Memoirs on Differential Equations and Mathematical Physics

Volume 90, 2023, 97–110

Yūki Naito

MULTIPLICITY OF SINGULAR SOLUTIONS TO A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS

Dedicated to Professor Kusano Takaŝi
on the occasion of his 90th birthday

Abstract. We consider the multiplicity of the singular radial solutions to the equation of the form $\rho^{-1}\operatorname{div}(\rho\nabla u)+\lambda u+u^p=0$ in $\mathbf{R}^N\setminus\{0\}$, where $N\geq 3$, $\rho\in C^1[0,\infty)$, $\rho>0$, on $[0,\infty)$, $\lambda\in C[0,\infty)$ and $p>N/(N-2)$.

2020 Mathematics Subject Classification. 35J61, 35A24, 35A02.

Key words and phrases. Singular solutions, uniqueness, infinitely many existences.

რეზიუმე. განხილულია $\mathbf{R}^N\setminus\{0\}$-ში $\rho^{-1}\operatorname{div}(\rho\nabla u)+\lambda u+u^p=0$ სახის განტოლების სინგულარული რადიალური ამონახსნების სიმრავლე, სადაც $N\geq 3$, $\rho\in C^1[0,\infty)$, $\rho>0$ on $[0,\infty)$, $\lambda\in C[0,\infty)$ and $p>N/(N-2)$.

1 Introduction

We consider singular solutions of the the following ordinary differential equation:

$$u'' + \Big(\frac{N-1}{r} + h(r)\Big)u' + \lambda(r)u + u^p = 0 \ \text{ for } \ r > 0, \tag{1.1}$$

where $N \geq 3$, $h, \lambda \in C[0,\infty)$ and $p > 1$. By a singular solution $u(r)$ of (1.1), we mean that $u(r)$ is a classical solution of (1.1) for $r > 0$ and it satisfies $u(r) \to \infty$ as $r \to 0$. This problem comes from the study of radial singular solutions of the semilinear elliptic equation

$$\rho(|x|)^{-1} \operatorname{div}\big(\rho(|x|)\nabla u\big) + \lambda(|x|)u + u^p = 0 \ \text{ in } \mathbf{R}^N \setminus \{0\}, \tag{1.2}$$

where $\rho \in C^1[0,\infty)$ satisfies $\rho(r) > 0$ for $r \geq 0$. Define $h(r) = \rho'(r)/\rho(r)$. Then the equation is reduced to (1.1). Eq. (1.2) was studied in [3, 10, 11]. Typical examples of the equation of form (1.1) are

$$u'' + \Big(\frac{N-1}{r} + \frac{r}{2}\Big)u' + \frac{1}{p-1}\,u + u^p = 0 \ \text{ for } \ r > 0 \tag{1.3}$$

and

$$u'' + \Big(\frac{N-1}{r} - \frac{r}{2}\Big)u' - \frac{1}{p-1}\,u + u^p = 0 \ \text{ for } \ r > 0. \tag{1.4}$$

Equations (1.3) and (1.4) appear in the study of self-similar solutions to the nonlinear heat equation

$$w_t = \Delta w + w^p \tag{1.5}$$

for $x \in \mathbf{R}^N$ and $t > 0$. Let $u(r)$ be a solution of (1.3) and put

$$w(x,t) = t^{-1/(p-1)} u\Big(\frac{|x|}{\sqrt{t}}\Big).$$

Then w solves (1.5) for $t > 0$, and w is called a forward self-similar solution. Now, let $u(r)$ be a solution of (1.4) and put

$$w(x,t) = t^{-1/(p-1)} u\Big(\frac{|x|}{\sqrt{T-t}}\Big)$$

with some $T > 0$. Then w solves (1.5) for $t < T$, and w is called a backward self-similar solution. It is well known that self-similar solutions play an important role in the study of the behavior of solutions to (1.5), and equations (1.3) and (1.4) have been widely studied by many authors (see, e.g., [1, 2, 4–9, 11] and the references therein).

Let us mention that, in the case $p > N/(N-2)$, both (1.3) and (1.4) possess the singular solution

$$U_A(r) = Ar^{-2/(p-1)}, \ \text{ where } \ A = \Big\{\frac{2}{p-1}\Big(N-2-\frac{2}{p-1}\Big)\Big\}^{1/(p-1)}. \tag{1.6}$$

It was shown by Quittner [7] that if u is a singular solution of (1.3) or (1.4) for $r > 0$, then $u(r) \equiv U_A(r)$ provided $p > (N+2)/(N-2)$. See also the previous result in [4] for (1.4). It was also shown by [7] that, in the case $p = (N+2)/(N-2)$, if u is a positive singular solution of (1.3) or (1.4) such that the number of sign changes of $u(r) - U_A(r)$ is finite, then $u(r) \equiv U_A(r)$. On the other hand, in the case $N/(N-2) < p < (N+2)/(N-2)$, the non-uniqueness of singular solutions was shown in [8, 9].

In this paper, we consider the multiplicity of singular solutions to the generalized equation (1.1) in the case $p > N/(N-2)$. First, we show the uniqueness of the singular solution to (1.1) in the case $p > (N+2)/(N-2)$. We note that $U_A(r)$ in (1.6) solves

$$u'' + \frac{N-1}{r}\,u' + u^p = 0 \ \text{ for } \ r > 0.$$

Then (1.1) has the singular solution $U_A(r)$ provided

$$h(r) = \frac{p-1}{2}\,\lambda(r)r \ \text{ for } \ r \geq 0, \tag{1.7}$$

since $U_A(r)$ satisfies $h(r)U_A'(r) + \lambda(r)U_A(r) = 0$ if (1.7) holds.

