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EXISTENCE AND REGULARITY FOR A DEGENERATE PROBLEM WITH SINGULAR GRADIENT LOWER ORDER TERM

Abstract. In this paper we study the existence and regularity results for nonlinear elliptic equation with degenerate coercivity and a singular gradient lower order term.

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1 Introduction

Consider the elliptic problem

$$\begin{cases} -\operatorname{div}\big(a(x,u)\widehat{a}(x,u,\nabla u)\big) + B\,\frac{|\nabla u|^p}{|u|^\theta} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$ $(N \geq 3)$, $B > 0$, f is a positive function belonging to $L^m(\Omega)$ with $m \geq 1$, and

$$0 < \theta < 1, \tag{1.2}$$

$$2 \leq p < N. \tag{1.3}$$

Here, we suppose that $a : \Omega \times \mathbb{R} \to \mathbb{R}$, is a Carathéodory function such that for a.e. $x \in \Omega$, for every $s \in \mathbb{R}$, we have

$$\frac{\alpha}{(1+|s|)^\gamma} \leq a(x,s) \leq \beta, \tag{1.4}$$

where α, β are strictly positive real numbers and $\gamma > 0$.

We suppose that $\widehat{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function satisfying a.e. $x \in \Omega$, $\forall\, s \in \mathbb{R}$ and $\xi, \xi' \in \mathbb{R}^N$, the following inequalities:

$$\widehat{a}(x,s,\xi)\cdot\xi \geq |\xi|^p, \tag{1.5}$$

$$|\widehat{a}(x,s,\xi)| \leq C_1|s|^{\frac{\theta(p-1)}{p}} + C_2|\xi|^{p-1}, \tag{1.6}$$

$$\big(\widehat{a}(x,s,\xi) - \widehat{a}(x,s,\xi')\big)(\xi-\xi') \geq C_3|\xi-\xi'|^p, \tag{1.7}$$

where C_1, C_2 and C_3 are positive real numbers.

As prototype examples, we consider the following models:

$$\begin{cases} -\operatorname{div}\Big(\dfrac{|u|^{\frac{\theta(p-1)}{p}}(1+|Du|)^{-1}Du + |Du|^{p-2}Du}{(1+|u|)^\gamma}\Big) + \dfrac{|\nabla u|^p}{|u|^\theta} = f & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \end{cases}$$

and

$$\begin{cases} a(x,s) = \dfrac{1}{(b(x)+|s|)^\gamma}, \;\; b(x) \in L^\infty(\Omega), \;\text{ and }\; b(x) \geq c > 0, \\ \widehat{a}(x,s,\xi) = |\xi|^{p-2}\big(1+|\xi|^{-\varepsilon}\big)\xi, \;\; \varepsilon \in (0,p-1). \end{cases}$$

The main difficulty in dealing with problem (1.1) is the fact that the lower order term has a quadratic growth with respect to the gradient and is singular in the variable u, and the differential operator

$$Au = \operatorname{div}\big(a(x,u)\widehat{a}(x,u,\nabla u)\big)$$

is well defined between $W_0^{1,p}(\Omega)$ and its dual, but it fails to be coercive if u is large. The corresponding results in the case $Au = \operatorname{div}(a(x,u)\nabla u)$ and $p = 2$ are developed in [6]. In the case where $\gamma = B = 0$, $f \in L^1(\Omega)$, the solution u of problem (1.1) belongs only to $W_0^{1,s}(\Omega)$ for every $s < \frac{N(p-1)}{N-1}$ (see [3, 4]). Once again, the lower order term improves the regularity of solutions of problem (1.1), since $\frac{N(p-1)}{N-1} < \frac{N(p-\theta)}{N-\theta}$ (due to the fact that $0 < \theta < 1$). In [1], under the assumptions $B \equiv 0$, $\gamma = \theta(p-1)$ and $\widehat{a}(x,s,\xi) = |\xi|^{p-2}\xi$, the authors proved only the existence of entropy solutions u of problem (1.1) belonging only to the Marcinkiewicz space $M^\tau(\Omega)$ for every $\tau = \frac{N(p-1)(1-\theta)}{N-p}$, with $|\nabla u| \in M^q(\Omega)$ for $q = \frac{N(p-1)(1-\theta)}{N-1-\theta(p-1)}$. The existence and regularity results for problem (1.1) have been obtained in [8] provided $\gamma > 0$, $1 \leq \theta \leq 2$ and $f \in L^m(\Omega)$ with $m > 1$.

To prove our main results, we approximate problem (1.1) by a sequence of non-degenerate and non-singular problems. Then we prove both a priori estimates and convergence results on the sequence of approximating solutions. Next, by the strong maximum principle [7], we prove that the weak limit of the approximate solutions is strictly positive in Ω. Finally, we pass to the limit in the approximate problems.

The paper is organized as follows. In Section 2, we introduce the main results. The approximate problem is presented in Section 3. Estimate uniforms are proved in Section 4. Theorems 2.1–2.4 are proved in Section 5.

2 Statement of main results

Definition. Let $f \in L^m(\Omega)$, $m \geq 1$. A measurable function u is said to be a solution in the sense of distributions to problem (1.1) if $u \in W_0^{1,1}(\Omega)$, $\widehat{a}(x,u,\nabla u) \in (L^1(\Omega))^N$, $\frac{|\nabla u|^p}{u^\theta} \in L^1(\Omega)$, $u > 0$ in Ω, and

$$\int\limits_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla\varphi\,dx + B\int\limits_\Omega \frac{|\nabla u|^p}{u^\theta}\,\varphi\,dx = \int\limits_\Omega f\varphi\,dx, \quad \forall\,\varphi \in C_0^1(\Omega). \tag{2.1}$$

Our main results are the following theorems.

Theorem 2.1. *Let $f \in L^1(\Omega)$ be a positive function and assume that (1.2)–(1.7) hold true. Then problem (1.1) has at least one distributional solution $u \in W_0^{1,\eta}(\Omega)$ with*

$$\eta = \frac{N(p-\theta)}{N-\theta}\,. \tag{2.2}$$

Remark 2.1. Hypothesis (1.3) implies that $\eta < p$. Since $p \geq 2 > 2 - \frac{1}{N}$, we can deduce that $\eta > 1$.

Theorem 2.2. *If hypotheses (1.2)–(1.7) hold and $f \in L^m(\Omega)$ is a positive function such that*

$$1 < m < \frac{pN}{pN - \theta(N-p)}\,, \tag{2.3}$$

then problem (1.1) has at least one distributional solution $u \in W_0^{1,\sigma}(\Omega)$ with

$$\sigma = \frac{mN(p-\theta)}{N-\theta m}\,. \tag{2.4}$$

Remark 2.2. Notice that condition (2.3) guarantees that $\sigma < p$.

