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VARIATIONAL APPROACH FOR A FOURTH-ORDER EQUATION ON A NONLINEAR ELASTIC FOUNDATION

Abstract. In this paper, we investigate the existence of positive weak solutions for a fourth-order differential equation with perturbed nonlinear term. This equation depends on two real parameters. The approach is based on variational methods and critical point theory.

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1 Introduction

The aim of the present paper is to establish the existence of positive solutions for the following fourth-order problem:

$$\begin{cases} u^{(iv)}(x) = \lambda\, \alpha(x) f(x, u(x)) + h(x, u(x)), \quad x \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu\, g(u(1)), \end{cases} \tag{1.1}$$

where λ and μ are positive parameters, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a non-negative L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a non-positive continuous function, $\alpha \in L^\infty([0,1])$, $\alpha(x) \geq 0$ for a.e. $x \in \mathbb{R}$, $\alpha \not\equiv 0$, and $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a non-negative Carathéodory function, and there exists $0 < L < 1$ such that $h(x,t) \leq L|t|$ for each $x \in [0,1]$ and $t \in \mathbb{R}$.

Problem (1.1) is related to the deflections of elastic beams based on nonlinear elasticity. In relation to problem (1.1), there is an interesting physical description.

Suppose an elastic beam of length $d = 1$, which is clamped at its left side $x = 0$ and resting on a kind of elastic bearing at its right side $x = 1$, is given by μg. Along its length, a load $\lambda\alpha f + h$ is added to cause deformations. If $u = u(x)$ denotes the configuration of the deformed beam, then since $u'''(1)$ represents the shear force at $x = 1$, the condition $u'''(1) = \mu g(u(1))$ means that the vertical force is equal to $\mu g(u(1))$ which denotes a relation, possibly nonlinear, between the vertical force and the displacement $u(1)$. In addition, since $u''(1) = 0$ indicates that there is no bending moment at $x = 1$, the beam is resting on the bearing μg.

Different models and their applications for problems such as (1.1) can be derived from [12]. We refer the reader to references [5] and [11] for a physical justification of this model. There is an increasing interest in studying the fourth-order boundary value problems, because the change of the static form beam or the support of a rigid body can be described by a fourth-order equation. Also, a model to study travelling waves in suspension bridges can be furnished by nonlinear fourth-order equations (for instance, see [7]). So the study of classical bending theory of elastic beams and especially fourth-order differential equations is very important in engineering sciences. Hence, several results concerning the existence of multiple solutions for the fourth-order boundary value problems are known. For example, using a variational methods, the existence of three solutions for special cases of problem (1.1) has been established in [4] and [13]. In [9], the author obtained the existence of at least two positive solutions for the problem

$$\begin{cases} u^{(iv)}(x) = f(x, u(x)), \quad x \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = g(u(1)), \end{cases} \tag{1.2}$$

based on the variational methods and maximum principle. It should be noted that the function f is assumed to be continuous. By assuming appropriate conditions on f and g, the author guarantees positive solutions to problem (1.2). Also, the existence and multiplicity results for this kind of problems were considered in [2, 6, 8]. In all these works, the critical point theory is applied.

Moreover, in [10], the authors considered numerical solutions for problem (1.2) with nonlinear boundary conditions.

In particular, using a variational methods, the existence of non-zero solutions for problem

$$\begin{cases} u^{(iv)}(x) = f(x, u(x)), \quad x \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu g(u(1)) \end{cases}$$

has been established in [2].

In the present paper, using a three critical points theorem obtained in [1], we establish the existence of at least three weak solutions for problem (1.1).

The paper is organized as follows. In Section 2, we establish all the preliminary results that we need, and in Section 3, we present our main results.

2 Preliminaries

In this section, we recall some basic facts and introduce the necessary notation.

Definition 2.1. A function $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function if:

(C_1) the function $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$;

(C_2) the function $t \to f(x,t)$ is continuous for a.e. $x \in [0,1]$.