Theorem 1.1. *Let $p > (N+2)/(N-2)$ and let $r_0 > 0$. Then (1.1) has at most one singular solution for $0 < r \le r_0$. In particular, if (1.7) holds, (1.1) has a unique singular solution $U_A(r)$ for $r > 0$.*

In the case $p = (N+2)/(N-2)$, we obtain the following Liouville type result for singular solutions.

Theorem 1.2. *Let $p = (N+2)/(N-2)$ and let $r_0 > 0$. Assume that h and λ satisfy (1.7). If u is a singular solution of (1.1) for $0 < r \le r_0$ such that the number of sign changes of $u(r) - U_A(r)$ is finite for $0 < r \le r_0$, then $u(r) \equiv U_A(r)$.*

Finally, we consider the case where $N/(N-2) < p < (N+2)/(N-2)$.

Theorem 1.3. *Let $N/(N-2) < p < (N+2)/(N-2)$ and let $r_0 > 0$. Then (1.1) has infinitely many singular solutions of (1.1) for $0 < r \le r_0$. Furthermore, any singular solution u satisfies*

$$u(r) = Ar^{-2/(p-1)}(1 + o(1)) \quad \text{as } r \to 0, \tag{1.8}$$

where A is the constant in (1.6).

The paper is organized as follows. In Section 2, we give some preliminary results, and in Section 3, we investigate the asymptotic behavior of singular solutions. Finally, in Sction4, we give the proof of Theorems 1.1, 1.2 and 1.3.

2 Preliminaries

2.1 Asymptotic estimates of singular solutions

In this subsection, we show the asymptotic estimates of singular solutions. Define $f(r,u)$ by

$$f(r,u) = \lambda(r)u + u^p \quad \text{for } r \ge 0, \quad u \ge 0. \tag{2.1}$$

Take any $r_0 > 0$ and fix it. Since $\lambda(r)$ is bounded on $[0, r_0]$, there exist the constants $u_0 > 0$, $C_f \ge c_f > 0$ such that

$$0 < c_f u^p \le f(r,u) \le C_f u^p \quad \text{for } 0 \le r \le r_0 \text{ and } u \ge u_0. \tag{2.2}$$

Define

$$H(r) = \int_0^r h(s)\,ds \quad \text{for } r \ge 0.$$

Since $h(r)$ is bounded on $[0, r_0]$, there exist the constants $C_H \ge c_H > 0$ such that

$$c_H \le H(r) \le C_H \quad \text{for } 0 \le r \le r_0. \tag{2.3}$$

We note here that (1.1) can be written as

$$\left(r^{N-1}e^{H(r)}u'\right)' + r^{N-1}e^{H(r)}f(r,u) = 0 \quad \text{for } r > 0. \tag{2.4}$$

First, we show the following results.

Lemma 2.1. *Let $p > N/(N-2)$ and let u be a singular solution of (1.1). Assume that*

$$u(r) \ge u_0 \quad \text{for } 0 < r \le r_0, \tag{2.5}$$

where u_0 and r_0 are the constants in (2.2). Then

$$u(r) \le C_1 r^{-2/(p-1)} \quad \text{and} \quad 0 < -u'(r) \le C_2 r^{-(p+1)/(p-1)} \quad \text{for } 0 < r \le r_0, \tag{2.6}$$

where theconstants C_1 and C_2 are independent of u. Furthermore, u satisfies

$$-r^{N-1}e^{H(r)}u'(r) = \int_0^r s^{N-1}e^{H(s)}f(s,u(s))\,ds \quad \text{for } 0 < r \le r_0. \tag{2.7}$$

Proof. First, we show that $u'(r) < 0$ for $0 < r \le r_0$. From (2.2), (2.4) and (2.5), we have

$$\big(r^{N-1}e^{H(r)}u'(r)\big)' = -r^{N-1}e^{H(r)}f(r,u(r)) < 0 \ \text{ for } \ 0 < r \le r_0.$$

Then $r^{N-1}e^{H(r)}u'(r)$ is decreasing in $r \in (0, r_0]$. Assume by a contradiction that there exists $r_1 \in (0, r_0]$ such that $u'(r_1) \ge 0$. Then we have $r^{N-1}e^{H(r)}u'(r) > 0$ for $0 < r < r_1$, and hence $u'(r) > 0$ for $0 < r < r_1$. This implies that $u(r) < u(r_1)$ for $0 < r < r_1$, which contradicts $u(r) \to \infty$ as $r \to 0$. Thus we obtain $u'(r) < 0$ for $0 < r \le r_0$.

Take $r_1 \in (0, r_0)$ arbitrarily. Integrating (2.4) on (r_1, r) with $r \le r_0$, we obtain

$$-r^{N-1}e^{H(r)}u'(r) = -r_1^{N-1}e^{H(r_1)}u'(r_1) + \int_{r_1}^{r} s^{N-1}e^{H(s)}f(s,u(s))\,ds > \int_{r_1}^{r} s^{N-1}e^{H(s)}f(s,u(s))\,ds.$$

Since $r_1 > 0$ is arbitrary, we obtain

$$-r^{N-1}e^{H(r)}u'(r) \ge \int_0^r s^{N-1}e^{H(s)}f(s,u(s))\,ds.$$

From (2.2) and (2.3) it follows that

$$-e^{C_H}r^{N-1}u'(r) \ge e^{c_H}c_f\int_0^r s^{N-1}u(s)^p\,ds \ge e^{c_H}c_f u(r)^p\int_0^r s^{N-1}\,ds = \frac{e^{c_H}c_f}{N}\,u(r)^p r^N.$$