Theorem 2.3. *Suppose that assumptions (1.2)–(1.7) hold and $f \in L^m(\Omega)$ is a positive function such that*

$$\frac{pN}{pN-\theta(N-p)} \leq m < \frac{N}{p}\,.$$

Then problem (1.1) has at least one distributional solution $u \in W_0^{1,p}(\Omega)$.

Theorem 2.4. *Let $0 < \gamma < p-1$. Suppose that assumptions (1.2)–(1.7) hold and $f \in L^m(\Omega)$ is a positive function such that*

$$m > \frac{N}{p}\,. \tag{2.5}$$

Then problem (1.1) has at least one distributional solution $u \in W_0^{1,p}(\Omega)L^\infty(\Omega)$.

3 The approximated problem

Hereafter, we denote by T_k the truncation function at the level $k > 0$ defined by

$$T_k(s) = \max\big\{-k, \min\{s,k\}\big\} \ \text{ for every } \ s \in \mathbb{R}.$$

Let (f_n) $(f_n > 0)$ be a sequence of bounded functions defined in Ω that converges to $f > 0$ in $L^1(\Omega)$, and verifies the inequalities $f_n \le n$ and $f_n \le f$ for every $n \ge 1$ (for example, $f_n = T_n(f)$). Consider the following non-degenerate and non-singular problem:

$$\begin{cases} -\operatorname{div}\big(a(x, T_n(u_n))\widehat{a}(x, u_n, \nabla u_n)\big) + B\,\dfrac{u_n|\nabla u_n|^p}{(|u_n| + \frac{1}{n})^{\theta+1}} = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Problem (3.1) admits at least one solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see [5]).

Proposition 3.1. *We have $u_n \ge 0$ almost everywhere in Ω.*

Proof. Taking $u_n^- = \min(u_n, 0)$ as a test function in (3.1), using (1.4) and (1.5), we find that

$$\frac{\alpha}{(1+n)^\gamma} \int_\Omega |\nabla u_n^-|^p\,dx \le \int_\Omega f_n u_n^-\,dx \le 0,$$

so, $u_n \ge 0$ almost everywhere in Ω. □

Therefore, Proposition 3.1 implies that u_n satisfies

$$\begin{cases} -\operatorname{div}\big(a(x, T_n(u_n))\widehat{a}(x, u_n, \nabla u_n)\big) + B\,\dfrac{u_n|\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

In the remainder of this paper, we denote by C various positive constants depending only on the data of the problem, but not on n.

4 A priori estimates

We are now going to prove some a priori estimates. The next lemma gives a control of the lower order term.

Lemma 4.1. *Let u_n be the solutions to problem (3.2). Then*

$$B \int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}}\,dx \le \int_\Omega f\,dx. \tag{4.1}$$

Proof. For any fixed $h > 0$, let us consider $\frac{T_h(u_n)}{h}$ as a test function in (3.2), and dropping the nonnegative first term, we obtain

$$B \int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} \frac{T_h(u_n)}{h}\,dx \le \int_\Omega f_n \frac{T_h(u_n)}{h}\,dx.$$

So,

$$B \int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} \frac{T_h(u_n)}{h}\,dx \le \int_\Omega f\,dx. \tag{4.2}$$

Letting h tend to 0 in (4.2), we deduce (4.1) by Fatou's lemma. □

Lemma 4.2. *Let $f \in L^1(\Omega)$ be a positive function. Then the sequence u_n is bounded in $W_0^{1,\eta}(\Omega)$, where η is given by (2.2), and $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ for every $k > 0$.*

Proof. Estimate (4.1) and the fact that $2^{\theta+1}u_n u_n^\theta \geq (u_n+\frac{1}{n})^{\theta+1}$ in $\{u_n \geq 1\}$ give

$$B\int_{\{u_n\geq 1\}} \frac{|\nabla u_n|^p}{u_n^\theta}\,dx \leq B\int_{\{u_n\geq 1\}} \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}\,dx \leq \|f\|_{L^1(\Omega)}. \tag{4.3}$$

Let $\eta < p$ and $G_1(t) = t - T_1(t)$. Using the Hölder inequality, we have

$$\begin{aligned}\int_\Omega |\nabla G_1(u_n)|^\eta\,dx &= \int_{\{u_n\geq 1\}} \frac{|\nabla G_1(u_n)|^\eta}{u_n^{\frac{\theta\eta}{p}}}\,u_n^{\frac{\theta\eta}{p}}\,dx \\ &\leq \Bigg(\int_{\{u_n\geq 1\}} \frac{|\nabla G_1(u_n)|^p}{u_n^\theta}\,dx\Bigg)^{\frac{\eta}{p}}\Bigg(\int_{\{u_n\geq 1\}} u_n^{\frac{\theta\eta}{p-\eta}}\,dx\Bigg)^{\frac{p-\eta}{p}}.\end{aligned} \tag{4.4}$$

By (4.3), (4.4) and due to $u_n \leq G_1(u_n)+1$, we obtain

$$\int_\Omega |\nabla G_1(u_n)|^\eta\,dx \leq \|f\|_{L^1(\Omega)}^{\frac{\eta}{p}}\Bigg(\int_{\{u_n\geq 1\}} u_n^{\frac{\theta\eta}{p-\eta}}\,dx\Bigg)^{\frac{p-\eta}{p}} \leq C\Bigg(\int_{\{u_n\geq 1\}} (G_1(u_n))^{\frac{\theta\eta}{p-\eta}}\,dx\Bigg)^{\frac{p-\eta}{p}} + C. \tag{4.5}$$

Inequality (2.2) implies that $\eta^* = \frac{\eta\theta}{p-\eta}$. By Sobolev embedding, we get

$$\Bigg(\int_{\{u_n\geq 1\}} (G_1(u_n))^{\eta^*}\,dx\Bigg)^{\frac{\eta}{\eta^*}} \leq C\int_{\{u_n\geq 1\}} |\nabla G_1(u_n)|^\eta \leq C\Bigg(\int_\Omega (G_1(u_n))^{\eta^*}\,dx\Bigg)^{\frac{\theta}{\eta^*}} + C. \tag{4.6}$$

Since $\theta < 1 < \eta$, inequality (4.6) implies that $G_1(u_n)$ is bounded in $L^{\eta^*}(\Omega)$. From (4.5) follows the boundedness of $G_1(u_n)$ in $W_0^{1,\eta}(\Omega)$. Using $T_k(u_n)$ as a test function in (3.2), one has

$$\int_{\{u_n\leq k\}} |\nabla T_k(u_n)|^p\,dx \leq Ck(k+1)^\gamma \tag{4.7}$$

for every $n \geq 1$. Taking $k = 1$ in (4.7), we deduce that $T_1(u_n)$ is bounded in $W_0^{1,p}(\Omega)$ and hence in $W_0^{1,\eta}(\Omega)$. Since $u_n = G_1(u_n) + T_1(u_n)$, we deduce that u_n is bounded in $W_0^{1,\eta}(\Omega)$. Moreover, (4.7) implies that

$$\|T_k(u_n)\|_{W_0^{1,p}(\Omega)} \leq Ck(k+1)^\gamma \tag{4.8}$$

for all $n \geq 1$. □

Lemma 4.3. *Suppose that the hypotheses of Theorem 2.2 are satisfied. Then the sequence u_n is bounded in $W_0^{1,\sigma}(\Omega)$, where σ is given by (2.3).*

