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OSCILLATION AND NONOSCILLATION FOR CERTAIN NONLINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to Professor Kusano Takaŝi
on the occasion of his 90th birthday anniversary

Abstract. The two-dimensional nonlinear system

$$u' = a(t)|v|^{1/\alpha}\operatorname{sgn} v, \quad v' = -b(t)|u|^{\alpha}\operatorname{sgn} u \tag{1.1}$$

is considered under the assumptions that $\alpha > 0$, $a, b \in C[t_0, \infty)$, $a(t) \geq 0$, $a(t) \not\equiv 0$ $(t \geq t_0)$. It is shown that, under certain conditions on $a(t)$ and $b(t)$, if system (1.1) is nonoscillatory, then the integral averages with the weight $a(t)$ of the functions $|c(t)|$ and $|c(t)|^{(\alpha+1)/\alpha}$ tend to 0 as $t \to \infty$. Here,

$$c(t) = \lim_{\tau\to\infty} \bigg(\int_t^\tau a(s)\,ds\bigg)^{-1} \int_t^\tau a(s)\bigg(\int_t^s b(r)\,dr\bigg)\,ds, \quad t \geq t_0.$$

Using this result, we can establish many kinds of Hartman–Wintner type oscillation criteria for (1.1).

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რეზიუმე. ორგანზომილებიანი არაწრფივი

$$u' = a(t)|v|^{1/\alpha}\operatorname{sgn} v, \quad v' = -b(t)|u|^{\alpha}\operatorname{sgn} u \tag{1.1}$$

სისტემა განხილულია იმ დაშვებით, რომ $\alpha > 0$, $a, b \in C[t_0, \infty)$, $a(t) \geq 0$, $a(t) \not\equiv 0$ $(t \geq t_0)$. ნაჩვენებია, რომ თუ $a(t)$ და $b(t)$-ზე დადებულ გარკვეულ პირობებში სისტემა (1.1) არაოსცილაციურია, მაშინ $|c(t)|$ and $|c(t)|^{(\alpha+1)/\alpha}$ ფუნქციების ინტეგრალური საშუალოები $a(t)$ წონით მიისწრაფიან 0-სკენ, როცა $t \to \infty$. აქ

$$c(t) = \lim_{\tau\to\infty} \bigg(\int_t^\tau a(s)\,ds\bigg)^{-1} \int_t^\tau a(s)\bigg(\int_t^s b(r)\,dr\bigg)\,ds, \quad t \geq t_0.$$

ამ შედეგის გამოყენებით შესაძლებელია (1.1) სისტემისთვის დავადგინოთ მრავალნაირი ჰარტმან-ვინტნერის ტიპის რხევის კრიტერიუმი.

1 Introduction

In this paper, we consider the two-dimensional nonlinear system of ordinary differential equations

$$\begin{cases} u' = a(t)|v|^{1/\alpha}\operatorname{sgn} v, \\ v' = -b(t)|u|^{\alpha}\operatorname{sgn} u, \end{cases} \tag{1.1}$$

where α is a positive constant, $a(t)$ and $b(t)$ are real-valued continuous functions on $[t_0, \infty)$ and

$$a(t) \geq 0 \ \text{ for } \ t \geq t_0, \ \text{ and } \ a(t) \not\equiv 0 \ \text{ on } \ [t_0^+, \infty) \ \text{ for any } \ t_0^+ \geq t_0.$$

By a solution $(u(t), v(t))$ of system (1.1) on an interval $I \subseteq [t_0, \infty)$ we mean that $u(t)$ and $v(t)$ are continuously differentiable on I and satisfy (1.1) at every point $t \in I$.

It is known (Mirzov [10, Lemma 2.1]) that all local solutions of (1.1) can be continued to t_0 and ∞, and so all solutions of (1.1) exist on the entire interval $[t_0, \infty)$. Following the paper by Dosoudilová, Lomtatidze and Šremr [3], we say that a solution $(u(t), v(t))$ of system (1.1) is *nontrivial* if $u(t) \not\equiv 0$ on any neighborhood of infinity, and that a nontrivial solution $(u(t), v(t))$ of (1.1) is *oscillatory* if $u(t)$ has a sequence of zeros tending to infinity, and *nonoscillatory* otherwise. It is worth noting that, for any nontrivial solution $(u(t), v(t))$ of (1.1), the sequence of zeros of $u(t)$ cannot have a finite cluster point (see Naito [13]). If $(u(t), v(t))$ is a solution of (1.1), then so is $(-u(t), -v(t))$. Therefore, without loss of generality, we can assume that a nonoscillatory solution $(u(t), v(t))$ of (1.1) satisfies $u(t) > 0$ for all large t.

It is also known (Mirzov [10, Theorem 1.1]) that an analogue of Sturm's separation theorem holds for system (1.1). In particular, if system (1.1) has an oscillatory [resp. nonoscillatory] solution, then any other nontrivial solution is also oscillatory [resp. nonoscillatory]. System (1.1) is said to be *oscillatory* [resp. *nonoscillatory*] if all its nontrivial solutions are oscillatory [resp. nonoscillatory].