This implies that

$$-\frac{u'(r)}{u(r)^p} \ge C_1 r,$$

where $C_1 = e^{c_H}c_f/(Ne^{C_H}) > 0$. Integrating the above on $[\rho, r]$ and letting $\rho \to 0$, we obtain $u(r)^{1-p} \ge \frac{(p-1)}{2}\,C_1 r^2$, and hence

$$u(r) \le C_2 r^{-2/(p-1)}, \tag{2.8}$$

where $C_2 = ((p-1)C_1/2)^{-1/(p-1)}$. Next, we show that

$$\liminf_{r\to 0}(-r^{N-1}u'(r)) = 0. \tag{2.9}$$

Assume to the contrary that $\liminf_{r\to 0}(-r^{N-1}u'(r)) = c > 0$. Then there exists $r_1 > 0$ such that

$$-r^{N-1}u'(r) \ge \frac{c}{2} \ \text{ for } \ 0 < r \le r_1.$$

Integrating the above on $[r, r_1]$, we obtain

$$u(r) \ge u(r_1) + \frac{c}{2(N-2)}\,(r^{2-N} - r_1^{2-N}). \tag{2.10}$$

Since $p > N/(N-2)$, we have $2/(p-1) < N-2$. Hence (2.10) contradicts (2.8). Thus we obtain (2.9).

By (2.9), there exists $r_k \to 0$ such that $r_k^{N-1}u'(r_k) \to 0$ as $k \to \infty$, Integrating (2.4) on $[r_k, r]$ and letting $k \to \infty$, we obtain (2.7). From (2.2), (2.3) and (2.8), we obtain

$$-e^{c_H}r^{N-1}u'(r) \le e^{C_H}C_f\int_0^r s^{N-1}u(s)^p\,ds \le e^{C_H}C_fC_2^p\int_0^r s^{N-1-2p/(p-1)}\,ds = C_3 r^{N-2p/(p-1)}$$

with $C_3 = e^{C_H}C_fC_2^p/(N-2p/(p-1))$. Thus we obtain (2.6). $\square$

2.2 Change of variables

Let u be a positive solution of (1.1) for $0 < r \le r_0$ with some $r_0 > 0$. Define

$$w(t) = r^{2/(p-1)}u(r) \ \text{ with } \ t = -\log r. \tag{2.11}$$

Then w satisfies

$$w'' - a(t)w' - b(t)w + w^p = 0 \ \text{ for } \ t \ge t_0, \tag{2.12}$$

where $t_0 = -\log r_0$,

$$a(t) = N - 2 - \frac{4}{p-1} - h(e^{-t})e^{-t} \tag{2.13}$$

and

$$b(t) = \frac{2}{p-1}\Big(N - 2 - \frac{2}{p-1}\Big) - \frac{2}{p-1}h(e^{-t})e^{-t} + \lambda(r^{-t})e^{-2t}. \tag{2.14}$$

3 Asymptotic behavior of the singular solution

In this section, we show the following result.

Proposition 3.1. *Let $p > N/(N-2)$ and let u be a singular solution of (1.1). Define $w(t)$ by (2.11).*

(i) *If $p \ne (N+2)/(N-2)$, then $w(t)$ satisfies*

$$\lim_{t\to\infty} w(t) = A \ \text{ and } \ \lim_{t\to\infty} w'(t) = 0, \tag{3.1}$$

where A is the constant in (1.6).

(ii) *Let $p = (N+2)/(N-2)$. If $w(t) - A$ has a finite number of zeros on $[t_0, \infty)$, then (3.1) holds.*

In the remaining part of this section we assume that $p > N/(N-2)$. To prove Proposition 3.1, we need a series of lemmas.

Lemma 3.1. *Let u be a singular solution of (1.1). Then*

$$\limsup_{r\to 0} r^{2/(p-1)}u(r) > 0. \tag{3.2}$$

Proof. Assume to the contrary that

$$\lim_{r\to 0} r^{2/(p-1)}u(r) = 0. \tag{3.3}$$

Define $f(r,u)$ by (2.1) and take any $r_0 > 0$. Then there exists $u_0 > 0$ such that (2.2) holds. Since $u(r) \to \infty$ as $r \to 0$, there exists $r_1 \le r_0$ such that $u(r) \ge u_0$ for $0 < r \le r_1$. Then we obtain

$$0 < c_f u(r)^p \le f(r, u(r)) \le C_f u(r)^p \ \text{ for } \ 0 < r \le r_1.$$

First, we show that

$$(r^{2/(p-1)}u(r))' > 0 \ \text{ for } \ 0 < r < r_2 \tag{3.4}$$

with some $r_2 \in (0, r_1]$. Define $w(t)$ by (2.11). Then w satisfies (2.12), where $a(t)$ and $b(t)$ are defined by (2.13) and (2.14), respectively. From (3.3) we have $w(t) \to 0$ as $t \to \infty$. Since $h(r)$ and $\lambda(r)$ are bounded, we have $b(t) \to A^{p-1} > 0$ as $t \to \infty$, where A is the constant in (1.6). Then there exists $t_1 \ge t_0$ such that

$$-b(t) + w(t)^{p-1} < 0 \ \text{ for } \ t \ge t_1.$$

From (2.12) we obtain $w''(t) - a(t)w'(t) > 0$ for $t \ge t_1$. This implies that

$$\Big(e^{-\int_0^t a(s)\,ds} w'(t)\Big)' > 0 \ \text{ for } \ t \ge t_1.$$

Hence $e^{-\int_0^t a(s)\,ds} w'(t)$ is increasing for $t \ge t_1$. Thus we see that either $w'(t) < 0$ for all $t \ge t_1$ or $w'(t) > 0$ for $t \ge t_2$ with some $t_2 \ge t_1$. Since $w(t) > 0$ and $w(t) \to 0$ as $t \to \infty$, the former case has to hold. Then $w'(t) < 0$ for $t \ge t_1$, which implies that (3.4) holds.