Proof. Take $\phi = (u_n+1)^{\theta+ps} - 1$ with

$$s = \frac{p^* - \theta m'}{pm' - p^*} \tag{4.9}$$

as a test function in problem (3.2) (note that $s < 0$ and $\theta + ps > 0$). We get

$$\begin{aligned}C&\int_\Omega a(x,T_n(u_n))\widehat{a}(x,u_n,\nabla u_n)\cdot\nabla u_n(u_n+1)^{\theta+ps-1}\,dx \\ &+ B\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}(u_n+1)^{\theta+ps}\,dx = B\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}\,dx + \int_\Omega f_n((u_n+1)^{\theta+ps}-1)\,dx.\end{aligned}$$

Using (1.4), (1.5), (4.1), $f_n \leq f$, and dropping the nonnegative first term, we find that

$$B\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}(u_n+1)^{\theta+ps}\,dx \leq \int_\Omega f(u_n+1)^{\theta+ps}\,dx + C.$$

Using the fact that

$$2u_n(u_n+1)^\theta \geq \Big(u_n+\frac{1}{n}\Big)^{\theta+1} \tag{4.10}$$

on the set $\{x\in\Omega, u_n\geq 1\}$, we write

$$\begin{aligned}\frac{B}{2(s+1)^p}\int\limits_{\{u_n\geq 1\}} \big|\nabla\big((u_n+1)^{s+1}-2^{s+1}\big)\big|^p\,dx &= \frac{B}{2}\int\limits_{\{u_n\geq 1\}} |\nabla u_n|^p(u_n+1)^{ps}\,dx\\ &\leq \int\limits_{\{u_n\geq 1\}} f(u_n+1)^{\theta+ps}\,dx + C \leq C\Big(\int\limits_{\{u_n\geq 1\}}(u_n+1)^{(ps+\theta)m'}\,dx\Big)^{\frac{1}{m'}}+C. \end{aligned}\tag{4.11}$$

Using (4.11) and the Sobolev's inequality on the left-hand side, we have

$$\begin{aligned}&\mathcal{S}\,\frac{B}{2(s+1)^p}\Big(\int\limits_{\{u_n\geq 1\}}\big|(u_n+1)^{s+1}-2^{s+1}\big|^{p^*}\,dx\Big)^{\frac{p}{p^*}}\\ &\leq \frac{B}{2(s+1)^p}\int\limits_{\{u_n\geq 1\}}\big|\nabla\big((u_n+1)^{s+1}-2^{s+1}\big)\big|^p\,dx \leq C\Big(\int\limits_{\{u_n\geq 1\}}(u_n+1)^{(ps+\theta)m'}\,dx\Big)^{\frac{1}{m'}}+C.\end{aligned}$$

We remark that (4.9) is equivalent to require $(s+1)p^*=(ps+\theta)m'$, moreover, $\frac{p}{p^*}>\frac{1}{m'}$, due to the hypotheses on m and θ. Hence

$$\Big(\int\limits_{\{u_n\geq 1\}}\big|(u_n+1)^{s+1}-2^{s+1}\big|^{p^*}\,dx\Big)^{\frac{p}{p^*}} \leq C\Big(\int\limits_{\{u_n\geq 1\}}(u_n+1)^{(s+1)p^*}\,dx\Big)^{\frac{1}{m'}}+C,$$

so,

$$\int\limits_{\{u_n\geq 1\}}(u_n+1)^{(s+1)p^*}\,dx\leq C. \tag{4.12}$$

Now, by Hölder's inequality, (2.4), (4.9) and the fact that $-s\,\frac{\sigma p}{p-\sigma}=(s+1)p^*$, we obtain

$$\begin{aligned}\int\limits_{\{u_n\geq 1\}}|\nabla u_n|^\sigma\,dx &= \int\limits_{\{u_n\geq 1\}}\frac{|\nabla u_n|^\sigma}{(u_n+1)^{-s\sigma}}(u_n+1)^{-s\sigma}\,dx\\ &\leq \Big(\int\limits_{\{u_n\geq 1\}}\frac{|\nabla u_n|^p}{(u_n+1)^{-ps}}\,dx\Big)^{\frac{\sigma}{p}}\Big(\int\limits_{\{u_n\geq 1\}}(u_n+1)^{-s\,\frac{\sigma p}{p-\sigma}}\,dx\Big)^{\frac{p-\sigma}{p}}\\ &\leq \Big(\int\limits_{\{u_n\geq 1\}}|\nabla u_n|^p(u_n+1)^{(ps+\theta)m'}\,dx\Big)^{\frac{\sigma}{p}}\Big(\int\limits_{\{u_n\geq 1\}}(u_n+1)^{(s+1)p^*}\,dx\Big)^{\frac{p-\sigma}{p}}.\end{aligned}\tag{4.13}$$

Using (4.12) and (4.13), we deduce

$$\int\limits_{\{u_n\geq 1\}}|\nabla u_n|^\sigma\,dx\leq C. \tag{4.14}$$

It remains to analyse the behaviour of ∇u_n on $\{u_n\leq 1\}$. Taking $T_1(u_n)$ as a test function in (3.2), using (1.4), (1.5), $f_n\leq f$ and dropping the non-negative lower order term, we get

$$\int\limits_{\{u_n\leq 1\}}|\nabla T_1(u_n)|^p\,dx\leq C. \tag{4.15}$$

As a consequence of estimates (4.14) and (4.15), the sequence $\{u_n\}_n$ is bounded in $W_0^{1,\sigma}(\Omega)$. $\square$

Lemma 4.4. *Suppose that the hypotheses of Theorem 2.3 are satisfied. Then the sequence u_n is bounded in $W_0^{1,p}(\Omega)$.*

Proof. Testing (3.2) with $\phi = (u_n+1)^\theta - 1$, we get

$$\theta \int_\Omega a(x, T_n(u_n))\widehat{a}(x, u_n, \nabla u_n) \cdot \nabla u_n (u_n+1)^{\theta-1}\, dx$$
$$+ B \int_\Omega \frac{|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} u_n (u_n+1)^\theta\, dx = B \int_\Omega \frac{u_n |\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}\, dx + \int_\Omega f_n((u_n+1)^\theta - 1)\, dx.$$

Using (1.4), (1.5), (4.1), $f_n \le f$ and dropping the non-negative first term, we obtain

$$B \int_\Omega \frac{u_n |\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}} (u_n+1)^\theta\, dx \le C \int_\Omega f u_n^\theta\, dx + C \le C \int_{\{u_n \ge 1\}} f(u_n-1)^\theta\, dx + C. \tag{4.16}$$