For the case where $a(t) > 0$ for $t \geq t_0$, the first component $u(t)$ of a solution $(u(t), v(t))$ of (1.1) satisfies the scalar differential equation

$$\left(a(t)^{-\alpha}|u'|^{\alpha}\operatorname{sgn} u'\right)' + b(t)|u|^{\alpha}\operatorname{sgn} u = 0.$$

Putting $p(t) = a(t)^{-\alpha}$ and $q(t) = b(t)$, we write the above equation in the form

$$\left(p(t)|u'|^{\alpha}\operatorname{sgn} u'\right)' + q(t)|u|^{\alpha}\operatorname{sgn} u = 0, \tag{1.2}$$

where $p(t)$ and $q(t)$ are continuous functions on $[t_0, \infty)$ and $p(t) > 0$ for $t \geq t_0$. If $u(t)$ is a solution of (1.2) and if c is a constant, then $cu(t)$ is also a solution of (1.2). Equation (1.2) is referred as "half-linear" equation. If $\alpha = 1$, then (1.2) becomes the linear equation

$$(p(t)u')' + q(t)u = 0. \tag{1.3}$$

In the last three decades, many results have been obtained in the theory of oscillatory and asymptotic behavior of solutions of the half-linear equation (1.2). It is known that basic results for the linear equation (1.3) can be generalized to the half-linear equation (1.2). The important works for (1.2) are summarized in the book by Došlý and Řehák [2]. For the recent results to the half-linear equation (1.2) see, for example, [4, 6, 7, 14, 16, 17]. For the results to the nonlinear system (1.1) (including the linear system) see, for example, [3, 5, 8–11, 13, 15].

For the scalar equation (1.2), it is usual to distinguish the cases

$$\int_{t_0}^{\infty} p(s)^{-1/\alpha}\, ds = \infty \ \text{ and } \ \int_{t_0}^{\infty} p(s)^{-1/\alpha}\, ds < \infty.$$

For system (1.1), these correspond to the cases

$$\int_{t_0}^{\infty} a(s)\, ds = \infty \ \text{ and } \ \int_{t_0}^{\infty} a(s)\, ds < \infty,$$

respectively. In the present paper, we focus our attention to the former case

$$\int_{t_0}^{\infty} a(s)\,ds = \infty. \tag{1.4}$$

An analogue of the Hartman–Wintner oscillation theorem for the linear equation (1.3) remains valid for the half-linear system (1.1). In fact, we have the following result.

Theorem 1.1 (Mirzov [11, Theorem 12.3]). *Consider the half-linear system* (1.1) *under condition* (1.4). *If*

$$\lim_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)\,ds = \infty$$

or

$$\begin{aligned}-\infty < \liminf_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)\,ds\\ < \limsup_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)\,ds,\end{aligned}$$

then system (1.1) *is oscillatory.*

An extension of Theorem 1.1 has been given by Dosoudilová, Lomtatidze and Šremr [3]. The special case $\lambda = 0$ and $\nu = 0$ of Corollary 2.5 in [3] becomes Theorem 1.1.

In the present paper, a result similar to Theorem 1.1 will be proved. Then the condition

$$\limsup_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-\kappa}\int_{t_0}^{t}\Big(\int_{s}^{t} a(r)\,dr\Big)^{\kappa} b(s)\,ds > -\infty \ \text{ for some } \ \kappa > \alpha \tag{1.5}$$

plays an important part.

Theorem 1.2. *Consider the half-linear system* (1.1) *under conditions* (1.4) *and* (1.5). *If*

$$\limsup_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big|\int_{t_0}^{s} b(r)\,dr\Big|\,ds = \infty \tag{1.6}$$

or

$$\begin{aligned}\liminf_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big|\int_{t_0}^{s} b(r)\,dr\Big|\,ds\\ < \limsup_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big|\int_{t_0}^{s} b(r)\,dr\Big|\,ds < \infty,\end{aligned} \tag{1.7}$$

then system (1.1) *is oscillatory.*

Further, we will prove the following theorem.

Theorem 1.3. *Consider the half-linear system* (1.1) *under conditions* (1.4) *and* (1.5). *If*

$$\limsup_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big|\int_{t_0}^{s} b(r)\,dr\Big|^{(\alpha+1)/\alpha}\,ds = \infty \tag{1.8}$$

or

$$\liminf_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big|\int_{t_0}^{s} b(r)\,dr\Big|^{(\alpha+1)/\alpha} ds$$
$$< \limsup_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big|\int_{t_0}^{s} b(r)\,dr\Big|^{(\alpha+1)/\alpha} ds < \infty, \tag{1.9}$$

then system (1.1) is oscillatory.

For the case $\alpha = 1$, $a(t) = 1$ and $b(t) = q(t)$, Theorem 1.3 reduces to the recent result in [12, Corollaries 1.2 and 1.3 (the linear case)]. Note that in the paper [12] nonlinear scalar equations of the form

$$x'' + q(t)f(x) = 0,\ \ t \ge t_0 \quad \big(xf(x) > 0 \ \text{ and } \ f'(x) > 0 \ \text{ for } \ x \ne 0\big)$$

are discussed.

Theorems 1.2 and 1.3 are actually derived from more general results for (1.1). The general results are stated and proved in Section 2. In the last part of Section 2, we give a few examples illustrating our results. In the final Section 3, further results for (1.1) are stated and proved. The results in Section 3 are slightly restrictive in the sense that, for example, Theorem 3.2 requires the condition $0 < \alpha \le 1$.