And $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is said to be an L^1-Carathéodory function if, in addition to conditions (C_1) and (C_2), the following condition is also satisfied:

(C_3) for every $\rho > 0$, there is a function $l_\rho \in L^1([0,1])$ such that $\sup_{|t|\le\rho} |f(x,t)| \le l_\rho(x)$ for almost every $x \in [0,1]$.

Denote

$$X := \left\{u \in H^2([0,1]) | u(0) = u'(0) = 0, \;\; u(1) \ge 0\right\}, \tag{2.1}$$

where $H^2([0,1])$ is the Sobolev space of all functions $u : [0,1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2([0,1])$. X is a Hilbert space with the usual norm

$$\|u\|_X = \left(\int_0^1 \left(|u''(x)|^2 + |u'(x)|^2 + |u(x)|^2\right) dx \right)^{1/2},$$

which is equivalent to the norm

$$\|u\| = \left(\int_0^1 |u''(x)|^2 \, dx \right)^{1/2}.$$

Also, according to [13], the embedding $X \hookrightarrow C^1([0,1])$ is compact and we have

$$\|u\|_{C^1([0,1])} = \max\left\{\|u\|_\infty, \|u'\|_\infty\right\} \le \|u\| \;\text{ for each }\; u \in X. \tag{2.2}$$

Remark 2.1. If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in X, then from the compact embedding $X \hookrightarrow C([0,1])$, it has a subsequence that pointwise converges to some $u \in X$ (this follows from the definition of compact embedding). Also, since X is reflexive space, then there exists a subsequence that weakly converges in X (see [3, Theorem 3.18]) and so, according to continuous embedding $X \to L^\infty([0,1])$, weakly converges in $L^\infty([0,1])$.

Put

$$F(x,t) = \int_0^t f(x,\xi)\, d\xi \;\text{ for all }\; (x,t) \in [0,1] \times \mathbb{R},$$

$$F^\theta = \int_0^1 \sup_{|\xi|\le\theta} F(x,\xi)\, dx \;\text{ for all }\; \theta > 0,$$

$$G(t) = \int_0^t g(\xi)\, d\xi \;\text{ for all }\; t \in \mathbb{R},$$

$$G_\eta = \min_{|t|\le\eta} G(t) = \inf_{|t|\le\eta} G(t) \;\text{ for all }\; \eta > 0$$

and

$$H(x,t) = \int_0^t h(x,\xi)\, d\xi \;\text{ for all }\; (x,t) \in [0,1] \times \mathbb{R}.$$

Let $\Phi,\ \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\int_0^1 |u''(x)|^2\,dx - \int_0^1 H(x,u(x))\,dx = \frac{1}{2}\|u\|^2 - \int_0^1 H(x,u(x))\,dx \tag{2.3}$$

and

$$\Psi(u) = \int_0^1 \alpha(x)F(x,u(x))\,dx - \frac{\mu}{\lambda}G(u(1)) \tag{2.4}$$

for every $u \in X$.

Now, according to (2.2), we observe that

$$\frac{(1-L)}{2}\|u\|^2 \le \Phi(u) \le \frac{(1+L)}{2}\|u\|^2 \tag{2.5}$$

for every $u \in X$. Similar to [2, p. 3], Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 \alpha(x)f(x,u(x))v(x)\,dx - \frac{\mu}{\lambda}g(u(1))v(1),$$

and Φ is continuously Gâteaux differentiable functional whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)\,dx - \int_0^1 h(x,u(x))v(x)\,dx \ \text{ for every } \ v \in X.$$

Definition 2.2. Let Φ and Ψ be defined as above and put $I_\lambda = \Phi - \lambda\Psi$. We say that $u \in X$ is a critical point of I_λ if $I'_\lambda(u) = 0_{\{X^*\}}$, that is,

$$I'_\lambda(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0 \ \text{ for all } \ v \in X.$$

Definition 2.3. A function $u \in X$ is a weak solution to problem (1.1) if

$$\int_0^1 u''(x)v''(x)\,dx - \lambda\int_0^1 \alpha(x)f(x,u(x))v(x)\,dx + \mu g(u(1))v(1) - \int_0^1 h(x,u(x))v(x)\,dx = 0$$

for every $v \in X$.