From (2.2) and (3.3) we have

$$\frac{r^2 f(r, u(r))}{u(r)} \le C_f r^2 u(r)^{p-1} = C_f \big(r^{2/(p-1)} u(r)\big)^{p-1} \to 0 \ \text{ as } \ r \to 0.$$

Then, for $\varepsilon > 0$ to be determined later, there exists $r_1 \in (0, r_0]$ such that

$$r^2 f(r, u(r)) < \varepsilon u(r) \ \text{ for } \ 0 < r \le r_1.$$

From (2.2), (2.3) and (3.4) we have

$$\begin{aligned}\int_0^r s^{N-1} e^{H(s)} f(s, u(s))\,ds &\le e^{C_H} \varepsilon \int_0^r s^{N-3} u(s)\,ds \\ &\le e^{C_H} \varepsilon r^{2/(p-1)} u(r) \int_0^r s^{N-3-2/(p-1)}\,ds \le \frac{e^{C_H}\varepsilon}{N-2-\frac{2}{p-1}}\, r^{N-2} u(r)\end{aligned}$$

for $0 < r \le r_1$. From (2.3) and (2.7) it follows that

$$-r^{N-1} e^{c_H} u'(r) \le \int_0^r s^{N-1} e^{H(s)} f(s, u(s))\,ds \le \frac{e^{C_H}\varepsilon}{N-2-\frac{2}{p-1}}\, r^{N-2} u(r) \tag{3.5}$$

for $0 < r \le r_1$. Put

$$\sigma = \frac{e^{C_H}\varepsilon}{e^{c_H}(N-2-\frac{2}{p-1})}$$

and take $\varepsilon > 0$ so small that $\sigma < 1/p$. From (3.5) it follows that

$$-r u'(r) \le \sigma u(r) \ \text{ for } \ 0 < r \le r_1.$$

This implies that $(r^\sigma u(r))' \ge 0$ for $0 < r \le r_1$, and hence $r^\sigma u(r) \le r_1^\sigma u(r_1)$ for $0 < r \le r_1$. Then we obtain $u(r) = O(r^{-\sigma})$ as $r \to 0$. From (2.2), (2.3) and (2.7) we obtain

$$-e^{c_H} r^{N-1} u'(r) \le e^{C_H} C_f \int_0^r s^{N-1} u(s)^p\,ds \le e^{C_H} C_f \int_0^r s^{N-1-p\sigma}\,ds = C r^{N-p\sigma}$$

with some constants $C > 0$. Thus $u'(r) = O(r^{1-p\sigma})$ as $r \to 0$. Since $\sigma < 1/p$, we have $u'(r) \to 0$ as $r \to 0$, and hence $\lim_{r \to 0} u(r) < \infty$. This is a contradiction. Thus we obtain (3.2). □

Define

$$a_0 = N - 2 - \frac{4}{p-1}\,, \quad \eta(t) = h(e^{-t})e^{-t} \ \text{ and } \ \mu(t) = \lambda(e^{-t})e^{-2t}.$$

Then $a(t)$ and $b(t)$ in (2.13) and (2.14) can be written as

$$a(t) = a_0 - \eta(t) \ \text{ and } \ b(t) = A^{p-1} - \frac{2}{p-1}\,\eta(t) + \mu(t),$$

where A is the constant in (1.6).

For a solution w of (2.12), define

$$E(w)(t) = \frac{1}{2}\, w'(t)^2 + \Phi(w(t)) \ \text{ for } \ t \ge t_0, \tag{3.6}$$

where

$$\Phi(v) = -\frac{A^{p-1}}{2}\, v^2 + \frac{1}{p+1}\, v^{p+1} \tag{3.7}$$

for $v \geq 0$. We note that

$$\Phi(v) \geq \Phi(A) = -\Big(\frac{1}{2} - \frac{1}{p+1}\Big)A^{p+1} \ \text{ for } \ v \geq 0. \tag{3.8}$$

Observe that $E(w)'(t) = (w''(t) - A^{p-1}w + w^p)w'(t)$ for $t > t_0$. We note here that (2.12) can be written in the form

$$w'' - a(t)w' - \Big(A^{p-1} - \frac{2}{p-1}\,\eta(t) + \mu(t)\Big)w + w^p = 0.$$

Then we have

$$E(w)'(t) = a(t)w'(t)^2 + \Big(-\frac{2}{p-1}\,\eta(t) + \mu(t)\Big)w(t)w'(t) \ \text{ for } \ t > t_0. \tag{3.9}$$

We obtain the following results.

Lemma 3.2. *The function w defined in Proposition 3.1 satisfies the following statements:*

(i) *$w(t)$, $w'(t)$ and $w''(t)$ are bounded on $[t_0, \infty)$.*

(ii) *One has*

$$\int_{t_0}^{\infty} a(s)w'(s)^2\, ds < \infty. \tag{3.10}$$

In particular, $w' \in L^2[t_0, \infty)$ if $p \neq (N+2)/(N-2)$.