2 General results

Suppose now that system (1.1) has a nonoscillatory solution $(u(t), v(t))$ such that $u(t) > 0$ for $t \ge T$ $(\ge t_0)$. Define the function $w(t)$ by

$$w(t) = \frac{v(t)}{u(t)^\alpha}\,,\ \ t \ge T. \tag{2.1}$$

Since $a(t)$ is nonnegative, it is clear that $w(t)$ satisfies either

$$\int_T^\infty a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds = \infty \tag{2.2}$$

or

$$\int_T^\infty a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds < \infty. \tag{2.3}$$

The results of this paper are based on the following theorem, in which it is shown that if (1.1) is nonoscillatory, then we must have either

$$\lim_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-\kappa}\int_{t_0}^{t}\Big(\int_{s}^{t} a(r)\,dr\Big)^{\kappa} b(s)\,ds = -\infty \ \text{ for any } \ \kappa > \alpha \tag{2.4}$$

or

$$\lim_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big) ds \ \text{ exists and is finite.} \tag{2.5}$$

Theorem 2.1. *Consider the half-linear system (1.1) under condition (1.4). Suppose that system (1.1) has a nonoscillatory solution $(u(t), v(t))$ such that $u(t) > 0$ for $t \ge T$ $(\ge t_0)$. Define $w(t)$ by (2.1).*

(I) *If (2.2) is satisfied, then (2.4) holds.*

(II) *If* (2.3) *is satisfied, then* (2.5) *holds.*

Proof. It is easy to see that $w(t)$ satisfies the generalized Riccati differential equation

$$w'(t) = -b(t) - \alpha a(t)|w(t)|^{(\alpha+1)/\alpha}, \quad t \geq T. \tag{2.6}$$

(I) Assume that (2.4) does not hold. This implies (1.5). Then, by Lemma 4.1 (the case $m = 1$, $\beta = 1$ and $\lambda = 0$) in the paper by Dosoudilová et al. [3], we see that (2.3) holds, that is, (2.2) does not hold. This proves (I).

(II) Suppose that (2.3) is satisfied. Integrating (2.6) from T to t, we have

$$w(t) = w(T) - \int_T^t b(s)\,ds - \alpha \int_T^t a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds, \quad t \geq T,$$

and so

$$\int_T^t a(s)w(s)\,ds = w(T)\int_T^t a(s)\,ds - \int_T^t a(s)\Big(\int_T^s b(r)\,dr\Big)\,ds - \alpha\int_T^t a(s)\Big(\int_T^s a(r)|w(r)|^{(\alpha+1)/\alpha}\,dr\Big)\,ds, \quad t \geq T. \tag{2.7}$$

By Hölder's inequality, we have

$$\int_T^t a(s)|w(s)|\,ds \leq \Big(\int_T^t a(s)\,ds\Big)^{1/(\alpha+1)}\Big(\int_T^t a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds\Big)^{\alpha/(\alpha+1)}, \quad t \geq T.$$

It follows from (1.4) that

$$\int_T^t a(s)\,ds > 0 \ \text{ for all large } t.$$

Hence, using (2.3), we get

$$\Big(\int_T^t a(s)\,ds\Big)^{-1}\int_T^t a(s)|w(s)|\,ds \leq \Big(\int_T^t a(s)\,ds\Big)^{-\alpha/(\alpha+1)}\Big(\int_T^\infty a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds\Big)^{\alpha/(\alpha+1)}$$

for all large t. Therefore, condition (1.4) implies

$$\lim_{t\to\infty}\Big(\int_T^t a(s)\,ds\Big)^{-1}\int_T^t a(s)|w(s)|\,ds = 0, \tag{2.8}$$

and, as a consequence,

$$\lim_{t\to\infty}\Big(\int_T^t a(s)\,ds\Big)^{-1}\int_T^t a(s)w(s)\,ds = 0.$$

Then, by (2.7), we easily find that

$$\lim_{t\to\infty}\Big(\int_T^t a(s)\,ds\Big)^{-1}\int_T^t a(s)\Big(\int_T^s b(r)\,dr\Big)\,ds \ \text{ exists and is finite} \tag{2.9}$$

and

$$0 = w(T) - \lim_{t\to\infty}\Big(\int_T^t a(s)\,ds\Big)^{-1}\int_T^t a(s)\Big(\int_T^s b(r)\,dr\Big)\,ds - \alpha\int_T^\infty a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds. \tag{2.10}$$

It is obvious that (2.9) implies (2.5). The proof of Theorem 2.1 is complete. □

For the case where (2.5) holds, we can define the function $c(t)$ by

$$c(t) = \lim_{\tau\to\infty}\Big(\int_t^\tau a(s)\,ds\Big)^{-1}\int_t^\tau a(s)\Big(\int_t^s b(r)\,dr\Big)\,ds,\quad t\ge t_0. \tag{2.11}$$

By (1.4), the function $c(t)$ can be written in the form

$$c(t) = \lim_{\tau\to\infty}\Big(\int_{t_0}^\tau a(s)\,ds\Big)^{-1}\int_t^\tau a(s)\Big(\int_t^s b(r)\,dr\Big)\,ds,\quad t\ge t_0.$$

For the case where

$$\lim_{t\to\infty}\int_{t_0}^t b(s)\,ds = \int_{t_0}^\infty b(s)\,ds \text{ exists and is finite,}$$

the function $c(t)$ is given by

$$c(t) = \int_t^\infty b(s)\,ds,\quad t\ge t_0.$$

Theorem 2.2. *Consider the half-linear system (1.1) under conditions (1.4) and (1.5). Suppose that system (1.1) has a nonoscillatory solution $(u(t), v(t))$ such that $u(t) > 0$ for $t \ge T$ $(\ge t_0)$. Define the function $w(t)$ by (2.1). Then (2.3) and (2.5) hold, and the function $c(t)$ can be defined by (2.11). Moreover, $w(t)$ satisfies the generalized Riccati integral equation*

$$w(t) = c(t) + \alpha\int_t^\infty a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds,\quad t\ge T. \tag{2.12}$$

Proof. Since (1.5) is assumed, (2.4) does not hold. Therefore, from (I) of Theorem 2.1, it is seen that (2.2) is not satisfied. Hence we have (2.3). Then, by (II) of Theorem 2.1, we obtain (2.5), and, as mentioned above, the function $c(t)$ can be defined by (2.11).

