Memoirs on Differential Equations and Mathematical Physics

Volume 90, 2023, 15–25

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LOCATION OF THE LIMIT CYCLE FOR A CLASS OF LIÉNARD SYSTEMS BY MEANS OF DULAC–CHERKAS FUNCTIONS

Abstract. Dulac–Cherkas functions can be used to estimate the number of limit cycles and to approximate their location. We consider a class of Liénard systems containing the van der Pol system as a special case and present two approaches to construct Dulac–Cherkas functions. By means of two Dulac–Cherkas functions, we improve the Poincaré–Bendixson annulus for the van der Pol system which has been derived in our previous paper [4].

2020 Mathematics Subject Classification. 34C05, 34C07.

Key words and phrases. Limit cycle, Liénard system, van der Pol system, Dulac–Cherkas function, Poincaré–Bendixson annulus.

რეზიუმე. დულაკ-ჩერკასის ფუნქციები შესაძლებელია გამოყენებულ იქნას ზღვრული ციკლების რაოდენობის შესაფასებლად და მათი მიახლოებითი მდებარეობის დასადგენად. ჩვენ განვიხილავთ ლენარდის სისტემების კლასს, რომელიც, როგორც განსაკუთრებულ შემთხვევას, შეიცავს ვან დერ პოლის სისტემას, და წარმოვადგენთ ორ მიდგომას დულაკ-ჩერკასის ფუნქციების ასაგებად. დულაკ-ჩერკასის ორი ფუნქციის საშუალებით ჩვენ ვაუმჯობესებთ პუანკარე-ბენდიქსონის რგოლს ვან დერ პოლის სისტემისთვის, რომელიც მიღებული იყო ჩვენს წინა ნაშრომში [4].

1 Introduction

Limit cycles as special limit sets play a fundamental role in the description of the global phase portrait of planar autonomous systems, as well as in modelling self-sustained oscillations in different fields of natural sciences. Their existence can be established by means of the bifurcation theory or the construction of a Poincaré–Bendixson annulus. The inner and the outer boundaries of a Poincaré–Bendixson annulus consist of simple closed curves in the phase plane with the property that if any trajectory of the considered planar system meets this boundary, it will enter the annulus either for increasing or decreasing time. They represent generalized curves without contact in the sense of A. A. Andronov et al. (see [1]). The crucial problem how to construct such curves depends essentially on the system under consideration and requires, in general, sophisticated geometric constructions. The goal of this paper is to describe a general method to construct generalized curves without contact by means of Dulac–Cherkas functions. This class of functions has been introduced by L. Cherkas in his seminal paper [2].

The structure of our paper is as follows. In Section 2, we introduce the class of Dulac–Cherkas functions and describe their fundamental properties. In Section 3, we construct Dulac–Cherkas functions for a class of Liénard systems and use these functions to derive conditions such that this class of autonomous systems has at most one limit cycle. In Section 4, we apply these results to the van der Pol system. By means of two different Dulac–Cherkas functions, we construct two generalized closed simple curves without contact as inner boundaries for a Poincaré–Bendixson annulus of the van der Pol system. Both boundaries can be used to construct an improved inner boundary. In Section 5, we use a Dulac–Cherkas function to improve an outer boundary for a Poincaré–Bendixson annulus of the van der Pol system, which has been derived in our paper [4] without a Dulac–Cherkas function. In this way we obtain a new global algebraic Poincaré–Bendixson annulus for the van der Pol system.

2 Dulac–Cherkas functions

We consider planar autonomous systems

$$\frac{dx}{dt} = P(x, y, \lambda), \quad \frac{dy}{dt} = Q(x, y, \lambda) \tag{2.1}$$

depending on a real parameter λ under the assumption

(A) $\qquad P, Q \in C^{\,1}_{(x,y)}\,{}^{0}_{\lambda}(\mathcal{G} \times \Lambda, \mathbb{R}),$

where $\mathcal{G}$ is an open region of the phase plane, Λ is some open interval. We denote by X the vector field defined by (2.1). First, we recall the definition of a Dulac function.

Definition 2.1. Suppose the assumption (A) to be valid. A function B belonging to the class $C^{\,1}_{(x,y)}\,{}^{0}_{\lambda}(\mathcal{G} \times \Lambda, \mathbb{R})$ is called a Dulac function of system (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$ if the expression

$$\operatorname{div}(BX) \equiv \frac{\partial(BP)}{\partial x} + \frac{\partial(BQ)}{\partial y} \equiv (\operatorname{grad} B, X) + B \operatorname{div} X$$

does not change sign in $\mathcal{G}$ and vanishes only on a set $\mathcal{N}_\lambda \subset \mathcal{G}$ of measure zero for $\lambda \in \Lambda$.