(iii) *$\lim_{t\to\infty} E(w)(t) = \zeta$ for some $\zeta \geq \Phi(A)$.*

Proof. From (2.11) we have

$$\frac{d}{dt}\, w(t) = -\frac{2}{p-1}\, r^{2/(p-1)}u(r) - r^{(p+1)/(p-1)}\, \frac{d}{dr}\, u(r).$$

Then by Lemma 2.1 $w(t)$ and $w'(t)$ are bounded on $[t_0, \infty)$. Note that $a(t)$ and $b(t)$ are bounded on $[t_0, \infty)$, since $\eta, \lambda \in C[0, r_0]$. From (2.12), $w''(t)$ is also bounded on $[t_0, \infty)$. Thus (i) holds. Integrating (3.9) on $[t_0, t]$, we have

$$E(w)(t) - E(w)(t_0) = \int_{t_0}^{t} a(s)w'(s)^2\, ds - \frac{2}{p-1}\int_{t_0}^{t} \eta(s)w(s)w'(s)\, ds + \int_{t_0}^{t} \mu(s)w'(s)w(s)\, ds. \tag{3.11}$$

Since $w(t)$ and $w'(t)$ are bounded on $[t_0, \infty)$, $E(w)(t)$ is bounded for $t \geq t_0$. By the definitions of η and μ, we have $\eta, \mu \in L^1[t_0, \infty)$. Then

$$\int_{t_0}^{\infty} \eta(s)w(s)w'(s)\, ds < \infty \ \text{ and } \ \int_{t_0}^{\infty} \mu(s)w'(s)w(s)\, ds < \infty. \tag{3.12}$$

Letting $t \to \infty$ in (3.11), we obtain (3.10). In the case $p \neq (N+2)/(N-2)$, since $a(t) \to a_0 \neq 0$ as $t \to \infty$, we obtain $w' \in L^2[t_0, \infty)$. Thus (ii) holds.

Letting $t \to \infty$ in (3.11) again, from (3.10) and (3.12) we see that the limit of $E(w)(t)$ as $t \to \infty$ exists and is finite. Put $\zeta = \lim_{t\to\infty} E(w)(t)$. From (3.6) and (3.8) we have

$$E(w)(t) \geq \Phi(w(t)) \geq \Phi(A) \ \text{ for } \ t \geq t_0.$$

Letting $t \to \infty$, we obtain $\zeta \geq \Phi(A)$. Thus (iii) holds. □

Lemma 3.3. *Let w be the function defined in Proposition 3.1. If w satisfies*

$$\lim_{t\to\infty} w(t) = \gamma \tag{3.13}$$

for some $\gamma > 0$, then (3.1) holds.

Proof. First, we show that

$$\lim_{t\to\infty} w'(t) = 0. \tag{3.14}$$

From (3.13) and Lemma 3.2 (iii) it follows that

$$\lim_{t\to\infty} \frac{w'(t)^2}{2} = \lim_{t\to\infty} \big(E(w)(t) - \Phi(w(t))\big) = \zeta - \Phi(\gamma). \tag{3.15}$$

Then it suffices to show that $\zeta = \Phi(\gamma)$. Since $E(w) \geq \Phi(w)$, we have $\zeta \geq \Phi(\gamma)$. Assume that $\zeta > \Phi(\gamma)$. Then from (3.15) we obtain

$$|w'(t)| > \frac{\sqrt{2(\zeta - \Phi(\gamma))}}{2} \quad \text{for} \quad t \geq t_1$$

with some $t_1 \geq t_0$, which implies that $\lim_{t\to\infty} |w(t)| = \infty$. This is a contradiction. Thus we obtain $\zeta = \Phi(\gamma)$, and hence (3.14) holds.

Next we show that $\gamma = A$. Assume to the contrary that $\gamma \neq A$. Letting $t \to \infty$ in (2.12), from (3.13) and (3.14) we obtain

$$\lim_{t\to\infty} w''(t) = -A^{p-1}\gamma + \gamma^p \neq 0.$$

Thus we obtain

$$|w''(t)| > \frac{1}{2}\,| - A^{p-1}\gamma + \gamma^p| \quad \text{for} \quad t \geq t_2$$

with some $t_2 \geq t_0$, which implies that $|w'(t)| \to \infty$ as $t \to \infty$. This is a contradiction. Thus we obtain $\gamma = A$ in (3.13). □

Lemma 3.4. *Let w be the function defined in Proposition 3.1. Assume that the limit of $w(t)$ as $t \to \infty$ does not exist. Then the following statements hold:*

(i) *One has $\zeta > \Phi(A)$, where ζ is the constant in Lemma 3.2 (iii).*

(ii) *There exists an infinite sequence $\tau_n \to \infty$ such that $w(\tau_n) = A$ for $n = 1, 2, \ldots$.*

Proof. Since $w(t)$ is positive and bounded for $t \geq t_0$, there exist $0 \leq \gamma_1 < \gamma_2$ such that

$$\liminf_{t\to\infty} w(t) = \gamma_1 \quad \text{and} \quad \limsup_{t\to\infty} w(t) = \gamma_2.$$

Then there exist the sequences $t_n \to \infty$ and $s_n \to \infty$ such that

$$w'(t_n) = w'(s_n) = 0, \quad \lim_{n\to\infty} w(t_n) = \gamma_1 \quad \text{and} \quad \lim_{n\to\infty} w(s_n) = \gamma_2.$$

Thus we obtain

$$E(w)(t_n) = \Phi(w(t_n)) \to \Phi(\gamma_1) \quad \text{and} \quad E(w)(s_n) = \Phi(w(s_n)) \to \Phi(\gamma_2) \quad \text{as} \quad n \to \infty.$$

By Lemma 3.2 (iii), we have $\Phi(\gamma_1) = \Phi(\gamma_2) = \zeta = \lim_{t\to\infty} E(w)(t)$. Since Φ is given by (3.7) and $0 \leq \gamma_1 < \gamma_2$, we conclude that

$$\gamma_1 < A < \gamma_2 \quad \text{and} \quad \Phi(A) < \zeta = \Phi(\gamma_1) = \Phi(\gamma_2).$$

Hence there exists an infinite sequence $\tau_n \to \infty$ such that $w(\tau_n) = A$ for $n = 1, 2, \ldots$. Thus (i) and (ii) hold. □

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1.