Since we have (2.3), the proof of (II) of Theorem 2.1 leads to equality (2.10). This implies

$$w(T) = c(T) + \alpha\int_T^\infty a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds.$$

Note that T is a number such that $u(t) > 0$ for $t \ge T$.

Now, let T_1 be an arbitrary number satisfying $T_1 \ge T$. It is trivial that $u(t) > 0$ for $t \ge T_1$. Therefore, we have

$$w(T_1) = c(T_1) + \alpha\int_{T_1}^\infty a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds.$$

Since T_1 $(\ge T)$ is arbitrary, this implies (2.12). The proof of Theorem 2.2 is complete. □

As shown in Theorem 2.2, if system (1.1) with (1.4) and (1.5) is nonoscillatory, then (2.5) holds and the function $c(t)$ can be defined by (2.11). The next theorem says furthermore that this $c(t)$ must satisfy

$$\lim_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)|c(s)|\,ds = 0 \tag{2.13}$$

and

$$\lim_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)|c(s)|^{(\alpha+1)/\alpha}\,ds = 0. \tag{2.14}$$

Theorem 2.3. *Consider the half-linear system (1.1) under conditions (1.4) and (1.5). If (1.1) is nonoscillatory, then (2.13) and (2.14) hold. Here, $c(t)$ is given by (2.11).*

Proof. Suppose that (1.1) is nonoscillatory. Let $(u(t), v(t))$ be a nonoscillatory solution of (1.1) and suppose that $u(t) > 0$ for $t \ge T$ $(\ge t_0)$, and define $w(t)$ by (2.1). By Theorem 2.2, we have (2.3), (2.5) and (2.12). Therefore,

$$|c(t)| \le |w(t)| + \alpha\int_{t}^{\infty} a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds,\quad t \ge T, \tag{2.15}$$

and hence

$$\begin{aligned}
&\Big(\int_{T}^{t} a(s)\,ds\Big)^{-1}\int_{T}^{t} a(s)|c(s)|\,ds\\
&\qquad\le \Big(\int_{T}^{t} a(s)\,ds\Big)^{-1}\int_{T}^{t} a(s)|w(s)|\,ds + \alpha\Big(\int_{T}^{t} a(s)\,ds\Big)^{-1}\int_{T}^{t} a(s)\Big(\int_{s}^{\infty} a(r)|w(r)|^{(\alpha+1)/\alpha}\,dr\Big)\,ds
\end{aligned}$$

for all large t. As in the proof of (II) of Theorem 2.1, we get (2.8). This implies that the first term of the right-hand side of the above inequality tends to 0 as $t \to \infty$. By (1.4) and (2.3), it is clear that the second term of the right-hand side of the above inequality also tends to 0 as $t \to \infty$. Hence we have (2.13).

To prove (2.14), we apply the general inequality

$$(\mu+\nu)^{\lambda} \le 2^{\lambda}(\mu^{\lambda}+\nu^{\lambda}),\quad \mu \ge 0,\quad \nu \ge 0,\quad \lambda > 0, \tag{2.16}$$

to the case

$$\mu = |w(t)|,\qquad \nu = \alpha\int_{t}^{\infty} a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds \ \text{ and } \ \lambda = \frac{\alpha+1}{\alpha}.$$

Then it follows from (2.15) that

$$|c(t)|^{(\alpha+1)/\alpha} \le 2^{(\alpha+1)/\alpha}|w(t)|^{(\alpha+1)/\alpha} + (2\alpha)^{(\alpha+1)/\alpha}\Big(\int_{t}^{\infty} a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds\Big)^{(\alpha+1)/\alpha}$$

for $t \ge T$. Hence

$$\begin{aligned}
&\Big(\int_{T}^{t} a(s)\,ds\Big)^{-1}\int_{T}^{t} a(s)|c(s)|^{(\alpha+1)/\alpha}\,ds\\
&\qquad\le 2^{(\alpha+1)/\alpha}\Big(\int_{T}^{t} a(s)\,ds\Big)^{-1}\int_{T}^{t} a(s)|w(s)|^{(\alpha+1)/\alpha}\,ds
\end{aligned}$$

$$+\,(2\alpha)^{(\alpha+1)/\alpha}\left(\int_T^t a(s)\,ds\right)^{-1}\int_T^t a(s)\left(\int_s^\infty a(r)|w(r)|^{(\alpha+1)/\alpha}\,dr\right)^{(\alpha+1)/\alpha} ds$$

for all large t. By (1.4) and (2.3), we easily find that the right-hand side of the above inequality tends to 0 as $t \to \infty$. This yields (2.14). The proof of Theorem 2.3 is complete. □

It is worth noting that condition (1.5) is satisfied provided

$$\liminf_{t\to\infty}\left(\int_{t_0}^t a(s)\,ds\right)^{-1}\int_{t_0}^t a(s)\left(\int_{t_0}^s b(r)\,dr\right) ds > -\infty. \tag{2.17}$$

In fact, if (2.17) holds, then

$$\liminf_{t\to\infty}\left(\int_{t_0}^t a(s)\,ds\right)^{-\kappa}\int_{t_0}^t \left(\int_s^t a(r)\,dr\right)^{\kappa} b(s)\,ds > -\infty \;\text{ for all }\; \kappa = 1,2,3,\ldots. \tag{2.18}$$