The class of Dulac functions has been generalized by L. A. Cherkas in 1997 (see [2]). The corresponding generalized Dulac function, which is called Dulac–Cherkas function nowerdays, is defined as follows.

Definition 2.2. Suppose the assumption (A) holds. A function $\Psi \in C^{1}_{(x,y)}\,{}^{0}_{\lambda}(\mathcal{G} \times \Lambda, \mathbb{R})$ is called a Dulac–Cherkas function of system (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$ if there exists a real number $\kappa \neq 0$ such that the function Φ satisfies

$$\Phi := (\operatorname{grad} \Psi, X) + \kappa \Psi \operatorname{div} X > 0 \ \ (< 0) \ \text{ in } \mathcal{G} \text{ for } \lambda \in \Lambda. \tag{2.2}$$

Remark 2.1. Condition (2.2) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set $\mathcal{N}_\lambda$ of measure zero, and that no simple closed curve of this set is a limit cycle.

Remark 2.2. In case $\kappa = 1$, Ψ is a Dulac function.

From relation (2.2) it immediately follows

Proposition 2.1. *If $\Psi(x, y, \lambda)$ is a Dulac–Cherkas function, then so is $c\Psi(x, y, \lambda)$, where c is any real number different from zero.*

Proposition 2.2. *The sign of the function Φ on the curve $\Psi = 0$ does not necessarily imply in which direction a trajectory of system (2.1) crosses the curve $\Psi = 0$ for increasing t.*

Proof. From (2.2) we get $\Phi_{|\Psi=0} = (\operatorname{grad}\Psi, X)_{|\Psi=0}$. If Ψ is a Dulac–Cherkas function, then by Proposition 2.1 so is $-\Psi$. Since

$$(\operatorname{grad}(-\Psi), X) = (\operatorname{grad}\Psi, -X)$$

holds, the claim is proved. □

In the sequel, we introduce the subset $\mathcal{W}_\lambda$ of $\mathcal{G}$ defined by

$$\mathcal{W}_\lambda := \big\{(x, y) \in \mathcal{G} : \ \Psi(x, y, \lambda) = 0\big\}.$$

From Definition 2.2 we immediately get

Lemma 2.1. *Suppose the assumption* (A) *to be valid. Let Ψ be a Dulac–Cherkas function of system (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$. Then any curve $\mathcal{K}_\lambda$ of $\mathcal{W}_\lambda$ having only a finite number of points, where $(\operatorname{grad}\Psi, X)$ vanishes, is a generalized curve without contact for system (2.1). Especially, if $\mathcal{K}_\lambda$ is a generalized simple closed curve without contact, it can be used as a boundary for a Poincaré–Bendixson annulus.*

The following theorem is a special case of a more general result established in [3].

Theorem 2.1. *Suppose the assumption* (A) *holds. Let $\mathcal{G}$ be a simply connected region, let Ψ be a Dulac–Cherkas function of (2.1) in $\mathcal{G}$ for $\lambda \in \Lambda$ such that $\mathcal{W}_\lambda$ contains exactly one simple closed curve $\mathcal{O}_\lambda$ in $\mathcal{G}$. Then in the case $\kappa < 0$ system (2.1) has for $\lambda \in \Lambda$ at most one limit cycle in $\mathcal{G}$, and if it exists, it surrounds $\mathcal{W}_\lambda$ and is hyperbolic.*

This theorem implies

Corollary. *Under the assumptions of Theorem 2.1, the simple closed curve $\mathcal{O}_\lambda$ can be used as an interior boundary for a Poincaré–Bendixson annulus of system (2.1), provided it is a generalized simple closed curve without contact.*

3 On the construction of Dulac–Cherkas functions

The key point in the construction of a Dulac–Cherkas function consists in finding a function $\Phi(x, y, \lambda)$ defined in (2.2) which is definite in some region of the phase plane for λ belonging to some interval. One possibility to simplify this problem is to look for a procedure such that Φ depends finally on only one phase variable. In what follows, we demonstrate this approach for a class of Liénard systems.