By (4.16) and (4.10), we deduce

$$\frac{B}{2} \int_{\{u_n \ge 1\}} |\nabla u_n|^p\, dx \le C \int_{\{u_n \ge 1\}} f(u_n-1)^\theta\, dx + C.$$

Using Sobolev's inequality (with exponent $\frac{p^*}{\theta}$ on the left-hand side) and Hölder's inequality (on the right-hand side), we obtain

$$\mathcal{S}\,\frac{B}{2}\Big(\int_{\{u_n \ge 1\}} (u_n-1)^{p^*}\Big)^{\frac{p}{p^*}} dx \le \frac{B}{2} \int_{\{u_n \ge 1\}} |\nabla u_n|^p\, dx \le C\Big(\int_{\{u_n \ge 1\}} (u_n-1)^{p^*}\, dx\Big)^{\frac{\theta}{p^*}} + C. \tag{4.17}$$

Since $\theta < p$, we have

$$\int_{\{u_n \ge 1\}} (u_n-1)^{p^*}\, dx \le C. \tag{4.18}$$

Inequalities (4.17) and (4.18) imply

$$\int_{\{u_n \ge 1\}} |\nabla u_n|^p\, dx \le C. \tag{4.19}$$

Let us search for the same kind of estimate in $\{u_n < 1\}$. Taking $T_1(u_n)$ as a test function in problem (3.2), using hypothesis (1.4) and dropping the non-negative lower order term, we get

$$\int_{\{u_n < 1\}} |\nabla T_1(u_n)|^p\, dx \le C. \tag{4.20}$$

As a consequence of (4.19) and (4.20), the sequence $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega)$. □

To prove the L^∞ a priori estimate, we will need the following result.

Lemma 4.5 ([2, Lemma 2.1])**.** *Let w be a function in $W_0^{1,\delta}(\Omega)$ such that for k greater than some k_0,*

$$\int_{A_k} |\nabla w|^\delta\, dx \le C k^{\frac{\gamma\delta}{p-1}} |A_k|^{\frac{\delta}{\delta^*}+\varepsilon_1},$$

where $\varepsilon_1 > 0$, $0 \le \gamma < p-1$, $\delta^* = \frac{N\delta}{N-\delta}$ and $A_k = \{x \in \Omega : \ w(x) > k\}$. Then the norm of w in $L^\infty(\Omega)$ is bounded by a constant which depends on C, γ, δ, N, ε_1, k_0 and $|\Omega|$.

Lemma 4.6. *Let $0 < \gamma < p-1$. Suppose that the hypotheses of Theorem 2.4 are satisfied. Then the sequence $\{u_n\}_n$ is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let us start with the estimate in $L^\infty(\Omega)$. For x in $\mathbb{R}$ and for $k > 0$, define

$$G_k(x) = (|x| - k)_+ sign(x) = x - T_k(x).$$

For $k > 0$, if we take $G_k(u_n)$ as a test function in (3.2), we get

$$\int_\Omega a(x, T_n(u_n))\widehat{a}(x, u_n, \nabla u_n) \cdot \nabla u_n G_k'(u_n)\, dx + B \int_\Omega \frac{u_n |\nabla u_n|^p}{(u_n + \frac{1}{n})^{\theta+1}} G_k(u_n)\, dx = \int_\Omega f_n G_k(u_n)\, dx.$$

By (1.4), (1.5), dropping the non-negative lower order term and using Hölder's inequality, we obtain

$$\alpha \int_{A_k} \frac{|\nabla u_n|^p}{(1+u_n)^\gamma}\, dx \le C \bigg(\int_{A_k} |G_k(u_n)|^{m'}\, dx \bigg)^{\frac{1}{m'}}, \tag{4.21}$$

where we have set $A_k = \{x \in \Omega : \ |u_n(x)| \ge k\}$. Let $\delta < p$, using Hölder's inequality and (4.21), we have

$$\begin{aligned} \int_{A_k} |\nabla u_n|^\delta\, dx &= \int_{A_k} \frac{|\nabla u_n|^\delta}{(1+u_n)^{\frac{\gamma\delta}{p}}} (1+u_n)^{\frac{\gamma\delta}{p}}\, dx \\ &\le C \bigg(\int_{A_k} |G_k(u_n)|^{m'}\, dx \bigg)^{\frac{\delta}{pm'}} \bigg(\int_{A_k} (1+u_n)^{\frac{\gamma\delta}{p-\delta}}\, dx \bigg)^{\frac{p-\delta}{p}} \end{aligned} \tag{4.22}$$

Choosing δ such that

$$\delta^* = \frac{\delta N}{N-\delta} = m' \Longleftrightarrow \delta = \frac{Nm}{Nm + m - N}, \tag{4.23}$$

it is easy to check that the hypotheses on m imply

$$\delta < \frac{N}{N-p+1} < p. \tag{4.24}$$

From (4.22), (4.23) and Sobolev's inequality, we obtain

$$\int_{A_k} |\nabla u_n|^\delta\, dx \le C \bigg(\int_{A_k} (1+u_n)^{\frac{\gamma\delta}{p-\delta}}\, dx \bigg)^{\frac{p-\delta}{p-1}}. \tag{4.25}$$

By (4.25) and die to $1 + u_n \le 2(k + G_k(u_n))$ on A_k if $k \ge 1$, we have

$$\int_{A_k} |\nabla u_n|^\delta\, dx \le C \bigg[k^{\frac{\gamma\delta}{p-1}} |A_k|^{\frac{p-\delta}{p-1}} + \bigg(\int_{A_k} G_k(u_n)^{\frac{\gamma\delta}{p-\delta}}\, dx \bigg)^{\frac{p-\delta}{p-1}} \bigg]. \tag{4.26}$$

Since $\gamma < p-1$ and (2.5) holds, with our choice of δ we have

$$\frac{\gamma\delta}{p-\delta} < \delta^*. \tag{4.27}$$

By (4.26), (4.27), Hôlder's, Sobolev's and Young's inequalities, one obtains

$$\begin{aligned} \int_{A_k} |\nabla u_n|^\delta\, dx &\le C \bigg[k^{\frac{\gamma\delta}{p-1}} |A_k|^{\frac{p-\delta}{p-1}} + \bigg(\int_{A_k} G_k(u_n)^{\delta^*}\, dx \bigg)^{\frac{\gamma\delta}{(p-1)\delta^*}} |A_k|^{\frac{p-\delta}{p-1} - \frac{\gamma\delta}{(p-1)\delta^*}} \bigg] \\ &\le C \bigg[k^{\frac{\gamma\delta}{p-1}} |A_k|^{\frac{p-\delta}{p-1}} + \bigg(\int_{A_k} |\nabla u_n|^\delta\, dx \bigg)^{\frac{\gamma}{p-1}} |A_k|^{\frac{p-\delta}{p-1} - \frac{\gamma\delta}{(p-1)\delta^*}} \bigg] \\ &\le C \bigg[k^{\frac{\gamma\delta}{p-1}} |A_k|^{\frac{p-\delta}{p-1}} + \varepsilon \int_{A_k} |\nabla u_n|^\delta\, dx + \varepsilon(p,\gamma) |A_k|^{\frac{(p-\delta)\delta^* - \gamma\delta}{\delta^*(p-1-\gamma)}} \bigg]. \end{aligned} \tag{4.28}$$