(i) By Lemma 3.1, we have $\limsup_{t\to\infty} w(t) > 0$. We show that (3.13) holds with some $\gamma > 0$. Assume to the contrary that the limit of $w(t)$ as $t \to \infty$ does not exist. Then by Lemma 3.4, $\zeta > \Phi(A)$ and there exists a sequence $\tau_n \to \infty$ such that $w(\tau_n) = A$ for $n = 1, 2, \ldots$. Observe that

$$\lim_{n\to\infty} E(w)(\tau_n) = \lim_{n\to\infty} \Big(\frac{w'(\tau_n)^2}{2} + \Phi(A)\Big) = \zeta.$$

Then it follows that

$$\lim_{n\to\infty} \frac{w'(\tau_n)^2}{2} = \zeta - \Phi(A) > 0.$$

Hence there exists an integer n_0 such that

$$|w'(\tau_n)|^2 \geq \zeta - \Phi(A) \ \text{ for } \ n \geq n_0.$$

Since $w''(t)$ is bounded for $t \geq t_0$ by Lemma 3.2 (i), there exists $\rho > 0$ such that

$$|w'(t)|^2 \geq \frac{\zeta - \Phi(A)}{2} \ \text{ for } \ \tau_n - \rho \leq t \leq \tau_n + \rho \ \text{ with } \ n \geq n_0.$$

This implies that

$$\int_{t_0}^{\infty} w'(t)^2\, dt = \infty,$$

which contradicts Lemma 3.2(ii). Thus (3.13) holds with some $\gamma > 0$. By Lemma 3.3, we obtain (3.1).

(ii) By Lemma 3.1, we have $\limsup_{t\to\infty} w(t) > 0$. We show that (3.13) holds for some $\gamma > 0$. Assume to the contrary that the limit of $w(t)$ as $t \to \infty$ does not exist. Then by Lemma 3.4 (ii) there exists an infinite sequence $\tau_n \to \infty$ such that $w(\tau_n) = A$ for $n = 1, 2, \ldots$. This is a contradiction. Thus (3.13) holds for some $\gamma > 0$. By Lemma 3.3, we obtain (3.1). □

4 Proof of Theorems 1.1, 1.2 and 1.3

4.1 Uniqueness of the singular solution

First, we show the following

Lemma 4.1. *Let $z(t)$ be a solution of*

$$z'' - p(t)z' + q(t)z = 0 \ \textit{ for } \ t \geq t_0, \tag{4.1}$$

where $p, q \in C[t_0, \infty)$ satisfy

$$\lim_{t\to\infty} p(t) = P \ \textit{ and } \ \lim_{t\to\infty} q(t) = Q \tag{4.2}$$

with the positive constants P and Q. If $z(t)$ is bounded for $t \geq t_0$, then $z(t) \equiv 0$.

Proof. Assume by contradiction that $z(t) \not\equiv 0$. Define

$$G(t) = \frac{1}{2} z'(t)^2 + \frac{Q}{2} z(t)^2 \ \text{ for } \ t \geq t_0.$$

Then, by the uniqueness of the initial value problem to (4.1), we have $z'(t)^2 + z(t)^2 \neq 0$ for $t \geq t_0$, and hence $G(t) > 0$ for all $t \geq t_0$. From (4.1) we see that

$$G'(t) = (z''(t) + Qz(t))z' = p(t)z'(t)^2 - (q(t) - Q)z(t)z'(t) \ \text{ for } \ t \geq t_0. \tag{4.3}$$

Then, by (4.2), there exists $t_1 \geq t_0$ such that

$$p(t) \geq \frac{P}{2} \ \text{ and } \ |q(t) - Q| \leq \frac{P}{2} \ \text{ for } \ t \geq t_1.$$

From (4.3) it follows that

$$G'(t) \geq \frac{P}{2}\left(z'(t)^2 - |z(t)|\,|z'(t)|\right) \ \text{ for } \ t \geq t_1.$$

By the Hölder inequality, we obtain

$$G'(t) \geq \frac{P}{2}\left(\frac{1}{2}\,z'(t)^2 - \frac{1}{2}\,z(t)^2\right) = \frac{P}{2}\,G(t) - \frac{P(Q+1)}{4}\,z(t)^2 \ \text{ for } \ t \geq t_1.$$

Since $z(t)$ is bounded, there exists $C_0 > 0$ such that

$$G'(t) \geq \frac{P}{2}\,G(t) - C_0 \ \text{ for } \ t \geq t_1.$$

Multiplying both sides by $e^{-\frac{P}{2}t}$, we get $(e^{-\frac{P}{2}t}G(t))' \geq -e^{-\frac{P}{2}t}C_0$ for $t \geq t_1$. Integrating both sides on $[t_1, t]$, we obtain

$$e^{-\frac{P}{2}t}G(t) \geq C_1 - C_2 e^{-\frac{P}{2}t} \ \text{ for } \ t \geq t_1,$$

where $C_1 = e^{-\frac{P}{2}t_1}G(t_1) > 0$ and $C_2 = \frac{2C_0}{P} > 0$. Thus we obtain $G(t) \geq C_1 e^{\frac{P}{2}t} - C_2 \to \infty$ as $t \to \infty$. Since $z(t)$ is bounded, we have $|z'(t)| \to \infty$ as $t \to \infty$, and hence $|z(t)| \to \infty$ as $t \to \infty$. This is a contradiction. Thus we obtain $z(t) \equiv 0$. □