Remark 2.2. We observe that the weak solutions of problem (1.1) are exactly the solutions of the equation $I'_\lambda(u)(v) = 0$. Also, if α, f and h are, in addition, continuous functions, then each weak solution of (1.1) is a classical solution.

Lemma 2.1. *If $u_0 \not\equiv 0$ is a weak solution for problem (1.1), then u_0 is non-negative a.e. in $[0,1]$.*

Proof. From Remark 2.2 one has $I'_\lambda(u_0)(v) = 0$ for all $v \in X$. Choosing $v(x) = \max\{-u_0(x), 0\}$ and setting $E = \{x \in [0,1] : u_0(x) < 0\}$, we have

$$\int_{E\cup E^c} u_0''(x)v''(x)\,dx = \lambda\int_0^1 \alpha(x)f(x,u_0(x))v(x)\,dx - \mu g(u_0(1))v(1) + \int_0^1 h(x,u_0(x))v(x)\,dx \ge 0,$$

that is,

$$-\int_E v''(x)v''(x)\,dx \ge 0,$$

which means that $-\|v\|^2 \ge 0$, and we have $v = 0$. Hence $-u_0 \le 0$ a.e. in $[0,1]$, that is, $u_0 \ge 0$ a.e. in $[0,1]$ and the proof is complete. $\square$

Below, we will present a non-standard state of the Palais–Smale condition introduced in [1].

Definition 2.4 (see [1])**.** Fix $r \in]-\infty, +\infty]$. A Gâteaux differentiable function $I : X \to \mathbb{R}$ satisfies the Palais–Smale condition cut off upper at r (in short, $(PS)^{[r]}$-condition) if any sequence $\{u_n\} \subseteq X$ such that

(a) $\{I(u_n)\}$ is bounded,

(b) $\lim\limits_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,

(c) $\Phi(u_n) < r \ \forall n \in \mathbb{N}$,

has a convergent subsequence.

Our main tool is the following critical point theorem.

Theorem 2.1 ([1, Theorem 7.3])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below and convex such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{u} \in X$ with $2r_1 < \Phi(\overline{u}) < \frac{r_2}{2}$ such that

$$(b_1) \qquad \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(\overline{u})}{\Phi(\overline{u})};$$

$$(b_2) \qquad \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(\overline{u})}{\Phi(\overline{u})}.$$

Assume also that for each

$$\lambda \in \Lambda = \left]\frac{3}{2}\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}, \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ satisfies the $(PS)^{[r_2]}$-condition and

$$\inf_{t\in[0,1]} \Psi(tu_1 + (1-t)u_2) \geq 0$$

for each $u_1, u_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$.

Then for each $\lambda \in \Lambda$, the functional $\Phi - \lambda\Psi$ admits at least three critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Now, we present one proposition that will be needed to prove the main Theorem of this paper.

Proposition 2.1. *Take $I_\lambda = \Phi - \lambda\Psi$ as in Definition 2.2. Then I_λ satisfies the $(PS)^{[r]}$-condition for any $r > 0$.*

Proof. Consider the sequence $\{u_n\} \subseteq X$ such that $\{I_\lambda(u_n)\}$ is bounded,

$$\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{X^*} = 0 \ \text{ and } \ \Phi(u_n) < r \ \text{ for all } \ n \in \mathbb{N}.$$