We consider the system

$$\begin{aligned} \frac{dx}{dt} &= -y, \\ \frac{dy}{dt} &= h_0(x) + h_1(x)y \end{aligned} \tag{3.1}$$

corresponding to the Liénard equation

$$\frac{d^2x}{dt^2} - h_1(x)\,\frac{dx}{dt} + h_0(x) = 0$$

and seek a Dulac–Cherkas function for system (3.1) in the form

$$\Psi(x,y) := \Psi_0(x) + \Psi_1(x)y + \Psi_2(x)y^2,$$

where we assume

$$\Psi_2(x) \neq 0 \ \text{ for all } x. \tag{3.2}$$

For the corresponding function Φ, from (2.2) we obtain

$$\begin{aligned}\Phi(x,y) = \Psi_1(x)h_0(x) + \kappa h_1(x)\Psi_0(x) + \big(-\Psi_0'(x) + 2\Psi_2(x)h_0(x) + \Psi_1(x)h_1(x)(1+\kappa)\big)y \\ + \big(\Psi_2(x)h_1(x)(2+\kappa) - \Psi_1'(x)\big)y^2 - \Psi_2'(x)y^3.\end{aligned} \tag{3.3}$$

In order to get Φ as a function depending only on x, we equal the coefficients before y, y^2 and y^3 to zero which yields the relations

$$\begin{aligned}\Psi_2' &= 0,\\ \Psi_1' &= (\kappa+2)h_1(x)\Psi_2,\\ \Psi_0' &= (\kappa+1)h_1(x)\Psi_1 + 2h_0(x)\Psi_2,\end{aligned} \tag{3.4}$$

representing a system of linear differential equations to determine Ψ_0, Ψ_1 and Ψ_2. For what follows, we assume

(H) The functions $h_0, h_1 : \mathbb{R} \to \mathbb{R}$ are continuous, h_0 is not identically zero.

Under the condition that relations (3.4) are satisfied, the function Φ defined in (3.3) reads as

$$\Phi(x,y) = \Psi_1(x)h_0(x) + \kappa h_1(x)\Psi_0(x). \tag{3.5}$$

In order to guarantee Φ to be a definite function of x, we have freedom in choosing κ and the constants c_i appearing in the process of integrating system (3.4), we can additionally impose the conditions on h_0 and h_1.

Our first approach to solve system (3.4) consists in choosing $\kappa = -2$. In what follows, we suppose

(B_1) The function h_1 has the form

$$h_1(x) = a\int^x h_0(s)\,ds + b, \tag{3.6}$$

where $a \neq 0$.

Then the following assertion is valid.

Theorem 3.1. *Under the assumptions* (H) *and* (B_1), *the function*

$$\Psi(x,y) = h_1(x) + \frac{a}{2}\,y^2$$

is a Dulac–Cherkas function for system (3.1) *in* $\mathbb{R}^2$.

Proof. From (3.4) and under condition (3.2) we obtain

$$\Psi_2(x) \equiv c_2 \neq 0, \quad \Psi_1(x) \equiv c_1.$$

Setting $c_1 = 0$, by (3.4) we have

$$\Psi_1(x) \equiv 0, \quad \Psi_2(x) \equiv c_2, \quad \Psi_0(x) = 2c_2\int^x h_0(s)\,ds + c_0, \tag{3.7}$$

and from (3.5) we get

$$\Phi(x,y) = -2h_1(x)\left(2c_2\int^x h_0(s)\,ds + c_0\right). \tag{3.8}$$

Using (3.6), from (3.8) we obtain

$$\Phi(x,y) = \Big(-2a \int^x h_0(s)\,ds - 2b \Big)\Big(2c_2 \int^x h_0(s)\,ds + c_0 \Big).$$

Setting $2c_2 = a$ and $c_0 = b$, we have

$$\Phi(x,y) = -2(h_1(x))^2 \le 0,$$

that is, Φ is a definite function in $\mathbb{R}^2$. Additionally, by (3.7) there holds

$$\Psi_0(x) = a \int^x h_0(s)\,ds + b = h_1(x), \quad \Psi_1(x) \equiv 0, \quad \Psi_2(x) = \frac{a}{2}\,.$$

Thus the assertion of Theorem 3.1 is valid. □

For what follows, we assume

(B_2) The set

$$\mathcal{W} := \big\{(x,y) \in \mathbb{R}^2 :\ \Psi(x,y) = 0\big\} = \Big\{(x,y) \in \mathbb{R}^2 :\ h_1(x) + \frac{a}{2}\,y^2 = 0\Big\}$$

contains exactly one simply closed curve $\mathcal{O}$.