If we choose $\varepsilon = \frac{1}{2C}$, then we can take on the right hand side in (4.28) the term containing the gradient, obtaining

$$\int_{A_k} |\nabla u_n|^\delta \, dx \le C\Big[k^{\frac{\gamma\delta}{p-1}} |A_k|^{\frac{p-\delta}{p-1}} + |A_k|^{\frac{(p-\delta)\delta^* - \gamma\delta}{\delta^*(p-1-\gamma)}}\Big]. \tag{4.29}$$

Now, (4.24) implies

$$\frac{(p-\delta)}{p-1} < \frac{(p-\delta)\delta^* - \gamma\delta}{\delta^*(p-1-\gamma)}. \tag{4.30}$$

Using (4.29), (4.30) and the fact that $|A_k| \le |\Omega|$, we get

$$\int_{A_k} |\nabla u_n|^\delta \, dx \le C_{22} k^{\frac{\gamma\delta}{p-1}} |A_k|^{\frac{p-\delta}{p-1}}.$$

Let now $\varepsilon_1 = \frac{p-\delta}{p-1} - \frac{\delta}{\delta^*} > 0$. Therefore, by Lemma 4.5, we find that $w = u_n$ is bounded in $L^\infty(\Omega)$.

The estimate in $W_0^{1,p}(\Omega)$ is now very easy. Taking u_n as a test function in (3.2), using hypotheses (1.4), (1.5), dropping the non-negative lower order term and using $\|u_n\|_{L^\infty(\Omega)} \le c$, we get

$$\int_\Omega |\nabla u_n|^p \, dx \le \frac{c\alpha}{(1+c)^\gamma} \int_\Omega f \, dx$$

and the right-hand side is trivially bounded, since f belongs to $L^1(\Omega)$. □

5 Proof of main results

5.1 Proof of Theorem 2.1

By Lemma 4.2, the sequence $\{u_n\}_n$ is bounded in $W_0^{1,\eta}(\Omega)$. Therefore, there exists a function $u \in W_0^{1,\eta}(\Omega)$ such that (up to a subsequence)

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,\eta}(\Omega), \\ u_n \to u & \text{a.e. in } \Omega. \end{cases} \tag{5.1}$$

Now, we have to prove

$$\nabla u_n \to \nabla u \ \text{ a.e. in } \Omega. \tag{5.2}$$

Let $h, k > 0$. We use $T_h(u_n - T_k(u))$ as a test function in (3.2), by hypothesis (1.4) and estimate (4.1), we get

$$\int_\Omega \frac{\alpha}{(1+T_n(u_n))^\gamma} \widehat{a}(x, u_n, \nabla u_n) \nabla T_h(u_n - T_k(u)) \, dx \le Ch, \tag{5.3}$$

whence by virtue of $\frac{1}{(1+u_n)^\gamma} \le \frac{1}{(1+T_n(u_n))^\gamma}$, (1.7) and (5.3), it follows that

$$\begin{aligned}
&\int_{\{|u_n - T_k(u)| \le h, |u| \le k\}} \frac{|\nabla(u_n - T_k(u))|^p}{(1+u_n)^\gamma} \, dx \\
&\le \int_{\{|u_n - T_k(u)| \le h, |u| \le k\}} C_p \frac{(\widehat{a}(x, u_n, \nabla u_n) - \widehat{a}(x, u_n, \nabla T_k(u))) \nabla T_h(u_n - T_k(u))}{(1+T_n(u_n))^\gamma} \, dx \\
&\le C_p \frac{Ch}{\alpha} - C_p \int_{\{|u_n - T_k(u)| \le h, |u| \le k\}} \frac{\widehat{a}(x, u_n, \nabla T_k(u)) \nabla T_h(u_n - u)}{(1+T_n(u_n))^\gamma} \, dx \\
&= -C_p \int_{\{|u_n - T_k(u)| \le h, |u| \le k\}} \frac{\widehat{a}(x, T_{h+k}(u_n), \nabla T_k(u)) \nabla T_h(u_n - u)}{(1+T_n(u_n))^\gamma} \, dx + C_p \frac{Ch}{\alpha}.
\end{aligned} \tag{5.4}$$

Combining (1.6), (4.8) and (5.1), we obtain

$$\begin{cases} \nabla T_h(u_n-u) \rightharpoonup 0 & \text{in } L^p(\Omega), \\ \widehat{a}(x, T_{k+h}(u_n), \nabla T_k(u)) \to \widehat{a}(x, u, \nabla T_k(u)) & \text{in } L^{p'}(\Omega). \end{cases} \tag{5.5}$$

According to (5.4) and (5.5), we have

$$\limsup_{n\to+\infty} \int\limits_{\{|u_n-T_k(u)|\le h, |u|\le k\}} \frac{|\nabla(u_n-T_k(u))|^p}{(1+u_n)^\gamma}\,dx \le Ch.$$

This implies

$$\limsup_{n\to+\infty} \int\limits_{\{|u_n-T_k(u)|\le h, |u|\le k\}} \big|\nabla(u_n-T_k(u))\big|^p\,dx \le Ch(1+k+h)^\gamma. \tag{5.6}$$

Let now τ be such that $1<\tau<\eta<p$. It is clear that

$$\int\limits_\Omega |\nabla(u_n-u)|^\tau\,dx \le \int\limits_{\{|u_n-u|\le h, |u|\le k\}} |\nabla(u_n-u)|^\tau\,dx + \int\limits_{\{|u_n-u|\le h, |u|> k\}} |\nabla(u_n-u)|^\tau\,dx + \int\limits_{\{|u_n-u|> h\}} |\nabla(u_n-u)|^\tau\,dx.$$

By Hölder's inequality and the fact that u_n is uniformly bounded in $W_0^{1,\eta}(\Omega)$, we obtain

$$\int\limits_\Omega |\nabla(u_n-u)|^\tau\,dx \le C\Big(\int\limits_{\{|u_n-u|\le h, |u|\le k\}} |\nabla(u_n-u)|^p\,dx\Big)^{\frac{\tau}{p}} + C\Big(\big(\mu(\{|u|>k\})\big)^{1-\frac{\tau}{\eta}} + \big(\mu(\{|u_n-u|>h\})\big)^{1-\frac{\tau}{\eta}}\Big). \tag{5.7}$$

Thus we deduce from (5.6) and (5.7) that

$$\limsup_{h\to 0}\limsup_{n\to\infty} \int\limits_\Omega |\nabla(u_n-u)|^\tau\,dx \le C\big(\mu(\{|u|>k\})\big)^{1-\frac{\tau}{\eta}}, \quad \forall\, k>0.$$

At the limit as $k\to+\infty$, $\mu(\{|u|>k\})$ converges to 0. Therefore (up to subsequences), $\nabla u_n \to \nabla u$ a.e. in Ω.