Since $\Phi(u_n) < r$, from (2.5) we see that $\{u_n\}$ is bounded in X. Therefore, without loss of generality, it can be assumed that $u_n(x) \to u(x)$, and there is $s > 0$ such that $|u_n(x)| \leq s$ for all $x \in [0,1]$ and for all $n \in \mathbb{N}$, and also $\{u_n\}$ weakly converges to u in $L^\infty([0,1])$ (see Remark 2.1). Now, according to Hölder's inequality and Lebesque's Dominated Convergence Theorem, since

$$\alpha(x)f(x,u_n(x)) \leq \alpha(x) \cdot \max_{|\xi|\leq s} f(x,\xi) \in L^1([0,1]) \ \text{ for all } \ n \in \mathbb{N}$$

and

$$f(x,u_n(x)) \to f(x,u(x)) \text{ for a.e. } x \in [0,1]$$

(f is L^1-Carathéodory function), one has that $\alpha f(x,u_n)$ strongly converges to $\alpha f(x,u)$ in $L^1([0,1])$. Now, since $u_n \rightharpoonup u$ in $L^\infty([0,1])$ and $\alpha f(x,u_n) \to \alpha f(x,u)$ in $L^1([0,1]) \subseteq (L^\infty([0,1]))^*$, from [3, Proposition 3.5(iv)], we have

$$\lim_{n\to+\infty} \int_0^1 \alpha(x) f(x,u_n(x))(u_n(x)-u(x))\,dx = 0. \tag{2.6}$$

Similarly, we have

$$\lim_{n\to+\infty} \int_0^1 h(x,u_n(x))(u_n(x)-u(x))\,dx = 0. \tag{2.7}$$

From

$$\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_X = 0,$$

there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0^+$ such that

$$\begin{aligned} \Big| \int_0^1 u''_n(x)v''(x)\,dx - \lambda \int_0^1 \alpha(x) f(x,u_n(x))v(x)\,dx \\ + \mu g(u_n(1))v(1) - \int_0^1 h(x,u_n(x))v(x)\,dx \Big| \le \varepsilon_n \end{aligned} \tag{2.8}$$

for all $n \in \mathbb{N}$ and for all $v \in X$ with $\|v\| \le 1$. Taking into account

$$v(x) = \frac{u_n(x)-u(x)}{\|u_n-u\|}\,,$$

from (2.8) we get

$$\begin{aligned} \Big| \int_0^1 u''_n(x)(u''_n(x)-u''(x))\,dx - \lambda \int_0^1 \alpha(x) f(x,u_n(x))(u_n(x)-u(x))\,dx \\ + \mu g(u_n(1))(u_n(1)-u(1)) - \int_0^1 h(x,u_n(x))(u_n(x)-u(x))\,dx \Big| \le \varepsilon_n \|u_n-u\| \end{aligned} \tag{2.9}$$

for all $n \in \mathbb{N}$. Now, according to the inequality

$$|a|\,|b| \le \frac{1}{2}\,|a|^2 + \frac{1}{2}\,|b|^2,$$

we have

$$\begin{aligned} \int_0^1 u''_n(x)(u''_n(x)-u''(x))\,dx &= \int_0^1 |u''_n(x)|^2\,dx - \int_0^1 u''_n(x)u''(x)\,dx \\ &\ge \|u_n\|^2 - \int_0^1 \Big(\frac{1}{2}\,|u''_n(x)|^2 + \frac{1}{2}\,|u''(x)|^2\Big)\,dx = \|u_n\|^2 - \frac{1}{2}\,\|u_n\|^2 - \frac{1}{2}\,\|u\|^2 = \frac{1}{2}\,\|u_n\|^2 - \frac{1}{2}\,\|u\|^2. \end{aligned}$$

Hence from (2.9), we get

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2 \le \lambda \int_0^1 \alpha(x) f(x, u_n(x))(u_n(x) - u(x))\,dx - \mu g(u_n(1))(u_n(1) - u(1)) + \int_0^1 h(x, u_n(x))(u_n(x) - u(x))\,dx + \varepsilon_n \|u_n - u\|,$$

that is,

$$\frac{1}{2}\|u_n\|^2 \le \frac{1}{2}\|u\|^2 + \lambda \int_0^1 \alpha(x) f(x, u_n(x))(u_n(x) - u(x))\,dx - \mu g(u_n(1))(u_n(1) - u(1)) + \int_0^1 h(x, u_n(x))(u_n(x) - u(x))\,dx + \varepsilon_n \|u_n - u\|. \quad (2.10)$$