Then, according to Theorem 2.1, the following theorem holds.

Theorem 3.2. *Under the assumptions* (H), (B_1) *and* (B_2), *system* (3.1) *has at most one limit cycle* Γ *in* $\mathbb{R}^2$. *If* Γ *exists, it surrounds* $\mathcal{O}$ *and is hyperbolic. The closed curve* $\mathcal{O}$ *can be used as an inner boundary for a Poincaré–Bendixson annulus of system* (3.1).

Our second approach to solve system (3.4) consists in choosing $\kappa = -1$. From (3.4) we obtain

$$\Psi_2(x) \equiv c_2 \ne 0, \quad \Psi_1(x) = c_2 \int^x h_1(s)\,ds + c_1, \quad \Psi_0(x) = 2c_2 \int^x h_0(s)\,ds + c_0.$$

Setting $c_1 = 0$, we get

$$\Phi(x,y) = c_2 h_0(x) \int^x h_1(s)\,ds - h_1(x)\Big(2c_2 \int^x h_0(s)\,ds + c_0 \Big). \tag{3.9}$$

Further, we suppose

(B_3) There are real numbers $c_2 = c_2^* \ne 0$ and $c_0 = c_0^*$ such that the expression

$$c_2^* h_0(x) \int^x h_1(s)\,ds - h_1(x)\Big(2c_2^* \int^x h_0(s)\,ds + c_0^* \Big)$$

is definite on $\mathbb{R}$.

This assumption implies that the function Φ in (3.9) is definite on $\mathbb{R}$ and the following theorem holds.

Theorem 3.3. *Under the assumptions* (H) *and* (B_3), *the function*

$$\Psi(x,y) = 2c_2^* \int^x h_0(s)\,ds + c_0^* + c_2^* \int^x h_1(s)\,ds\, y + c_2^* y^2$$

is a Dulac–Cherkas function for system (3.1) *in* $\mathbb{R}^2$.

If we additionally assume

(B_4) The set

$$\mathcal{W} := \left\{(x,y) \in \mathbb{R}^2 : \ 2c_2^* \int^x h_0(s)\,ds + c_0^* + c_2^* \int^x h_1(s)\,ds\, y + c_2^* y^2 = 0\right\}$$

contains exactly one simply closed curve $\mathcal{O}$,

then, according to Theorem 2.1, the following theorem holds.

Theorem 3.4. *Under the assumptions* (H), (B_3) *and* (B_4), *system* (3.1) *has at most one limit cycle* Γ *in* $\mathbb{R}^2$. *If* Γ *exists, it surrounds* $\mathcal{O}$ *and is hyperbolic. The closed curve* $\mathcal{O}$ *can be used as an inner boundary for a Poincaré–Bendixson annulus of system* (3.1).

In the next section, we apply the results obtained in this section to a qualitative study of the van der Pol equation. Especially, we derive an improved Poincaré–Bendixson annulus to include the unique limit cycle for all parameter values.

4 Dulac–Cherkas functions and the qualitative study of the van der Pol system

We consider the differential equation

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\frac{dx}{dt} + x = 0 \tag{4.1}$$

depending on the real parameter μ. This equation was introduced by the Dutch engineer and physicist Balthasar van der Pol [6] in 1926 to describe self-oscillations in a triod circuit. The parameter μ characterizes the damping force of the oscillations. If we replace t by $-t$ and μ by $-\mu$, then equation (4.1) remains invariant. Thus, to study the phase portrait of equation (4.1), we can restrict ourselves to the case $\mu \geq 0$. We note that (4.1) can be rewritten as the Liénard system

$$\begin{aligned} \frac{dx}{dt} &= -y, \\ \frac{dy}{dt} &= x - \mu(x^2-1)y. \end{aligned} \tag{4.2}$$

It is well-known (see, e.g., [5]) that the van der Pol equation (4.1) has for $\mu > 0$ a unique limit cycle Γ_μ which is orbitally stable and hyperbolic. For $\mu = 0$, the phase portrait of system (4.2) consists of a continuum of concentric circles centered at the origin. When the parameter μ passes zero, the limit cycle Γ_μ bifurcates from the circle of radius two. For small μ, the periodic solutions $p(t,\mu)$ describing the limit cycle Γ_μ behaves like the solution of the harmonic oscillator. For very large values of the parameter μ, the nature of oscillations is characterized by intermittent jumps that occur every time when the system becomes unstable, and the period of these oscillations is determined by the duration of the capacitive discharge, which is called as the relaxation time. It means that for large μ, the solution $p(t,\mu)$ represents a relaxation oscillation.

