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The Marcinkiewicz Integral in Lebesgue Weighted Spaces with Variable Exponent

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In the present report we set force one result on the boundedness of the Marcinkiewicz integral in Lebesgue spaces with a variable exponent with power weight.

Let $p : R^n \to R$ be a measurable function such that the conditions:

$$1 \le p_0 \le p(x) \le p < \infty, \quad x \in R^n \tag{1.1}$$

and

$$\big|p(x) - p(y)\big| \le \frac{A}{\ln \frac{1}{|x-y|}}, \ \ |x-y| \le \frac{1}{2}, \quad x, y \in R^n \tag{1.2}$$

are fulfilled.

By $L^{p(\cdot)}(R^n)$ we denote a space of functions on R^n for which

$$A_p(f) = \int\limits_{R^n} \Big|\frac{f(x)}{\lambda}\Big|^{p(x)} dx < \infty$$

for some $\lambda > 0$.

This is the Banach functional space with respect to the norm

$$\|f\|_{L^{p(\cdot)}} = \inf\Big\{\lambda > 0 : \ A_p\Big(\frac{f}{\lambda}\Big) \le 1\Big\}.$$

Under the condition (1.1) the space $L^{(p(\cdot)}$ coincides with the space

$$\Big\{f(x) : \Big|\int\limits_{R^n} f(x)\varphi(x)\,dx\Big| < \infty\Big\} \ \text{ for all } \ \varphi(x) \in L^{q(\cdot)}(R^n),$$

where $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$ up to the equivalence of the norm

$$\|f\|_{L^{p(\cdot)}} \approx \sup_{\|f\|_{L^{q(\cdot)}} \le 1} \Big|\int\limits_{R^n} f(x)\varphi(x)\,dx\Big| \approx \sup_{A_q(\varphi) \le 1} \Big|\int\limits_{R^n} f(x)\varphi(x)\,dx\Big|.$$

Let ρ be a measurable, almost everywhere positive integrable function on R^n. The weighted Lebesgue space $L^{p(\cdot)}_\rho = L^{p(\cdot)}(R^n, \rho)$ is defined as a set of all measurable functions for which

$$\|f\|_{L^{p(\cdot)}_\rho} = \|\rho f\|_{L^{p(\cdot)}} < \infty.$$

$L^{p(\cdot)}(R^n, \rho)$ is the Banach space. In the sequel, we will consider the weight function $\rho(x) = |x - x_0|^\alpha$, where $x_0 \in R^n$.

Let P be a closed set of the space R^n. Introduce the notation

$$\delta(y) = \operatorname{dist}(y, P) = \inf_{z \in P} |y - z|.$$

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Assume that λ is some positive number, and we consider the following integral transformation:

$$Jf(x) = \int\limits_{CP} \frac{(\delta(y))^{\lambda}}{|x-y|^{n+\lambda}}\, f(y)\, dy.$$

This integral has been introduced by Yu. Marcinkiewicz [1] and is of importance in different fields of the theory of functions, in particular, in the theory of singular and hyper-singular integrals, in the theory of trigonometric Fourier series, and so on.

Later on, L. Corleson [2] and A. Zygmund [3] considered the modified Marcinkiewicz integral, namely, the integral

$$J^{*}f(x) = \int\limits_{R^n} \frac{(\delta(y))^{\lambda}}{(|x-y|+\delta(y))^{n+\lambda}}\, f(y)\, dy.$$

Obviously, J and J^{*} coincide on the set P.

In what follows, along with the conditions (1.1) and (1.2) it will be assumed that $p(x)$ satisfies the following condition: there exists

$$\lim_{x\to\infty} p(x) = p(\infty) \ \text{ and } \ |p(x) - p(\infty)| \le \frac{A}{\ln(1+|x|)}\,. \tag{1.3}$$

The following theorem is valid.

Theorem 1. *Let $p(x)$ satisfy the conditions* (1.1), (1.2) *and* (1.3). *For the operator J to be bounded in $L^{p(\cdot)}_{|x-x_0|^{\alpha}}$, it is necessary and sufficient that the condition*

$$-\frac{1}{p(x_0)} < \alpha < \frac{1}{p'(x_0)}\,,$$

where $\frac{1}{p(x_0)} + \frac{1}{p'(x_0)} = 1$ be fulfilled.

A. Calderon [4] has introduced the more general integral than that of Marcinkiewicz one.

Let $\varphi(\rho,t)$ be a non-negative measurable function of two variables $\rho>0$ and $t\ge 0$, satisfying the following conditions:

1. For every t, $(\rho+t)^{-n}\varphi(\rho,t)$ decreases with respect to ρ, and $\lim\limits_{\rho\to\infty} \frac{\varphi(\rho,t)}{(\rho+t)^n} = 0$.

2. There exists $c>0$ such that $\int\limits_0^\infty \rho^{n-1}(\rho+t)^{-n}\varphi(\rho,t)d\rho \le c$ for any t.

Let the function $\psi(y)$ and also the function $\varphi(|x-y|,\psi(y))$ be measurable.

The Calderon's integral has the form

$$Kf(x) = \int\limits_{R''} \frac{\varphi(|x-y|,\psi(y))}{(|x-y|+\psi(y))^n}\, f(y)\, dy.$$

If in the expression $Kf(x)$ we put $\varphi(\rho,t) = \frac{t^{\lambda}}{(\rho+t)^{\lambda}} f(y)$ and $\psi(y) = \delta(y)$, then we will obtain the modified Marcinkiewicz integral.

The following theorem is valid.

Theorem 2. *For the operator K to be bounded in $L^{p(\cdot)}_{|x-x_0|^{\alpha}}$, it is sufficient that the condition*

$$-\frac{1}{\rho(x_0)} < \alpha < \frac{1}{\rho'(x_0)}$$

be fulfilled.

References

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