It is clear that (2.18) implies (1.5). In what follows, we show that (2.17) implies (2.18). If (2.17) is satisfied, then there are constants $L_1 > 0$ and $M_1 > 0$ such that

$$\int_{t_0}^t \left(\int_s^t a(r)\,dr\right) b(s)\,ds = \int_{t_0}^t a(s)\left(\int_{t_0}^s b(r)\,dr\right) ds \geq -L_1 - M_1\int_{t_0}^t a(s)\,ds,\;\; t \geq t_0.$$

Multiplying the above inequality by $a(t)$ and integrating from t_0 to t, we see that there are constants $L_2 > 0$ and $M_2 > 0$ such that

$$\int_{t_0}^t \left(\int_s^t a(r)\,dr\right)^2 b(s)\,ds \geq -L_2 - M_2\left(\int_{t_0}^t a(s)\,ds\right)^2,\;\; t \geq t_0.$$

Repeating this procedure, we deduce that there are constants $L_\kappa > 0$ and $M_\kappa > 0$ such that

$$\int_{t_0}^t \left(\int_s^t a(r)\,dr\right)^{\kappa} b(s)\,ds \geq -L_\kappa - M_\kappa\left(\int_{t_0}^t a(s)\,ds\right)^{\kappa},\;\; t \geq t_0,\;\; \kappa = 1,2,3,\ldots.$$

This gives (2.18).

Condition (2.5) clearly implies (2.17) and, as mentioned above, (2.17) implies (1.5). Therefore, the following corollary is derived from Theorem 2.3.

Corollary 2.1. *Consider the half-linear system (1.1) under conditions (1.4) and (2.5). If (1.1) is nonoscillatory, then (2.13) and (2.14) hold. Here, $c(t)$ is given by (2.11).*

Lemma 2.1. *Suppose that (1.4) and (2.5) hold. Then the function $c(t)$ defined by (2.11) satisfies*

$$c(t) = c(t_0) - \int_{t_0}^t b(s)\,ds,\;\; t \geq t_0, \tag{2.19}$$

and

$$\lim_{t\to\infty}\left(\int_{t_0}^t a(s)\,ds\right)^{-1}\int_{t_0}^t a(s)c(s)\,ds = 0. \tag{2.20}$$

Proof. Let $t_0 \le t < \tau$, and suppose that τ is sufficiently large. We have

$$\begin{aligned}
&\Big(\int_t^\tau a(s)\,ds\Big)^{-1}\int_t^\tau a(s)\Big(\int_t^s b(r)\,dr\Big)\,ds\\
&\quad= \Big(\int_{t_0}^\tau a(s)\,ds\Big)\Big(\int_t^\tau a(s)\,ds\Big)^{-1}\Big(\int_{t_0}^\tau a(s)\,ds\Big)^{-1}\int_{t_0}^\tau a(s)\Big(\int_{t_0}^s b(r)\,dr\Big)\,ds\\
&\qquad - \Big(\int_t^\tau a(s)\,ds\Big)^{-1}\int_{t_0}^t a(s)\Big(\int_{t_0}^s b(r)\,dr\Big)\,ds - \int_{t_0}^t b(s)\,ds.
\end{aligned}$$

In the above equality, fix t ($\ge t_0$) and let $\tau \to \infty$. Then, by (1.4), (2.5) and (2.11), we obtain (2.19). Equality (2.20) is clear from the equalities (2.11) and (2.19). $\square$

Equality (2.19) implies that $c(t)$ is continuously differentiable on $[t_0,\infty)$ and $c'(t) = -b(t)$ for $t \ge t_0$.

Theorem 2.4. *Consider the half-linear system* (1.1) *under conditions* (1.4) *and* (1.5). *If system* (1.1) *is nonoscillatory, then* $c(t)$ *can be defined by* (2.11) *and*

$$\lim_{t\to\infty}\Big(\int_{t_0}^t a(s)\,ds\Big)^{-1}\int_{t_0}^t a(s)\Big|\int_{t_0}^s b(r)\,dr\Big|\,ds = |c(t_0)| \tag{2.21}$$

and

$$\lim_{t\to\infty}\Big(\int_{t_0}^t a(s)\,ds\Big)^{-1}\int_{t_0}^t a(s)\Big|\int_{t_0}^s b(r)\,dr\Big|^{(\alpha+1)/\alpha}\,ds = |c(t_0)|^{(\alpha+1)/\alpha}. \tag{2.22}$$

Proof. Suppose that (1.1) is nonoscillatory. Then condition (2.5) is satisfied, and $c(t)$ can be defined by (2.11) (see Theorem 2.2). By Lemma 2.1, we have (2.19). It is seen that for all large t,

$$\begin{aligned}
&\Big|\Big(\int_{t_0}^t a(s)\,ds\Big)^{-1}\int_{t_0}^t a(s)\Big|\int_{t_0}^s b(r)\,dr\Big|\,ds - |c(t_0)|\,\Big|\\
&\quad= \Big(\int_{t_0}^t a(s)\,ds\Big)^{-1}\Big|\int_{t_0}^t a(s)|c(t_0)-c(s)|\,ds - \int_{t_0}^t a(s)|c(t_0)|\,ds\Big|\\
&\quad\le \Big(\int_{t_0}^t a(s)\,ds\Big)^{-1}\int_{t_0}^t a(s)\big|\,|c(t_0)-c(s)| - |c(t_0)|\,\big|\,ds \le \Big(\int_{t_0}^t a(s)\,ds\Big)^{-1}\int_{t_0}^t a(s)|c(s)|\,ds.
\end{aligned}$$

By Theorem 2.3, we have (2.13). Therefore, we obtain (2.21).