In what follows, we show that the theorems proved in Section 3 can be applied to system (4.2). For this purpose, we use the notation

$$h_0(x) \equiv x, \quad h_1(x) \equiv -\mu(x^2-1) \tag{4.3}$$

due to comparing systems (3.1) and (4.2). From (4.3) we find that the hypothesis (H) is valid and (B_1) is fulfilled with $a = -2\mu$ and $b = \mu$.

Applying Theorem 3.1 and Proposition 2.1, we get the following result.

Theorem 4.1. *The function* $\Psi_1(x,y,\mu) := x^2 + y^2 - 1$ *for* $\mu > 0$ *is a Dulac–Cherkas function for the van der Pol system* (4.2).

Remark 4.1. The Dulac–Cherkas function $\Psi_1(x,y,\mu)$ has been derived originally by L. Cherkas in [2].

We note that the set $\mathcal{W}_\mu := \{(x,y) \in \mathbb{R}^2 : \ \Psi_1(x,y,\mu) = 0\}$ consists for any $\mu > 0$ of the unit circle $\mathcal{E}$, and that a trajectory of system (4.2), starting in the set bounded by the unit circle, will leave the unit circle for increasing t. Thus the unit circle can be used as an interior boundary of a Poincaré–Bendixson annulus and, by Theorem 3.2, we have

Theorem 4.2. *System (4.2) has at most one limit cycle Γ_μ in $\mathbb{R}^2$. If Γ_μ exists, it surrounds the unit circle, is hyperbolic and asymptotically orbitally stable.*

The Dulac–Cherkas function Ψ_1 does not depend on the parameter μ. For a better approximation of the limit cycle Γ_μ, we need a Dulac–Cherkas function depending on μ. To reach this goal, we have to use Theorem 3.3. This requires that the supposition (B_3) is fulfilled. Using (4.3) we have

$$\Phi(x,y,\mu) \equiv c_2 h_0(x) \int^x h_1(s)\,ds - h_1(x)\Big(2c_2 \int^x h_0(s)\,ds + c_0\Big) \equiv \mu\Big(\frac{2}{3}\,c_2 x^4 + c_0 x^2 - c_0\Big).$$

If we assume

$$c_2 > 0, \quad c_0 < 0, \quad 8c_2 \geq -3c_0,$$

then we have

$$\Phi(x,y,\mu) \geq 0 \ \text{ for } \ \mu > 0. \tag{4.4}$$

From Theorem 3.3 and Proposition 2.1, for $8c_2 = -3c_0$, we obtain

Theorem 4.3. *The polynomial*

$$\Psi_2(x,y,\mu) := x^2 + y^2 - \frac{8}{3} + \mu\Big(x - \frac{x^3}{3}\Big)y$$

is a Dulac–Cherkas function for system (4.2) in $\mathbb{R}^2$ for $\mu > 0$.

Now, we study the set

$$\mathcal{W}_\mu := \Big\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - \frac{8}{3} + \mu\Big(x - \frac{x^3}{3}\Big)y = 0\Big\}. \tag{4.5}$$

For $\mu = 0$, the set $\mathcal{W}_\mu$ consists of the circle $x^2 + y^2 = 8/3$. Further, we note that the intersection of the set $\mathcal{W}_\mu$ with the straight lines $x = \pm\sqrt{3}$ is empty. Thus the set $\mathcal{W}_\mu$ consists of three curves which are symmetric to the origin and separated by the straight lines $x = \pm\sqrt{3}$. The curve surrounding the origin is a simply closed curve $\mathcal{O}_\mu$ located in the region bounded by the straight lines $x = \pm\sqrt{3}$ and tending to the circle $x^2 + y^2 = 8/3$ as μ tends to zero. The other two curves $\mathcal{K}_\mu^1$ and $\mathcal{K}_\mu^3$ look like hyperbolas located in the first and the third quadrants (see Fig. 1).