Now, according to (2.6), (2.7) and (2.10), as $\varepsilon_n \to 0^+$, we have

$$\limsup_{n\to+\infty} \|u_n\| \le \|u\|.$$

Thus [3, Proposition 3.32] ensures that $u_n \to u$ strongly converges in X and the proof is complete. □

3 Main results

Fix three positive constants θ_1, θ_2 and δ such that

$$\frac{12(1+L)(\frac{2}{3})^3\pi^4\delta^2}{\int_{3/4}^1 \alpha(x)F(x,\delta)\,dx} < (1-L)\min\left\{\frac{{\theta_1}^2}{\|\alpha\|_\infty F^{\theta_1}}, \frac{{\theta_2}^2}{2\|\alpha\|_\infty F^{\theta_2}}\right\},$$

where $\|\alpha\|_\infty = \|\alpha\|_{L^\infty}$, and take

$$\lambda \in \Lambda := \left]\frac{6(1+L)(\frac{2}{3})^3\pi^4\delta^2}{\int_{3/4}^1 \alpha(x)F(x,\delta)\,dx}, \min\left\{\frac{(1-L){\theta_1}^2}{2\|\alpha\|_\infty F^{\theta_1}}, \frac{(1-L){\theta_2}^2}{4\|\alpha\|_\infty F^{\theta_2}}\right\}\right[.$$

Set

$$\eta_{\lambda,g} := \min\left\{\frac{2\lambda\|\alpha\|_\infty F^{\theta_1} - (1-L){\theta_1}^2}{2G_{\theta_1}}, \frac{4\lambda\|\alpha\|_\infty F^{\theta_2} - (1-L){\theta_2}^2}{4G_{\theta_2}}\right\}, \quad (3.1)$$

where G_{θ_1} and G_{θ_2} are assumed to be negative.

It is easy to show that $\eta_{\lambda,g} > 0$. Now, we formulate the main results.

Theorem 3.1. *Assume that there exist three positive constants δ, θ_1 and θ_2 with*

$$\frac{3}{4\pi^2}\sqrt{\frac{3}{2}}\,\theta_1 < \delta < \frac{3}{8\pi^2}\sqrt{\frac{3(1-L)}{2(1+L)}}\,\theta_2$$

such that

(i) $$\frac{F^{\theta_1}}{{\theta_1}^2} < \frac{(1-L)\int_{3/4}^1 \alpha(x)F(x,\delta)\,dx}{12\left(\frac{2}{3}\right)^3(1+L)\|\alpha\|_\infty \pi^4\delta^2},$$

(ii) $$\frac{F^{\theta_2}}{{\theta_2}^2} < \frac{(1-L)\int\limits_{3/4}^{1} \alpha(x)F(x,\delta)\,dx}{24(\frac{2}{3})^3(1+L)\|\alpha\|_\infty \pi^4\delta^2}\,.$$

Then for each $\lambda \in \Lambda$ and for every non-positive continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\eta_{\lambda,g} > 0$ given by (3.1) such that for each $\mu \in]0, \eta_{\lambda,g}[$, problem (1.1) admits at least three positive weak solutions u_i, $i = 1, 2, 3$, in X such that $0 \le u_i(x) < \theta_2$ $\forall\, x \in [0,1]$, $i = 1, 2, 3$.

Proof. Our aim is to apply Theorem 2.1 to problem (1.1). Fix λ, as in the conclusion. Let X, Φ and Ψ be defined by (2.1), (2.3) and (2.4), respectively.

We observe that the regularity assumptions of Theorem 2.1 on Φ and Ψ are satisfied. Also, according to Proposition 2.1, the functional I_λ satisfies the $(PS)^{[r]}$-condition for all $r > 0$.