Proof of Theorem 1.1. Let $u_1(r)$ and $u_2(r)$ be singular solutions of (1.1) for $0 < r \leq r_0$. Define $w_i(t) = r^{2/(p-1)}u_i(r)$ with $t = -\log r$ for $i = 1, 2$. Then $w = w_i(t)$ satisfies (2.12) with $t_0 = -\log s_0$ for $i = 1, 2$. By Proposition 3.1, we obtain

$$\lim_{t\to\infty} w_1(t) = \lim_{t\to\infty} w_2(t) = A. \tag{4.4}$$

Put $z(t) = w_1(t) - w_2(t)$. Then z satisfies $\lim\limits_{t\to\infty} z(t) = 0$ and

$$z'' - a(t)z' + B(t)z = 0 \ \text{ for } \ t \geq t_0,$$

where $a(t)$ is given by (2.13) and

$$B(t) = -A^{p-1} + \frac{2}{p-1}\,\eta(t) - \mu(t) + \frac{w_1(t)^p - w_2(t)^p}{w_1(t) - w_2(t)}\,.$$

From (4.4) we have

$$\lim_{t\to\infty} \frac{w_1(t)^p - w_2(t)^p}{w_1(t) - w_2(t)} = pA^{p-1}.$$

Then we obtain

$$\lim_{t\to\infty} B(t) = (p-1)A^{p-1} > 0.$$

Lemma 4.1 implies that $z(t) \equiv 0$, and hence the singular solution of (1.1) is unique. □

4.2 Liouville property

Proof of Theorem 1.2. Let $p = (N+2)/(N-2)$ and let $h(r)$ and $\lambda(r)$ satisfy (1.7). In this case, (2.12) can be written as

$$w'' + \eta(t)w' - A^{p-1}w + w^p = 0 \ \text{ for } \ t \geq t_0.$$

Define $E(w)$ by (3.6). Then we have

$$E(w)'(t) = -\eta(t)w'(t)^2 \ \text{ for } \ t > t_0. \tag{4.5}$$

Assume that $w(t) - A$ has a finite number of zeros on $[t_0, \infty)$. Then, by Proposition 3.1, we obtain $w(t) \to A$ and $w'(t) \to 0$ as $t \to \infty$. Integrating (4.5) on $[t, \tau]$ and letting $\tau \to \infty$, we obtain

$$\frac{1}{2}\,w'(t)^2 \leq E(w)(t) - \Phi(A) = \int_t^\infty \eta(s)w'(s)^2\,ds.$$

Put

$$U(t) = \int_t^\infty \eta(s) w'(s)^2 \, ds.$$

Then $U'(t) = -\eta(t) w'(t)^2 \geq -2\eta(t) U(t)$ for $t \geq 0$. This implies that

$$\frac{d}{dt}\Big(e^{2\int_{t_0}^t \eta(s)\,ds} U(t)\Big) \geq 0 \ \text{ for } \ t \geq t_0.$$

Integrating the above on $[t, \tau]$ and letting $\tau \to \infty$, we obtain $U(t) \leq 0$ for $t \geq 0$. This implies that $U(t) \equiv 0$, and hence $w'(t) \equiv 0$. It follows that $w(t) \equiv A$. Thus we obtain $u \equiv U_A$. □

4.3 Infinitely many existences of singular solutions

In order to prove Theorem 1.3, we consider the initial value problem

$$\begin{cases} w'' - a(t)w' - b(t)w + w^p = 0 \ \text{ for } \ t \geq t_0, \ \ t \geq t_0, \\ \qquad\qquad w(t_0) = \alpha, \quad w'(t_0) = \beta, \end{cases} \tag{4.6}$$

where $\alpha, \beta \in \mathbf{R}$ and $a(t)$ and $b(t)$ are defined by (2.13) and (2.14), respectively.

Define Φ by (3.7), and define a constant B by

$$B = \Big(\frac{(p+1)A^{p-1}}{2}\Big)^{1/(p-1)}.$$

Then we see that $\Phi(v) \leq 0$ if and only if $0 \leq v \leq B$. Recall that $\Phi(A) < 0$ and $\Phi(v) \geq \Phi(A)$ for $v \geq 0$. For any $\delta \in (0, -\Phi(A))$, there exist the constants $\Gamma_\delta > \gamma_\delta > 0$ such that $\gamma_\delta \leq v \leq \Gamma_\delta$ if $\Phi(v) \leq -\delta$.

Lemma 4.2. *Define $E(w)$ by (3.6). Let w be a solution of (4.6) for $t_0 \leq t \leq t_1$ such that*

$$E(w)(t) \leq 0 \ \text{ for } \ t_0 \leq t \leq t_1.$$

Then $w(t) \leq B$ and $|w'(t)| \leq \sqrt{-2\Phi(A)}$ for $t_0 \leq t \leq t_1$.

Proof. Since

$$E(w)(t) = \frac{1}{2} w'(t)^2 + \Phi(w(t)) \leq 0 \ \text{ for } \ t_0 \leq t \leq t_1,$$

we have $\Phi(w(t)) \leq 0$ for $t_0 \leq t \leq t_1$. This implies that $w(t) \leq B$ for $t_0 \leq t \leq t_1$. We also have

$$\frac{1}{2} w'(t)^2 + \Phi(A) \leq E(w)(t) \leq 0 \ \text{ for } \ t_0 \leq t \leq t_1.$$

Then $w'(t)^2 \leq -2\Phi(A)$ for $t_0 \leq t \leq t_1$, which implies that $|w'(t)| \leq \sqrt{-2\Phi(A)}$ for $t_0 \leq t \leq t_1$, □

Lemma 4.3. *Take any $\delta > 0$ such that $2\delta < -\Phi(A)$. Then there exists $t_0 > 0$ such that if $E(w)(t_0) < -2\delta$, then the solution $w(t)$ of (4.6) exists for all $t \geq t_0$ and satisfies $E(w)(t) < -\delta$ for all $t > t_0$.*

