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WEIGHTED INEQUALITIES CRITERIA FOR LITTLEWOOD–PALEY FUNCTIONS IN ORLICZ CLASSES

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Let $x \in R^n$ and let $y > 0$. Poisson integral of the function $f : R^n \to R^1$ is defined by

$$u_f(x,y) = \int\limits_{R^n} f(t)P(x-t,y)dt,$$

where

$$P(x,z) = \frac{c_n z}{(|z|^2+|y|^2)^{-\frac{n+1}{2}}}, \quad x \in R^n, \ \ z \in R^n.$$

The aim of the present note is to give the criteria of various modular inequalities for the Littlewood-Paley function g_f in weighted Orlicz classes. This function is defined by

$$g_f(x) = \bigg(\int\limits_0^\infty y\big|\nabla u(x,y)\big|^2 dy \bigg)^{1/2},$$

where

$$\nabla u(x,y) = \Big(\frac{\partial u}{\partial x_1}(x,y), \ \ \frac{\partial u}{\partial x_2}(x,y), \dots, \frac{\partial u}{\partial y}(x,y) \Big)$$

is full gradient of $u(x,y)$.

For our purpose we need some basic definitions. Let Φ be a class of functions $\varphi : R^1 \to R^1$ nonnegative, even and increasing on $(0,\infty)$, $\varphi(0+) = 0$, $\lim\limits_{t\to\infty} \varphi(t) = \infty$.

A function φ is called quasiconvex if there exist a Young function ω and a constant $c > 1$ such that the chain of inequalities

$$\omega(t) \le \varphi(t) \le \omega(ct), \quad t \ge 0$$

holds.

To each quasiconvex function φ we can put into correspondence its complementary function $\widetilde{\varphi}$ defined by

$$\widetilde{\varphi}(t) = \sup_{s\ge 0} \big(st - \varphi(s)\big).$$

By definition, the function φ satisfies the global condition Δ_2 ($\varphi \in \Delta_2$) if there is $c > 0$ such that

$$\varphi(2t) \le c\varphi(t), \quad t > 0.$$

Further, for each quasiconvex function φ we define the number $p(\varphi)$ as follows:

$$\frac{1}{p(\varphi)} = \inf\{\alpha : \varphi^\alpha \quad \text{is quasiconvex}\}.$$

We need also the definition of the well-known Muckenhoupt's class of weight functions.

An almost everywhere locally integrable function $\omega : R^1 \to R^1$ will be called a weight function.

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By definition, the weight function $\omega \in A_s(R^n)$ $(1 < s < \infty)$ if

$$\sup_B \left(\frac{1}{|B|} \int_B \omega(x)\, dx \right) \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{1}{s-1}}\, dx \right)^{s-1} < \infty,$$

where supremum is taken over all balls $B \subset R^n$.

Now we are ready to formulate the main results of our note.

Theorem 1. *Let $\varphi \in \Phi$. The modular inequality*

$$\int_{R^n} \varphi\big(g_f(x)\big)\omega(x)\, dx \le c \int_{R^n} \varphi\big(f(x)\big)\omega(x)\, dx$$

holds if and only if $\varphi \in \Delta_2$, φ^α is quasiconvex for some α, $0 < \alpha < 1$ and $\omega \in A_{p(\varphi)}$.

Theorem 2. *Let $\varphi \in \Phi$. Then the inequality*

$$\int_{R^n} \widetilde{\varphi}\Big(\frac{g_f(x)}{\omega(x)}\Big)\omega(x) dx \le c \int_{R^n} \widetilde{\varphi}\Big(\frac{f(x)}{\omega(x)}\Big)\omega(x)\, dx$$

holds if and only if the conditions of Theorem 1 are satisfied.

Theorem 3. *Let $\varphi \in \Phi$. Then the following conditions are equivalent:*

i) *there exists $c > 0$ such that*

$$\int_{R^n} \varphi\big(g_f(x)\big)\omega(x)\, dx \le c \int_{R^n} \varphi\big(f(x)\big)\omega(x)\, dx$$

for all f with the finite integral on the right side,

ii) *$\varphi \in \Delta_2$, φ^α is quasiconvex for some α, $0 < \alpha < 1$, $\omega^{p(\varphi)} \in A_{p(\varphi)}$ and $\omega^{-p(\varphi)} \in A_{p(\widetilde{\varphi})}$.*

For the maximal functions and singular integrals criteria of modular weighted inequalities are presented in [1], [2]. For unweighted case for g_f function see [3].

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