Put
$$r_1 := \frac{(1-L)}{2}\,\theta_1^2, \quad r_2 := \frac{(1-L)}{2}\,\theta_2^2$$
and
$$w(x) := \begin{cases} 0 & \text{if } x \in \left[0, \frac{3}{8}\right], \\ \delta\cos^2\left(\dfrac{4\pi x}{3}\right) & \text{if } x \in \left]\frac{3}{8}, \frac{3}{4}\right[, \\ \delta & \text{if } x \in \left[\frac{3}{4}, 1\right]. \end{cases}$$
We observe that $w \in X$ and
$$\|w\|^2 = 8\pi^4\delta^2\left(\frac{2}{3}\right)^3.$$
In particular, from (2.5) we have
$$4(1-L)\pi^4\delta^2\left(\frac{2}{3}\right)^3 \le \Phi(w) \le 4(1+L)\pi^4\delta^2\left(\frac{2}{3}\right)^3. \tag{3.2}$$
Therefore, using the condition
$$\frac{3}{4\pi^2}\sqrt{\frac{3}{2}}\,\theta_1 < \delta < \frac{3}{8\pi^2}\sqrt{\frac{3(1-L)}{2(1+L)}}\,\theta_2,$$
we find that
$$2r_1 < \Phi(w) < \frac{r_2}{2}\,.$$

Now, for each $u \in X$ and bearing (2.2) in mind, we see that
$$\begin{aligned}\Phi^{-1}(]-\infty, r_i]) &= \{u \in X;\ \Phi(u) \le r_i\} \\ &= \left\{u \in X;\ \frac{(1-L)}{2}\,\|u\|^2 \le r_i\right\} \subseteq \{u \in X;\ |u(x)| \le \theta_i \text{ for each } x \in [0,1]\},\end{aligned}$$
and it follows that
$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r_i[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r_i[)} \left(\int_0^1 \alpha(x)F(x,u(x))\,dx - \frac{\mu}{\lambda}\,G(u(1))\right) \\ &\le \int_0^1 \alpha(x)\sup_{|\xi|\le\theta_i} F(x,\xi)\,dx - \frac{\mu}{\lambda}\,G_{\theta_i} \le \|\alpha\|_\infty F^{\theta_i} - \frac{\mu}{\lambda}\,G_{\theta_i}.\end{aligned}$$

Hence, since $\mu < \eta_{\lambda,g}$, from (3.1) we have
$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} \le \frac{\|\alpha\|_\infty F^{\theta_1} - \frac{\mu}{\lambda}G_{\theta_1}}{\frac{(1-L)}{2}\,\theta_1^2} < \frac{1}{\lambda}\,. \tag{3.3}$$

On the other hand, from (3.2), since $\lambda \in \Lambda$, we get

$$\frac{2}{3}\,\frac{\Psi(w)}{\Phi(w)} \geq \frac{2}{3}\,\frac{\int_0^1 \alpha(x)F(x,w(x))\,dx - \frac{\mu}{\lambda}\,G(w(1))}{4(1+L)\pi^4\delta^2(\frac{2}{3})^3}$$
$$\geq \frac{2}{3}\,\frac{\int_{3/4}^1 \alpha(x)F(x,\delta)\,dx - \frac{\mu}{\lambda}\,G(\delta)}{4(1+L)\pi^4\delta^2(\frac{2}{3})^3} \geq \frac{2}{3}\,\frac{\int_{3/4}^1 \alpha(x)F(x,\delta)\,dx}{4(1+L)\pi^4\delta^2(\frac{2}{3})^3} > \frac{1}{\lambda}\,. \tag{3.4}$$

Now, from (3.3) and (3.4) we have

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \frac{2}{3}\,\frac{\Psi(w)}{\Phi(w)}\,.$$

Analogously, from (3.4) we get

$$\frac{2\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} \leq 2\,\frac{\|\alpha\|_\infty^{\theta_2} - \frac{\mu}{\lambda}\,G_{\theta_2}}{\frac{(1-L)}{2}\,\theta_2^2} < \frac{1}{\lambda} < \frac{2}{3}\,\frac{\Psi(w)}{\Phi(w)}$$

which means that

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \frac{1}{3}\,\frac{\Psi(w)}{\Phi(w)}\,.$$

Hence (b_1) and (b_2) of Theorem 2.1 are established.