Similarly, we see that for all large t,

$$\begin{aligned}
&\Big|\Big(\int_{t_0}^t a(s)\,ds\Big)^{-1}\int_{t_0}^t a(s)\Big|\int_{t_0}^s b(r)\,dr\Big|^{(\alpha+1)/\alpha}\,ds - |c(t_0)|^{(\alpha+1)/\alpha}\Big|\\
&\quad\le \Big(\int_{t_0}^t a(s)\,ds\Big)^{-1}\int_{t_0}^t a(s)\Big|\,|c(t_0)-c(s)|^{(\alpha+1)/\alpha} - |c(t_0)|^{(\alpha+1)/\alpha}\Big|\,ds.
\end{aligned}$$

By the mean value theorem, there is $\theta \in (0,1)$ such that

$$|\mu_1|^\beta - |\mu_2|^\beta = \beta(\mu_1-\mu_2)\big(\mu_1-\theta(\mu_1-\mu_2)\big)^{(\beta-1)*} \ \text{ for } \ \mu_1,\mu_2\in\mathbb{R},\ \ \beta>1,$$

where the asterisk notation $\xi^{\gamma *} = |\xi|^{\gamma}\operatorname{sgn}\xi$ ($\xi \in \mathbb{R}$, $\gamma > 0$) is used. Applying the above equality to the case $\mu_1 = c(t_0) - c(s)$, $\mu_2 = c(t_0)$ and $\beta = (\alpha+1)/\alpha$ and making use of (2.16) with $\mu = |c(t_0)|$, $\nu = 2|c(s)|$ and $\lambda = 1/\alpha$, we have

$$\begin{aligned}
&\Big| |c(t_0) - c(s)|^{(\alpha+1)/\alpha} - |c(t_0)|^{(\alpha+1)/\alpha} \Big| \\
&\qquad \le \frac{\alpha+1}{\alpha} |c(s)| \big\{ |c(t_0) - c(s)| + |c(s)| \big\}^{1/\alpha} \le \frac{\alpha+1}{\alpha} |c(s)| \big\{ |c(t_0)| + 2|c(s)| \big\}^{1/\alpha} \\
&\qquad\qquad \le \frac{\alpha+1}{\alpha} |c(s)| \Big\{ 2^{1/\alpha} |c(t_0)|^{1/\alpha} + 2^{2/\alpha} |c(s)|^{1/\alpha} \Big\} = k_1 |c(s)| + k_2 |c(s)|^{(\alpha+1)/\alpha},
\end{aligned}$$

where

$$k_1 = \frac{\alpha+1}{\alpha} 2^{1/\alpha} |c(t_0)|^{1/\alpha} \ \text{ and } \ k_2 = \frac{\alpha+1}{\alpha} 2^{2/\alpha}.$$

Thus we are led to

$$\begin{aligned}
&\left| \left(\int_{t_0}^{t} a(s)\, ds \right)^{-1} \int_{t_0}^{t} a(s) \left| \int_{t_0}^{s} b(r)\, dr \right|^{(\alpha+1)/\alpha} ds - |c(t_0)|^{(\alpha+1)/\alpha} \right| \\
&\qquad \le k_1 \left(\int_{t_0}^{t} a(s)\, ds \right)^{-1} \int_{t_0}^{t} a(s) |c(s)|\, ds + k_2 \left(\int_{t_0}^{t} a(s)\, ds \right)^{-1} \int_{t_0}^{t} a(s) |c(s)|^{(\alpha+1)/\alpha} ds
\end{aligned}$$

for all large t. By Theorem 2.3, we have (2.13) and (2.14). Therefore, we obtain (2.22). The proof of Theorem 2.4 is complete. □

It is obvious that if condition (1.6) or condition (1.7) is satisfied, then (2.21) does not hold. Therefore, Theorem 1.2 is directly derived from Theorem 2.4. Theorem 1.3 is also derived from Theorem 2.4.

Example 2.1. Consider system (1.1) for the case

$$a(t) = 1 \ \text{ and } \ b(t) = \sin t \ \text{ for } \ t \ge t_0. \tag{2.23}$$

In this case, we have

$$\lim_{t\to\infty} \left(\int_{t_0}^{t} a(s)\, ds \right)^{-1} \int_{t_0}^{t} a(s) \left(\int_{t_0}^{s} b(r)\, dr \right) ds = \lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} \left(\int_{t_0}^{s} \sin r\, dr \right) ds = \cos t_0,$$

and so, (2.5) holds and the function $c(t)$ defined by (2.11) is equal to $\cos t$. Then it is easy to see that

$$\lim_{t\to\infty} \left(\int_{t_0}^{t} a(s)\, ds \right)^{-1} \int_{t_0}^{t} a(s) |c(s)|\, ds = \lim_{t\to\infty} \frac{1}{t - t_0} \int_{t_0}^{t} |\cos s|\, ds = \frac{2}{\pi} \ne 0.$$

Since (2.13) does not hold, we conclude by Corollary 2.1 that (1.1) with (2.23) is oscillatory.