Remark 5.1. The technique used for the proof of (5.2) under the hypotheses of Theorem 2.1 is the same as compared to Theorem 2.2 and Theorem 2.3.

Now, we are going to prove the strict positivity of the weak limit u of the sequence of approximated solutions u_n.

Lemma 5.1. *Let $0<\theta<1$. Let u_n and u be as in (5.2). Then $u>0$.*

Proof. For $s\ge 0$, define

$$H_n(s) = \int\limits_0^s \frac{t(1+T_n(t))^\gamma}{\alpha(t+\frac{1}{n})^{\theta+1}}\,dt, \quad H_\infty(s) = \int\limits_0^s \frac{(1+t)^\gamma}{\alpha t^\theta}\,dt.$$

Observe that H is well-defined, since $\theta<1$. Let $\phi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ be a positive function. We choose $e^{-BH_n(u_n)}\phi$ as a test function in (3.1). Using hypotheses (1.4), (1.5) and $f_n\ge T_1(f), \forall\, n\ge 1$, we get

$$\beta\int\limits_\Omega \widehat{a}(x,u_n,\nabla u_n)\cdot\nabla\phi e^{-BH_n(u_n)}\,dx \ge \int\limits_\Omega T_1(f)e^{-BH_n(u_n)}\phi\,dx. \tag{5.8}$$

Now, let us define for $\lambda > 0$, the function

$$\psi_\lambda(t) = \begin{cases} 1 & \text{if } 0 \le t < 1, \\ -\dfrac{1}{\lambda}(t-1-\lambda) & \text{if } 1 \le t < \lambda+1, \\ 0 & \text{if } \lambda+1 \le t, \end{cases}$$

and fix a function φ in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \ge 0$. Taking $\phi = \psi_\lambda(u_n)\varphi$ in (5.8), using (1.5) and $\psi_\lambda'(t) \le 0$, we obtain

$$\int_\Omega T_1(f)\varphi\psi_\lambda(u_n)e^{-BH_n(u_n)}\chi_{\{0\le u_n\le\lambda+1\}}\,dx \le \beta\int_\Omega \widehat{a}(x,u_n,\nabla u_n)\cdot\nabla\varphi\psi_\lambda(u_n)e^{-BH_n(u_n)}\chi_{\{0\le u_n\le\lambda+1\}}\,dx. \tag{5.9}$$

Then, letting $\lambda \to 0$, Lebesgue's theorem yields

$$\psi_\lambda(u_n)\chi_{\{0\le u_n\le\lambda+1\}} \to \chi_{\{0\le u_n\le 1\}} \quad \text{in } L^1(\Omega). \tag{5.10}$$

Equations (5.9) and (5.10) imply that

$$\beta\int_{\{0\le u_n\le 1\}} \widehat{a}\big(x,T_1(u_n),\nabla T_1(u_n)\big)\cdot\nabla\varphi e^{-BH_n(T_1(u_n))}\,dx \ge \int_{\{0\le u_n\le 1\}} T_1(f)\varphi e^{-BH_n(T_1(u_n))}\,dx. \tag{5.11}$$

According to (1.6), (4.8), (5.1) and (5.2), we have

$$\widehat{a}\big(x,T_1(u_n),\nabla T_1(u_n)\big) \rightharpoonup \widehat{a}\big(x,T_1(u),\nabla T_1(u)\big) \quad \text{in } L^{p'}(\Omega). \tag{5.12}$$

Now, we pass to the limit as $n \to +\infty$ in (5.11) and deduce from (5.12) that

$$\int_\Omega \widehat{a}\big(x,T_1(u),\nabla T_1(u)\big)\cdot\nabla\varphi e^{-BH_\infty(T_1(u))}\,dx \ge \frac{1}{\beta}\int_{\{0\le u\le 1\}} T_1(f)\varphi e^{-BH_\infty(T_1(u))}\,dx \tag{5.13}$$

for all φ in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi \ge 0$, and then, by density, for every non-negative φ in $W_0^{1,p}(\Omega)$. Now, we define the function

$$P(t) = \int_0^t e^{-BH_\infty(s)}\,ds,$$

inequality (5.13) is equivalent to

$$\int_\Omega M(x,v,\nabla v)\cdot\nabla\varphi\,dx \ge \frac{1}{\beta}\int_\Omega g(x)\varphi\,dx,$$

where

$$M(x,s,\xi) = e^{-BH_\infty(T_1(u))}\,\widehat{a}\left(x, sT_1(u)\left(\int_0^{T_1(u)} e^{-BH_\infty(s)}\,ds\right)^{-1}, \frac{\xi}{e^{-BH_\infty(T_1(u))}}\right),$$

$$g(x) = \frac{1}{\beta}T_1(f)e^{-BH_\infty(1)}\chi_{\{0\le u(x)\le 1\}}, \quad v = P(T_1(u)).$$

The comparison principle in $W_0^{1,p}(\Omega)$ says that $v(x) \ge z(x)$ (see [7]), where z is the bounded weak solution of

$$-\operatorname{div}(M(x,z,\nabla z)) = g(x), \quad z \in W_0^{1,p}(\Omega).$$

Indeed, using (1.5)–(1.7), for almost every $x \in \Omega$ and for every ξ, ξ' in $\mathbb{R}^N$, M satisfies

$$M(x,s,\xi)\cdot\xi \geq |\xi|^p, \quad |M(x,s,\xi)| \leq C_1|s|^{p-1} + C_2|\xi|^{p-1} + C_3,$$
$$\big(\widehat{a}(x,s,\xi) - \widehat{a}(x,s,\xi')\big)(\xi-\xi') \geq 0.$$

To check this, from (1.5), $p \geq 2$ and that $u \geq 0$ a.e. in Ω, we have

$$\begin{aligned} M(x,s,\xi)\cdot\xi &= \frac{e^{-2BH_\infty(T_1(u))}}{e^{-BH_\infty(T_1(u))}}\,\widehat{a}\Bigg(x, sT_1(u)\Bigg(\int\limits_0^{T_1(u)} e^{-BH_\infty(s)}\,ds\Bigg)^{-1}, \frac{\xi}{e^{-BH_\infty(T_1(u))}}\Bigg)\cdot\xi \\ &\geq e^{(p-2)BH_\infty(T_1(u))}|\xi|^p \geq |\xi|^p. \end{aligned}$$