Now, if $u_1, u_2 \in X$ are two local minima of the functional $I_\lambda = \Phi - \lambda\Psi$, with $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, then according to Lemma 2.1, u_1 and u_2 are nonnegative, and we get

$$\inf_{t\in[0,1]} \Psi(tu_1 + (1-t)u_2) \geq 0.$$

Finally, for every

$$\lambda \in \Lambda \subseteq \left]\frac{3}{2}\,\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)},\ \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right\}\right[,$$

since the weak solutions of problem (1.1) are exactly the solutions of the equation $I'_\lambda(u) = 0$, Theorem 2.1 (with $\overline{u} = w$) and Lemma 2.1 guarantee the conclusion. □

Remark 3.1. In Theorem 3.1, if $f(x,0) \neq 0$ or $h(x,0) \neq 0$, then problem (1.1) has at least three non-trivial and non-negative weak solutions.

As an example, we give the following consequence of Theorem 3.1.

Corollary 3.1. *Let $\int_{3/4}^1 \alpha(x)\,dx \neq 0$ and $f : \mathbb{R} \to [0,+\infty[$ be a continuous and nonzero function such that*

$$\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi} = \lim_{\xi\to+\infty} \frac{f(\xi)}{\xi} = 0.$$

Then for each $\lambda > \lambda^*$, where

$$\lambda^* = \inf\left\{\frac{6(1+L)(\frac{2}{3})^3\pi^4}{\int_{3/4}^1 \alpha(x)\,dx}\,\frac{\delta^2}{\int_0^\delta f(\xi)\,d\xi}\ :\ \delta > 0,\ \int_0^\delta f(\xi)\,d\xi > 0\right\},$$

and for every non-positive continuous function $g : \mathbb{R} \to \mathbb{R}$*, there exists* $\eta_{\lambda,g} > 0$ *such that, for each* $\mu \in]0, \eta_{\lambda,g}[$ *, the problem*

$$\begin{cases} u^{(iv)}(x) = \lambda\, \alpha(x) f(u(x)) + h(x, u(x)), \quad x \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu\, g(u(1)) \end{cases}$$

admits at least three distinct non-negative weak solutions.

Proof. Since $\int\limits_{3/4}^{1} \alpha(x)\, dx \neq 0$ and $f \not\equiv 0$, we see that

$$\frac{6(1+L)\left(\frac{2}{3}\right)^3 \pi^4}{\int\limits_{3/4}^{1} \alpha(x)\, dx} \frac{\delta^2}{\int\limits_0^{\delta} f(\xi)\, d\xi} < +\infty.$$

Suppose that $\lambda > \lambda^*$ is fixed. Let $\delta > 0$ such that $\int\limits_0^{\delta} f(\xi)\, d\xi > 0$ and

$$\lambda > \frac{6(1+L)\left(\frac{2}{3}\right)^3 \pi^4}{\int\limits_{3/4}^{1} \alpha(x)\, dx} \frac{\delta^2}{\int\limits_0^{\delta} f(\xi)\, d\xi}.$$

According to $F(x,t) = \int\limits_0^t f(x,\xi)\, d\xi$ for all $(x,t) \in [0,1] \times \mathbb{R}$, we can consider $F(t) = \int\limits_0^t f(\xi)\, d\xi$ for all $t \in \mathbb{R}$. Our hypotheses on f guarantee that $F \in C^1(\mathbb{R})$, and since $F'(t) = f(t) \geq 0$ for all $t \in \mathbb{R}$, $F(t)$ is non-decreasing and so

$$F^{\theta} = \int_0^1 \sup_{|\xi| \leq \theta} F(\xi)\, dx = \sup_{|\xi| \leq \theta} F(\xi) = F(\theta) \text{ for all } \theta > 0.$$