Thus we have

Lemma 4.1. *The set $\mathcal{W}_\mu$ consists of a simple closed curve $\mathcal{O}_\mu$ surrounding the origin and of two unbounded curves $\mathcal{K}_\mu^1$ and $\mathcal{K}_\mu^3$ located in the first and third quadrants, respectively. All curves are symmetric with respect to the origin.*

If we denote by $\mathcal{I}_\mu$ the finite region bounded by the closed curve $\mathcal{O}_\mu$, then it is easy to verify that a trajectory of system (4.2) starting in $\mathcal{I}_\mu$ will leave this region for increasing t. Using this fact and Lemma 4.1, from Theorem 3.4 we obtain

Theorem 4.4. *The van der Pol system (4.2) has at most one limit cycle Γ_μ in $\mathbb{R}^2$. If Γ_μ exists, it surrounds $\mathcal{O}_\mu$ and is orbitally stable. The closed curve $\mathcal{O}_\mu$ can be used as an interior boundary of a Poincaré–Bendixson annulus.*

In what follows, we compare the closed curve $\mathcal{O}_\mu$ with the unit circle $\mathcal{E}$ as interior boundaries of a Poincaré–Bendixson annulus for the van der Pol system (4.2). It is obvious that for sufficiently small μ the closed curve $\mathcal{O}_\mu$ surrounds the unit circle $\mathcal{E}$ and is a better interior boundary of a Poincaré–Bendixson annulus for system (4.2) than the unit circle $\mathcal{E}$. The problem of intersection of the closed curves $\mathcal{E}$ and $\mathcal{O}_\mu$ is equivalent to the existence of roots of the equation $f(y) = 5/\mu$ in $(0,1)$, where

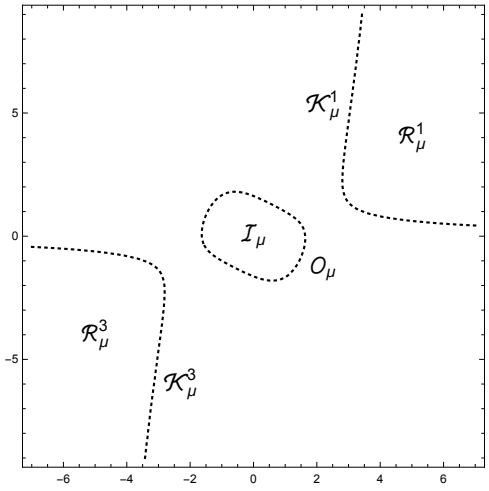
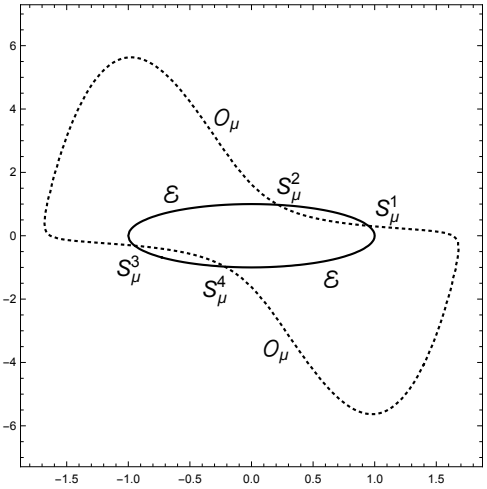
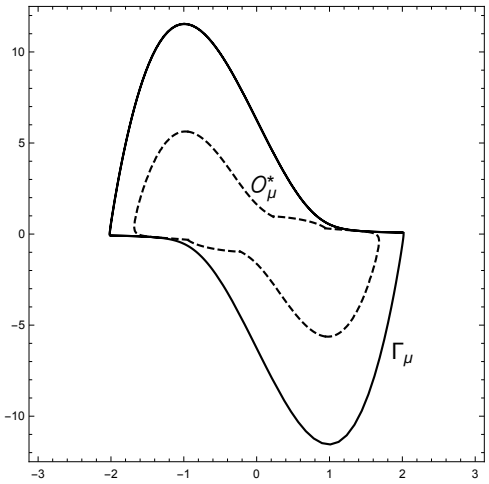