In view of (1.6) and $\theta < p$, we get

$$\begin{aligned} |M(x,s,\xi)| &= e^{-BH_\infty(T_1(u))}\Bigg|\widehat{a}\Bigg(x, sT_1(u)\Bigg(\int\limits_0^{T_1(u)} e^{-BH_\infty(s)}\,ds\Bigg)^{-1}, \frac{\xi}{e^{-BH_\infty(T_1(u))}}\Bigg)\Bigg| \\ &\leq Ce^{-BH_\infty(T_1(u))}|T_1(u)|^{\frac{\theta(p-1)}{p}}\Bigg(\int\limits_0^{T_1(u)} e^{-BH_\infty(s)}\,ds\Bigg)^{\frac{-\theta(p-1)}{p}}|s|^{\frac{\theta(p-1)}{p}} + C'e^{(p-2)BH_\infty(T_1(u))}|\xi|^{p-1} \\ &\leq Ce^{-BH_\infty(T_1(u))}\,\frac{|T_1(u)|^{\frac{\theta(p-1)}{p}}}{\Big(\int\limits_0^{T_1(u)} e^{-BH_\infty(s)}\,ds\Big)^{\frac{\theta(p-1)}{p}}}\,(|s|^{p-1}+1) + C'e^{(p-2)BH_\infty(T_1(u))}|\xi|^{p-1}. \end{aligned} \tag{5.14}$$

Since H_∞ is increasing, we obtain

$$\frac{|T_1(u)|^{\frac{\theta(p-1)}{p}}}{\Big(\int\limits_0^{T_1(u)} e^{-BH_\infty(s)}\,ds\Big)^{\frac{\theta(p-1)}{p}}} \leq \frac{1}{\big(e^{-BH_\infty(T_1(u))}\big)^{\frac{\theta(p-1)}{p}}}.$$

By this inequality and (5.14), we obtain

$$|M(x,s,\xi)| \leq C_1|s|^{p-1} + C_2|\xi|^{p-1} + C_3.$$

Finally, thanks to (1.7), we can write

$$\big(M(x,s,\xi) - M(x,s,\xi')\big)\cdot(\xi-\xi') \geq C_p e^{(p-2)BH_\infty(T_1(u))}|\xi-\xi'| \geq 0.$$

Since g is non-negative and not identically zero, the weak Harnack inequality (see [7]) yields $z > 0$ in Ω and so, $v > 0$. Since $T_1(u) \geq v$, we conclude that $T_1(u) > 0$ in Ω, which then implies that $u > 0$ in Ω. □

Corollary 5.1. *Let $0 < \theta < 1$. We have $\frac{|\nabla u|^p}{u^\theta} \in L^1(\Omega)$.*

In fact, by passing to the limit in (4.1), we deduce from (5.1), Lemma 5.1 and Fatou's lemma that

$$B\int\limits_\Omega \frac{|\nabla u|^p}{u^\theta}\,dx \leq \int\limits_\Omega f\,dx. \tag{5.15}$$

5.2 Passage to the limit

Let us define

$$H_{\frac{1}{n}}(t) = \int\limits_0^t \frac{B(1+s)^\gamma}{\alpha(s+\frac{1}{n})^\theta}\,ds, \quad H_0(t) = \int\limits_0^t \frac{B(1+s)^\gamma}{\alpha s^\theta}\,ds, \quad t \geq 0, \quad n \in \mathbb{N}.$$

For $k \in \mathbb{N}$, we use

$$R_k(s) = \begin{cases} 1 & \text{if } s \le k, \\ k+1-s & \text{if } k \le s \le k+1, \\ 0 & \text{if } s > k+1. \end{cases}$$

Consider

$$v = e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n)\varphi,$$

where $j \in \mathbb{N}$ and φ is a positive $C_0^1(\Omega)$ function, as a test function in (3.2). Then

$$\begin{aligned}
&\int_\Omega a(x,T_n(u_n))\widehat{a}\big(x, T_{k+1}(u_n), \nabla T_{k+1}(u_n)\big) \cdot \nabla\varphi e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n)\, dx \\
&\qquad + \frac{B}{\alpha} \int_\Omega a(x,T_n(u_n))\widehat{a}\big(x, T_{k+1}(u_n), \nabla T_{k+1}(u_n)\big) \nabla T_j(u)\, \frac{(1+T_j(u))^\gamma}{(T_j(u)+\frac{1}{j})^\theta} \\
&\qquad\qquad \times e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n)\varphi\, dx \\
&= \frac{B}{\alpha} \int_\Omega \Big[a(x,T_n(u_n))\widehat{a}\big(x, T_{k+1}(u_n), \nabla T_{k+1}(u_n)\big) \cdot \nabla u_n \frac{(1+u_n)^\gamma}{(u_n+\frac{1}{n})^\theta} - \alpha \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}\Big] \\
&\qquad\qquad \times e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n)\varphi\, dx \\
&\qquad + \int_\Omega T_n(f) e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u_n)\varphi\, dx \\
&\qquad - \int_\Omega a(x,T_n(u_n))\widehat{a}(x, u_n, \nabla u_n) \cdot \nabla u_n e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k'(u_n)\varphi\, dx.
\end{aligned} \tag{5.16}$$

Using (1.4), (1.5) and $R_k'(s) \le 0$, we have

$$-\int_\Omega a(x,T_n(u_n))\widehat{a}(x, u_n, \nabla u_n) \cdot \nabla u_n e^{-H_{\frac{1}{n}}(u_n)} e^{H_{\frac{1}{j}}(T_j(u))} R_k'(u_n)\varphi\, dx \ge 0. \tag{5.17}$$

Combining (1.4) and (1.5), we get

$$\begin{aligned}
&\Big[a(x,T_n(u_n))\widehat{a}(x, u_n, \nabla u_n) \cdot \nabla u_n \, \frac{(1+u_n)^\gamma}{(u_n+\frac{1}{n})^\theta} - \alpha\, \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}\Big] \\
&\qquad\qquad \ge \alpha\, \frac{|\nabla u_n|^p}{(u_n+\frac{1}{n})^\theta} \Big(1 - \frac{u_n}{u_n+\frac{1}{n}}\Big) \ge 0.
\end{aligned} \tag{5.18}$$

Letting $n \to +\infty$, using (4.8), (5.16)–(5.18) and Fatou's lemma, we have

$$\begin{aligned}
&\int_\Omega a(x,u)\widehat{a}(x, u, \nabla u) \cdot \nabla\varphi e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\, dx \\
&\qquad + \frac{B}{\alpha} \int_\Omega a(x,u)\widehat{a}(x, u, \nabla u)\nabla T_j(u)\, \frac{(1+T_j(u))^\gamma}{(T_j(u)+\frac{1}{j})^\theta}\, e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\varphi\, dx \\
&\ge \frac{B}{\alpha} \int_\Omega a(x,u)\widehat{a}(x, u, \nabla u) \cdot \nabla u\, \frac{(1+u)^\gamma}{u^\theta}\, e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\varphi\, dx \\
&\qquad - B \int_\Omega \frac{|\nabla u|^p}{u^\theta}\, e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\varphi\, dx + \int_\Omega f e^{-H_0(u)} e^{H_{\frac{1}{j}}(T_j(u))} R_k(u)\varphi\, dx.
\end{aligned} \tag{5.19}$$

