0+} \sup \big(p(x) - p(0)\big) \log \frac{1}{x} < \infty.$$

Assume that $0 < \alpha < \frac{1}{p'(0)}$.

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If $f \in L^{p(\cdot)}_{x^\alpha}(0,\pi)$ *and*

$$f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx,$$

then the trigonometric series (3) *is the Fourier series of some function* $F \in L^{p(\cdot)}_{x^\alpha}$, *and*

$$\|F\|_{L^{p(\cdot)}_{x^\alpha}(0,\pi)} \le c|f|_{L^{p(\cdot)}_{x^\alpha}(0,\pi)},$$

where the positive constant c *does not depend on* f.

Let now $p : (-\pi,\pi) \to R'$ *be the measurable function, such that*

$$1 < P \le p(x) \le p < \infty,$$

where $P = \operatorname*{ess\,inf}_{x\in(-\pi,\pi)} p(x)$ *and* $p = \operatorname*{ess\,sup}_{x\in(-\pi,\pi)} p(x)$.

By $L^{p(\cdot)}(-\pi,\pi)$ *we denote a set of those functions* $f : (-\pi,\pi)$ *for which*

$$A^\lambda_p(f) = \int_{-\pi}^{\pi} \Big|\frac{f(x)}{\lambda}\Big|^{p(x)}\,dx < \infty$$

for some $\lambda > 0$. *This set is in fact the Banach space with respect to the norm*

$$\|f\|_{L^{p(\cdot)}} = \inf\{\lambda > 0 : A^\lambda_p(f) \le 1\}.$$

The weight space $L^{p(\cdot)}_\rho$ *is defined as a set of all measurable functions for which*

$$\|f\|_{L^{p(\cdot)}_\rho} = \|\rho f\|_{L^{p(\cdot)}} < \infty.$$

The space $L^{p(\cdot)}_\rho$ *is likewise the Banach space. In the sequel, the use will be made of the following notation:*

$$\rho_0(x) = \operatorname*{ess\,inf}_{y\in(0,x)} \rho(y), \quad 0 < x < \pi.$$

The following theorem is valid.

Theorem 3. *Let* $\rho(x)$ *and* $q(x)$ *be measurable functions defined on* $(0,\pi)$. *Furthermore, we assume that*

$$1 < P \le \rho_0(x) \le q(x) < q < \infty, \quad x \in (0,\pi).$$

Let the condition

$$\sup_{0<x<\pi} \int_x^1 \Big(\frac{u(t)}{t}\,dt\Big)^{q(t)} \Big(\int_0^x w^{(p_0)'(t)}(\tau)\,d\tau\Big)^{\frac{q(t)}{(p_0)'(t)}}\,dt < \infty$$

be fulfilled. Then for any $f \in L^{p(\cdot)}_w$, *the trigonometric series* (3) *together with the Fourier series* (2) *is the Fourier series of some function* $F \in L^q_v$, *where*

$$\|F\|_{L^q_v(0,\pi)} \le c\|f\|_{L^p_w(0,\pi)}.$$

Note that similar statements are valid for the Fourier sine-series.

The properties of Bellman transforms in different functional classes have been considered by many authors (see, for e.g., [2], [3]). It should be noted that when proving Theorem 3 we have, to a certain extent, used the results obtained in [4].

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