Proof. We note that if $p < (N+2)/(N-2)$, then $N - 2 - 4/(p-1) < 0$ and $\lim_{t\to\infty} a(t) < 0$. Take $t_0 > 0$ such that $a(t) < 0$ for $t \geq t_0$ and

$$B\sqrt{-2\Phi(A)} \int_{t_0}^\infty \Big(\frac{2}{p-1} |\eta(s)| + |\mu(s)|\Big)\, ds < \delta.$$

Assume by contradiction that there exists $t_1 > t_0$ such that

$$E(w)(t) < -\delta \ \text{ for } \ t_0 \leq t < t_1 \ \text{ and } \ E(w)(t_1) = -\delta.$$

Then, by Lemma 4.2, we have $w(t) \le B$ and $|w'(t)| \le \sqrt{-2\Phi(A)}$ for $t_0 \le t \le t_1$. From (3.9) we have

$$E'(w)(t) < B\sqrt{-2\Phi(A)}\left(\frac{2}{p-1}\,|\eta(t)| + |\mu(t)|\right)$$

for $t_0 \le t \le t_1$. Integrating the above on $[t_0, t_1]$, we get

$$E(w)(t_1) - E(w)(t_0) < B\sqrt{-2\Phi(A)}\int_{t_0}^{t_1}\left(\frac{2}{p-1}\,|\eta(s)| + |\mu(s)|\right)ds < \delta.$$

This implies that $-\delta = E(w)(t_1) < E(w)(t_0) + \delta < -\delta$. This is a contradiction. Thus we obtain $E(w)(t) < -\delta$ for all $t \ge t_0$. Lemma 4.2 implies that $w(t)$ and $w'(t)$ are bounded for $t \ge t_0$, and hence the solution $w(t)$ of (4.6) exists for all $t \ge t_0$. $\square$

Proof of Theorem 1.3. Take any $\delta > 0$ such that $2\delta < -\Phi(A)$. Since $\Phi(A) < -2\delta$, we can take $\alpha, \beta \in \mathbf{R}$ in (4.6) such that

$$\frac{1}{2}\,\beta^2 + \Phi(\alpha) < -2\delta. \tag{4.7}$$

Then, by Lemma 4.3, the solution $w(t)$ of (4.6) exists for all $t \ge t_0$ and satisfies $E(w)(t) < -\delta$ for $t \ge t_0$. This implies that $\Phi(w(t)) < -\delta$ for all $t \ge t_0$. Note here that there exists a constant $\gamma_\delta > 0$ such that $v > \gamma_\delta$ if $\Phi(v) < -\delta$. Then we obtain $w(t) > \gamma_\delta$ for all $t \ge t_0$. This implies that $u(r) > \gamma_\delta r^{-2/(p-1)}$ for $0 < r \le r_0$, and hence u is a singular solution of (1.1). By Proposition 3.1, $u(r)$ satisfies (1.8). Since there are infinitely many $\alpha, \beta \in \mathbf{R}$ satisfying (4.7), we have infinitely many singular solutions of (1.1). $\square$

Acknowledgements

The author was supported by JSPS KAKENHI Grant Number 23K03167. This work was also supported by Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

References

[1] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions of the heat equation. *Nonlinear Anal.* **11** (1987), no. 10, 1103–1133.

[2] A. Haraux and F. B. Weissler, Nonuniqueness for a semilinear initial value problem. *Indiana Univ. Math. J.* **31** (1982), no. 2, 167–189.

[3] J. C. Kurtz, Weighted Sobolev spaces with applications to singular nonlinear boundary value problems. *J. Differential Equations* **49** (1983), no. 1, 105–123.

[4] N. Mizoguchi, On backward self-similar blow-up solutions to a supercritical semilinear heat equation. *Proc. Roy. Soc. Edinburgh Sect. A* **140** (2010), no. 4, 821–831.

[5] Y. Naito, An ODE approach to the multiplicity of self-similar solutions for semi-linear heat equations. *Proc. Roy. Soc. Edinburgh Sect. A* **136** (2006), no. 4, 807–835.

[6] L. A. Peletier, D. Terman and F. B. Weissler, On the equation $\Delta u + \frac{1}{2}\,x \cdot \nabla u + f(u) = 0$. *Arch. Rational Mech. Anal.* **94** (1986), no. 1, 83–99.

[7] P. Quittner, Uniqueness of singular self-similar solutions of a semilinear parabolic equation. *Differential Integral Equations* **31** (2018), no. 11-12, 881–892.

[8] S. Sato, A singular solution with smooth initial data for a semilinear parabolic equation. *Nonlinear Anal.* **74** (2011), no. 4, 1383–1392.

[9] S. Sato and E. Yanagida, Singular backward self-similar solutions of a semilinear parabolic equation. Discrete Contin. *Dyn. Syst. Ser. S* **4** (2011), no. 4, 897–906.

[10] N. Shioji and K. Watanabe, Uniqueness and nondegeneracy of positive radial solutions of $\mathrm{div}(\rho\nabla u)+\rho(-gu+hu^p)=0$. *Calc. Var. Partial Differential Equations* **55** (2016), no. 2, Art. 32, 42 pp.

[11] F. B. Weissler, Rapidly decaying solutions of an ordinary differential equation with applications to semilinear elliptic and parabolic partial differential equations. *Arch. Rational Mech. Anal.* **91** (1985), no. 3, 247–266.

(Received 28.09.2023; accepted 20.09.2023)

Author's address:

Yūki Naito

Department of Mathematics, Hiroshima University, Higashi-Hiroshima 739-8526, Japan.

E-mail: yunaito@hiroshima-u.ac.jp