Now, from $\lim\limits_{\xi \to 0^+} \dfrac{f(\xi)}{\xi} = 0$ we have

$$\lim_{\xi \to 0^+} \frac{\int\limits_0^{\xi} f(t)\, dt}{\xi^2} = \lim_{\xi \to 0^+} \frac{F(\xi)}{\xi^2} = 0,$$

and according to the definition of the right-side limit as ξ tends to 0, there is $\theta_1 > 0$ such that $\frac{3}{4\pi^2} \sqrt{\frac{3}{2}}\, \theta_1 < \delta$ and

$$\frac{\int\limits_0^{\theta_1} f(t)\, dt}{{\theta_1}^2} = \frac{F(\theta_1)}{{\theta_1}^2} = \frac{F^{\theta_1}}{{\theta_1}^2} < \frac{1-L}{2\lambda \|\alpha\|_{\infty}} \left(\text{an arbitrary positive upper bound for } \frac{F^{\theta_1}}{{\theta_1}^2}\right).$$

Also, from $\lim\limits_{\xi \to +\infty} \frac{f(\xi)}{\xi} = 0$ we have

$$\lim_{\xi \to +\infty} \frac{\int\limits_0^{\xi} f(t)\, dt}{\xi^2} = \lim_{\xi \to +\infty} \frac{F(\xi)}{\xi^2} = 0,$$

and according to the definition of the limit at infinity, there is $\theta_2 > 0$ such that

$$\delta < \frac{3}{8\pi^2}\sqrt{\frac{3(1-L)}{2(1+L)}}\,\theta_2$$

and

$$\frac{\int\limits_0^{\theta_2} f(t)\,dt}{{\theta_2}^2} = \frac{F(\theta_2)}{{\theta_2}^2} = \frac{F^{\theta_2}}{{\theta_2}^2} < \frac{1-L}{4\lambda\|\alpha\|_\infty} \ \Big(\text{an arbitrary positive upper bound for } \frac{F^{\theta_2}}{{\theta_2}^2}\Big).$$

Now, we can apply Theorem 3.1 and the conclusion follows. □

Next, we will present an example to illustrate Corollary 3.1.

Example. Let $f(t) = t^2 e^{-t^3}$ and hence

$$\int\limits_0^{\delta} f(\xi)\,d\xi = \frac{1}{3}\,(1 - e^{-\delta^3}) \ \text{ for all } \ \delta > 0.$$

Also, suppose that

$$\alpha(x) := \begin{cases} 4 & \text{if } x \in \Big[\dfrac{3}{4}, 1\Big], \\ 1 & \text{otherwise.} \end{cases}$$

Hence $\int\limits_{3/4}^{1} \alpha(x)\,dx = 1$. Now if, for example, we consider $h(x,t) = \frac{1}{2}x|t|$ for each $x \in [0,1]$ and $t \in \mathbb{R}$ with $L = \frac{1}{2}$, then according to Corollary 3.1 for each

$$\lambda > \inf\Big\{\frac{8\pi^4\delta^2}{1-e^{-\delta^3}}, \delta > 0\Big\}$$

and for every non-positive continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\eta_{\lambda,g} > 0$ such that for each $\mu \in]0, \eta_{\lambda,g}[$, the problem

$$\begin{cases} u^{(iv)}(x) = \begin{cases} 4\lambda u(x)^2 e^{-u(x)^3} + \dfrac{1}{2}\,x|u(x)| & \text{if } x \in \big[\frac{3}{4}, 1\big], \\ \lambda u(x)^2 e^{-u(x)^3} + \dfrac{1}{2}\,x|u(x)| & \text{if } x \in \big[0, \frac{3}{4}\big), \end{cases} \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \ \ u'''(1) = \mu\, g(u(1)) \end{cases}$$

admits at least three non-negative weak solutions.

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