Example 2.2. Consider system (1.1) for the case

$$a(t) = 1 \ \text{ and } \ b(t) = \frac{d}{dt}\left(\sqrt{t+1} \sin t \right) \ \text{ for } \ t \ge t_0 = 0. \tag{2.24}$$

First, observe that

$$\int_{t_0}^{t} a(s) \left(\int_{t_0}^{s} b(r)\, dr \right) ds = \int_{0}^{t} \sqrt{s+1} \sin s\, ds = 1 - \sqrt{t+1}\cos t + \frac{1}{2} \int_{0}^{t} \frac{\cos s}{\sqrt{s+1}}\, ds = O\big(\sqrt{t}\,\big)$$

as $t \to \infty$, and so,

$$\lim_{t\to\infty} \Big(\int_{t_0}^{t} a(s)\, ds \Big)^{-1} \int_{t_0}^{t} a(s) \Big(\int_{t_0}^{s} b(r)\, dr \Big)\, ds = 0. \tag{2.25}$$

Therefore, (2.17) is satisfied, and hence, as mentioned above, (1.5) is satisfied. Moreover, (1.6) is also satisfied. Indeed, if $t = (5/6)\pi + i\pi$ $(i = 1, 2, 3, \dots)$, then

$$\begin{aligned}
\int_{t_0}^{t} a(s) \Big| \int_{t_0}^{s} b(r)\, dr \Big|\, ds &= \int_{0}^{t} \Big| \sqrt{s+1} \sin s \Big|\, ds = \int_{0}^{(5/6)\pi + i\pi} \Big| \sqrt{s+1} \sin s \Big|\, ds \\
&\ge \sum_{j=0}^{i} \int_{(1/6)\pi + j\pi}^{(5/6)\pi + j\pi} \Big| \sqrt{s+1} \sin s \Big|\, ds \ge \frac{1}{2} \sum_{j=0}^{i} \int_{(1/6)\pi + j\pi}^{(5/6)\pi + j\pi} \sqrt{s+1}\, ds \ge \frac{\pi}{3} \sum_{j=1}^{i} \sqrt{\frac{1}{6}\pi + j\pi + 1} \\
&\ge \frac{\pi}{3} \int_{0}^{i} \sqrt{\frac{1}{6}\pi + s\pi + 1}\, ds = \frac{\pi}{3}\, \frac{2}{3\pi} \Big[\Big(\frac{1}{6}\pi + i\pi + 1 \Big)^{3/2} - \Big(\frac{1}{6}\pi + 1 \Big)^{3/2} \Big].
\end{aligned}$$

Then it is easily seen that (1.6) is satisfied. Consequently, by Theorem 1.2, system (1.1) with (2.24) is oscillatory.

Example 2.3. Consider system (1.1) for the case

$$\alpha = 3; \quad a(t) = \frac{1}{t+1} \quad \text{and} \quad b(t) = \frac{d}{dt} \big((t+1) \sin^3 t \big) \quad \text{for} \quad t \ge t_0 = 0. \tag{2.26}$$

Then

$$\int_{t_0}^{t} a(s) \Big(\int_{t_0}^{s} b(r)\, dr \Big)\, ds = \int_{0}^{t} \sin^3 s\, ds = \frac{2}{3} + \frac{1}{12} \cos 3t - \frac{3}{4} \cos t,$$

and so, the same equality as (2.25) holds. Therefore, (2.17) is satisfied and hence, as mentioned above, (1.5) is satisfied. Moreover, we get

$$\begin{aligned}
\int_{t_0}^{t} a(s) \Big| \int_{t_0}^{s} b(r)\, dr \Big|^{(\alpha+1)/\alpha} ds &= \int_{0}^{t} \frac{1}{s+1} \big| (s+1) \sin^3 s \big|^{4/3}\, ds \\
&= \int_{0}^{t} (s+1)^{1/3} \sin^4 s\, ds \ge \int_{0}^{t} \sin^4 s\, ds = \frac{1}{32} \sin 4t - \frac{1}{4} \sin 2t + \frac{3}{8} t,
\end{aligned}$$

which gives (1.8). Consequently, by Theorem 1.3, system (1.1) with (2.26) is oscillatory.

3 Further results

Theorem 3.1. *Consider the half-linear system (1.1) under condition (1.4). Suppose that*

$$\int_{t_0}^{t} b(s)\, ds \ \ \textit{is bounded on} \ \ [t_0, \infty). \tag{3.1}$$

If system (1.1) is nonoscillatory, then $c(t)$ can be defined by (2.11) and for any integer $n = 1, 2, 3, \dots$,

$$\lim_{t\to\infty} \Big(\int_{t_0}^{t} a(s)\, ds \Big)^{-1} \int_{t_0}^{t} a(s) \Big(\int_{t_0}^{s} b(r)\, dr \Big)^{n} ds = c(t_0)^n. \tag{3.2}$$

Proof. Suppose that (1.1) is nonoscillatory. Condition (3.1) implies (2.17) and hence (1.5) is satisfied. The claim that $c(t)$ can be defined by (2.11) follows from Theorem 2.2. The proof of (3.2) is done by induction. From (2.19) and (3.1) it is clear that $c(t)$ is bounded on $[t_0, \infty)$. By Theorem 2.3, we have (2.13). Therefore,

$$\lim_{t\to\infty}\left(\int_{t_0}^{t} a(s)\,ds\right)^{-1}\int_{t_0}^{t} a(s)c(s)^n\,ds = 0 \tag{3.3}$$

for all $n = 1, 2, 3, \ldots$.