Figure 1: Location of the curves $\mathcal{K}^1_\mu$, $\mathcal{K}^3_\mu$, $\mathcal{O}_\mu$ for $\mu = 1$

f is defined by $f(y) := \sqrt{1-y^2}y(2+y^2)$. Since the function f takes its maximum in $(0,1)$ at $y_m = \sqrt{\frac{-1+\sqrt{33}}{8}}$, we get that the closed curves $\mathcal{E}$ and $\mathcal{O}_\mu$ intersect for $\mu > 5/f(y_m) =: \mu_1 \approx 3.925$. Thus, for $0 < \mu < \mu_1$, the closed curve $\mathcal{O}_\mu$ is a better interior boundary of a Poincaré–Bendixson annulus than the unit circle $\mathcal{E}$. For $\mu > \mu_1$, the curve $\mathcal{O}_\mu$ intersects the unit circle $\mathcal{E}$ at four point denoted by $S^1_\mu, S^2_\mu, S^3_\mu, S^4_\mu$ (see Fig. 2(a)). Thus the arcs of the unit circle $\mathcal{E}$ bounded by the points S^1_μ, S^2_μ and S^3_μ, S^4_μ represent the curves without contact with respect to system (4.2) and can be used to improve the closed curve $\mathcal{O}_\mu$ as an interior boundary for system (4.2). The improved interior boundary $\mathcal{O}^*_\mu$ is represented in Fig. 2(b) together with the limit cycle Γ_μ.



(a) Intersection of the unit circle $\mathcal{E}$ (solid curve) and the closed curve $\mathcal{O}_\mu$ (dashed curve) for $\mu = 8$



(b) Improved interior boundary $\mathcal{O}^*_\mu$ (dashed curve) together with the limit cycle Γ_μ (solid curve) for $\mu = 8$

Figure 2

Thus we can conclude that the use of different Dulac–Cherkas functions helps to improve the interior boundary of a Poincaré–Bendixson annulus for the van der Pol system (4.2).

If we denote by $\mathcal{R}^1_\mu$ ($\mathcal{R}^3_\mu$) the region in the first (third) quadrant bounded by the curve $\mathcal{K}^1_\mu$ ($\mathcal{K}^3_\mu$) containing all points of the straight line $y = x$ for a sufficiently large positive (negative) x (see Fig. 1), then we get from (4.4) that a trajectory of system (4.2) starting in $\mathcal{R}^1_\mu$ ($\mathcal{R}^3_\mu$) will leave this region for

increasing t. Thus the curves $\mathcal{K}_\mu^1$ and $\mathcal{K}_\mu^3$ help to improve the location of a possible limit cycle Γ_μ of the van der Pol system (4.2). As we already mentioned, for $\mu = 0$ the phase portrait of system (4.2) consists of a continuum of circles centered at the origin and the limit cycle Γ_μ bifurcates at $\mu = 0$ from the circle with radius 2.

5 Explicit global algebraic Poincaré–Bendixson annulus for the van der Pol system

In our paper [4], we presented a procedure to construct an outer boundary of a Poincaré–Bendixson annulus $\mathcal{A}_\mu$ for the van der Pol system (4.2) without using a Dulac–Cherkas function. Our result reads as follows.

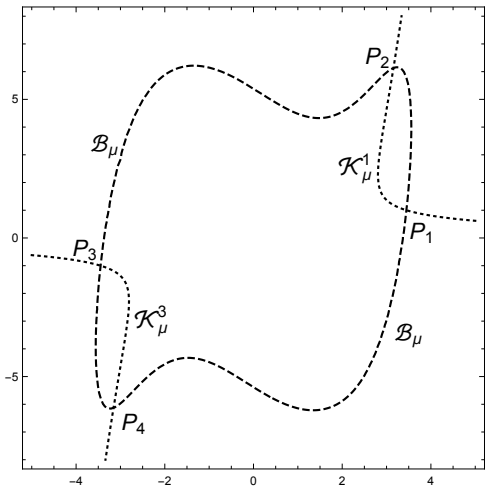
Theorem 5.1. *The simple closed curve $\mathcal{B}_\mu$ defined by*

$$\mathcal{B}_\mu := \left\{(x,y) \in \mathbb{R}^2 :\ y^2 + \mu y x(2 - x^{2/3}) + (1+\mu^2)x^2 - \frac{7\mu^2}{12}\,x^4 + \frac{\mu^2}{18}\,x^6 - 8 - 3\mu - 18\mu^2 = 0\right\} \quad (5.1)$$