Let $j > k+1$. Using (1.4), (1.6), (5.15) and $R_k(u) = 0$ on $\{u > k+1\}$, we get

$$\begin{aligned}
&\Big| a(x,u)\widehat{a}(x,u,\nabla u)\nabla T_j(u)\,\frac{(1+T_j(u))^\gamma}{(T_j(u)+\frac{1}{j})^\theta}\,e^{-H_0(u)}e^{H_{\frac{1}{j}}(T_j(u))}R_k(u)\varphi\Big| \\
&\qquad \le \beta\Big(C|u|^{\frac{\theta(p-1)}{p}} + C'|\nabla u|^{p-1}\Big)|\nabla u|\,\frac{(1+u)^\gamma}{(u+\frac{1}{j})^\theta}\,R_k(u)\varphi \\
&\qquad \le \beta\Big(C|u|^{\theta} + C'|\nabla u|^{p-1}\Big)|\nabla u|\,\frac{(1+u)^\gamma}{u^\theta}\,R_k(u)\varphi \\
&\qquad \le \beta\Big(C|\nabla u| + C'\,\frac{|\nabla u|^p}{u^\theta}\Big)(1+u)^\gamma R_k(u)\varphi \in L^1(\Omega).
\end{aligned} \tag{5.20}$$

Passing first to the limit as $j \to \infty$ in (5.19), using that $e^{-H_0(u)}e^{H_{\frac{1}{j}}(T_j(u))} \le 1$ (since $H_{\frac{1}{j}}(T_j(u)) \le H_{\frac{1}{j}}(u) \le H_0(u)$), (5.20) and Lebesgue's theorem, and then to the limit as $k \to +\infty$, we obtain

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla\varphi\,dx + B\int_\Omega \frac{|\nabla u|^p}{u^\theta}\varphi\,dx \ge \int_\Omega f\varphi\,dx. \tag{5.21}$$

To prove the opposite inequality, we choose $\varphi \in C_0^1(\Omega)$ with $\varphi \ge 0$ as a test function in (3.2), to obtain

$$\int_\Omega a(x,T_n(u_n))\widehat{a}(x,u_n,\nabla u_n)\nabla\varphi\,dx + B\int_\Omega \frac{u_n|\nabla u_n|^p}{(u_n+\frac{1}{n})^{\theta+1}}\,\varphi\,dx = \int_\Omega f_n\varphi\,dx. \tag{5.22}$$

From (1.6), (5.1), (5.2) and Lemma 4.3, we have

$$\widehat{a}(x,u_n,\nabla u_n) \to \widehat{a}(x,u,\nabla u) \ \text{ in } L^\delta(\Omega), \ \ \forall\,\delta \in \Big(1, \frac{\eta}{p-1}\Big). \tag{5.23}$$

Therefore, (5.22), (5.23) and Fatou's lemma imply

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla\varphi\,dx + B\int_\Omega \frac{|\nabla u|^p}{u^\theta}\,\varphi\,dx \le \int_\Omega f\varphi\,dx. \tag{5.24}$$

Combining (5.21) and (5.24), we deduce that

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla\varphi\,dx + \int_\Omega \frac{|\nabla u|^p}{u^\theta}\,\varphi\,dx = \int_\Omega f\varphi\,dx \tag{5.25}$$

for every φ in $C_0^1(\Omega)$ with $\varphi \ge 0$. Now, let φ be any function from $C_0^1(\Omega)$ and $\varepsilon > 0$. We define $\varphi_\pm^\varepsilon = \rho^\varepsilon * \varphi_\pm$ as the convolution of a modifier ρ^ε with $\varphi_\pm$. Then $\varphi_\pm^\varepsilon$ is a positive $C_0^1(\Omega)$ function for ε sufficiently small. By (5.25), we have

$$\int_\Omega a(x,u)\widehat{a}(x,u,\nabla u)\nabla(\varphi_-^\varepsilon - \varphi_+^\varepsilon)\,dx + \int_\Omega \frac{|\nabla u|^p}{u^\theta}\,(\varphi_-^\varepsilon - \varphi_+^\varepsilon)\,dx = \int_\Omega f(\varphi_-^\varepsilon - \varphi_+^\varepsilon)\,dx.$$

Since $\varphi_-^\varepsilon - \varphi_+^\varepsilon \to \varphi$ uniformly in Ω and in $W_0^{1,\eta}(\Omega)$ for every $\eta \ge 1$, as $\varepsilon \to 0$, the results follow.

5.3 Proof of Theorem 2.2

By virtue of Lemma 4.2 and Lemma 4.3, there exists a function u belonging to $W_0^{1,\sigma}(\Omega)$ such that, up to the subsequences,

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,\sigma}(\Omega), \ \text{ and a.e. in } \Omega\,, \\ T_k(u_n) \rightharpoonup T_k(u) & \text{in } W_0^{1,p}(\Omega), \ \text{ and a.e. in } \Omega\,. \end{cases} \tag{5.26}$$

Moreover, by repeating the same technique used in the proof of Lemma 5.1, it follows that $u > 0$ in Ω. Corollary 5.1 ensures that $\frac{|\nabla u|^p}{u^\theta}$ belongs to $L^1(\Omega)$. From Remark 5.1, using (5.26) and

$$\widehat{a}(x, u_n, \nabla u_n) \to \widehat{a}(x, u, \nabla u) \ \text{ in } L^\delta(\Omega), \ \ \forall\, \delta \in \Big(1, \frac{\sigma}{p-1}\Big), \tag{5.27}$$

we can pass to the limit in (2.1) exactly as in the proof of Theorem 2.1 to conclude that u is a distributional solution of problem (1.1).

5.4 Proof of Theorems 2.3 and 2.4

In order to prove these theorems, we modify the proof of Theorem 2.1. We replace (5.26) by

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,p}(\Omega), \text{ and a.e. in } \Omega\,, \\ T_k(u_n) \rightharpoonup T_k(u) & \text{in } W_0^{1,p}(\Omega), \text{ and a.e. in } \Omega\,, \end{cases}$$

and (5.27) by

$$\widehat{a}(x, u_n, \nabla u_n) \to \widehat{a}(x, u, \nabla u) \ \text{ in } L^\delta(\Omega), \ \ \forall\, \delta \in \Big(1, \frac{p}{p-1}\Big).$$

Using the last convergence and (5.2), we can pass to the limit in (5.22) to obtain (2.1). Thus Theorem 2.3 and Theorem 2.4 are proved.

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