For the case $n = 1$, assertion (3.2) is true. In fact, (3.2) with $n = 1$ is nothing but (2.11) with $t = t_0$. Let $k = 1, 2, 3, \ldots$ be an arbitrary integer and suppose that (3.2) is true for $n = 1, 2, \ldots, k$. We will show that (3.2) is true for $n = k+1$. By (2.19), we can calculate as follows:

$$\begin{aligned}\int_{t_0}^{t} a(s)c(s)^{k+1}\,ds &= \int_{t_0}^{t} a(s)\left(c(t_0) - \int_{t_0}^{s} b(r)\,dr\right)^{k+1} ds\\
&= \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}c(t_0)^{k+1-i}\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\,dr\right)^{i} ds\\
&= \sum_{i=0}^{k}(-1)^i\binom{k+1}{i}c(t_0)^{k+1-i}\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\,dr\right)^{i} ds + (-1)^{k+1}\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\,dr\right)^{k+1} ds.\end{aligned}$$

Therefore, from equality (3.3) and the assumption of induction it follows that

$$\begin{aligned}(-1)^{k+1}\lim_{t\to\infty}\left(\int_{t_0}^{t} a(s)\,ds\right)^{-1}\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\,dr\right)^{k+1} ds &= -\sum_{i=0}^{k}(-1)^i\binom{k+1}{i}c(t_0)^{k+1-i}c(t_0)^i\\
&= -c(t_0)^{k+1}\left\{\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i} - (-1)^{k+1}\binom{k+1}{k+1}\right\} = (-1)^{k+1}c(t_0)^{k+1}.\end{aligned}$$

This shows that (3.2) is true for $n = k+1$. The proof of Theorem 3.1 is complete. □

The following corollary is a consequence of Theorem 3.1.

Corollary 3.1. *Consider the half-linear system* (1.1) *under conditions* (1.4) *and* (3.1). *Suppose that for some* $n = 1, 2, 3, \ldots$,

$$\limsup_{t\to\infty}\left(\int_{t_0}^{t} a(s)\,ds\right)^{-1}\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\,dr\right)^{n} ds = \infty$$

or

$$\begin{aligned}&\liminf_{t\to\infty}\left(\int_{t_0}^{t} a(s)\,ds\right)^{-1}\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\,dr\right)^{n} ds\\
&\qquad < \limsup_{t\to\infty}\left(\int_{t_0}^{t} a(s)\,ds\right)^{-1}\int_{t_0}^{t} a(s)\left(\int_{t_0}^{s} b(r)\,dr\right)^{n} ds < \infty.\end{aligned}$$

Then system (1.1) *is oscillatory.*

In Theorem 3.1, if (1.5) is satisfied and if $n = 2$ and $0 < \alpha \le 1$, then condition (3.1) is unnecessary.

Theorem 3.2. *Consider the half-linear system* (1.1) *under conditions* (1.4) *and* (1.5)*. Let* $0 < \alpha \le 1$*. If system* (1.1) *is nonoscillatory, then* $c(t)$ *is well-defined by* (2.11) *and*

$$\lim_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)^2 ds = c(t_0)^2. \tag{3.4}$$

Proof. Suppose that (1.1) is nonoscillatory. Then, by Theorem 2.2, the function $c(t)$ can be defined by (2.11). If $0 < \alpha < 1$, then Hölder's inequality gives

$$\int_{t_0}^{t} a(s)|c(s)|^2\,ds \le \Big(\int_{t_0}^{t} a(s)\,ds\Big)^{(-\alpha+1)/(\alpha+1)}\Big(\int_{t_0}^{t} a(s)|c(s)|^{(\alpha+1)/\alpha}\,ds\Big)^{2\alpha/(\alpha+1)} \tag{3.5}$$

for $t \ge t_0$. If $\alpha = 1$, then (3.5) is clear. Therefore, in either case,

$$\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)c(s)^2\,ds \le \Big[\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)|c(s)|^{(\alpha+1)/\alpha}\,ds\Big]^{2\alpha/(\alpha+1)}$$

for all large t. By Theorem 2.3, equality (2.14) holds. Therefore, we get

$$\lim_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)c(s)^2\,ds = 0. \tag{3.6}$$

Using (2.19), we have

$$\begin{aligned}\int_{t_0}^{t} a(s)c(s)^2\,ds &= \int_{t_0}^{t} a(s)\Big(c(t_0) - \int_{t_0}^{s} b(r)\,dr\Big)^2 ds\\ &= c(t_0)^2\int_{t_0}^{t} a(s)\,ds - 2c(t_0)\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)ds + \int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)^2 ds\end{aligned}$$

for $t \ge t_0$. Then, by (2.11) and (3.6), it is clear that

$$0 = c(t_0)^2 - 2c(t_0)c(t_0) + \lim_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)^2 ds,$$

which yields (3.4). The proof of Theorem 3.2 is is complete. □

Corollary 3.2. *Consider the half-linear system* (1.1) *under conditions* (1.4) *and* (1.5)*. Let* $0 < \alpha \le 1$*. If*

$$\limsup_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)^2 ds = \infty \tag{3.7}$$

or

$$\begin{aligned}&\liminf_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)^2 ds\\ &\qquad < \limsup_{t\to\infty}\Big(\int_{t_0}^{t} a(s)\,ds\Big)^{-1}\int_{t_0}^{t} a(s)\Big(\int_{t_0}^{s} b(r)\,dr\Big)^2 ds < \infty,\end{aligned} \tag{3.8}$$

then system (1.1) *is oscillatory.*

Note that, for the case $\alpha = 1$, Corollary 3.2 coincides with Theorem 1.3.

In [1], Butler, Erbe and Mingarelli showed that, for the case $\alpha = 1$, $a(t) = 1$ and $b(t) = q(t)$, namely, for the linear equation

$$x'' + q(t)x = 0, \tag{3.9}$$

conditions (2.17) and (3.7) with $a(t) = 1$ and $b(t) = q(t)$ are sufficient for the oscillation of (3.9). They found the result in connection with the study of oscillation theory for the second order $n \times n$ matrix linear differential systems.

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