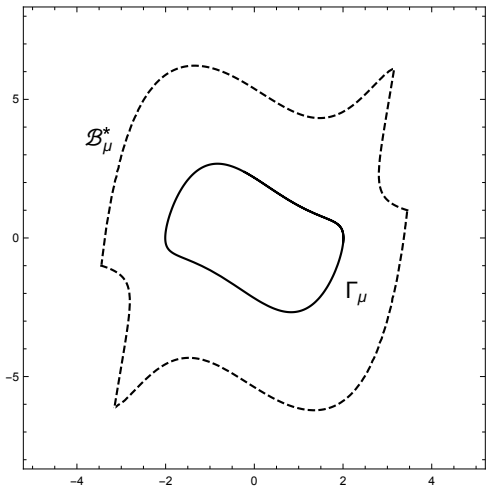
is centrosymmetric and represents an outer boundary for a Poincaré–Bendixson annulus for the van der Pol system (4.2).

If we denote by $\mathcal{I}_{\mathcal{B}_\mu}$ the finite region bounded by $\mathcal{B}_\mu$, then it is easy to show that any trajectory of system (4.2) meeting $\mathcal{B}_\mu$ will enter $\mathcal{I}_{\mathcal{B}_\mu}$ for increasing t. Using the inner boundary $\widetilde{\mathcal{O}}_\mu$, we have the following result.

Theorem 5.2. *The simple closed curves $\mathcal{B}_\mu$ and $\widetilde{\mathcal{O}}_\mu$ form a global algebraic Poincaré–Bendixson annulus for the van der Pol system (4.2) containing for all $\mu > 0$ the orbitally stable limit cycle Γ_μ.*



(a) Location of the curve $\mathcal{B}_\mu$ (dashed curve) and the curves $\mathcal{K}_\mu^1$ and $\mathcal{K}_\mu^3$ (dotted curves) for $\mu = 1$



(b) Improved exterior boundary $\mathcal{B}_\mu^*$ (dashed curve) together with the limit cycle Γ_μ (solid curve) for $\mu = 1$

Figure 3

In what follows, we show that the outer boundary $\mathcal{B}_\mu$ can be improved by the curves $\mathcal{K}_\mu^1$ and $\mathcal{K}_\mu^3$ belonging to the zero-level set $\mathcal{W}_\mu$ of the Dulac–Cherkas function $\Psi_2(x,y,\mu)$. For this purpose, we consider the mutual location of the simple closed curve $\mathcal{B}_\mu$ and of the curves without contact $\mathcal{K}_\mu^1$ and $\mathcal{K}_\mu^3$. From (5.1) and (4.5), it follows that for a sufficiently small μ, the intersection of these curves is empty. If μ increases, there is a unique value μ_2, $\mu_2 \approx 0.6012$ such that the curve $\mathcal{K}_{\mu_2}^1$ touches the closed curve $\mathcal{B}_{\mu_2}$ at exactly one point in the first quadrant and the curve $\mathcal{K}_{\mu_2}^3$ touches the closed curve

$\mathcal{B}_{\mu_2}$ at exactly one point in the third quadrant. For $\mu > \mu_2$, the curve $\mathcal{K}_\mu^1$ intersects the closed curve $\mathcal{B}_\mu$ at exactly two points P_1 and P_2. The arc of the curve $\mathcal{K}_\mu^1$ between the points P_1 and P_2 can be used to improve the closed curve $\mathcal{B}_\mu$ as an outer boundary of a Poincaré–Bendixson annulus for the van der Pol system (4.2). The corresponding result is valid for the curve $\mathcal{K}_\mu^3$ between the points P_3 and P_4 (see Fig. 3(a)). The improved outer closed curve is denoted by $\mathcal{B}_\mu^*$ (see Fig. 3(b)).

Consequently, we have

Theorem 5.3. *The closed curves $\widetilde{\mathcal{B}}_\mu$ and $\widetilde{\mathcal{O}}_\mu$ form a global algebraic Poincaré–Bendixson annulus for the van der Pol system (4.2) containing for all $\mu > 0$ the orbitally stable limit cycle Γ_μ.*

In this way, we have shown that the application of different Dulac–Cherkas functions can be used to improve the inner boundary, as well as the outer boundary of a Poincaré–Bendixson annulus for the van der Pol system (4.2). It is clear that this approach can also be used for the construction of Poincaré–Bendixson annuli of other dynamical systems.

Acknowledgments

The first author acknowledgments the financial support by Belarusian Republican Foundation for Fundamental Research (Project # F23U-008).

The authors are grateful to the referee for some valuable hints that improve the paper.

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(Received 22.06.2023; accepted 08.07.2023)

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