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AN ANALYSIS OF THE PRESSURE TERM IN THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS WITH BOUNDED INITIAL DATA

Abstract. In this paper, we consider the Cauchy problem for the incompressible Navier–Stokes equations in $\mathbb{R}^n$, $n \geq 3$, for nondecaying initial data. First, this paper provides an analysis of the nondecaying (BMO) pressure term in the incompressible Navier–Stokes equations that appears in the paper [11] by H. O. Kreiss and J. Lorenz. Next, this paper considers a smooth periodic initial data and formally derives a periodic pressure term to analyze a relationship between these two pressure terms in the Cauchy problems with two slightly different initial data. This overall phenomenon is interesting, since these two pressure terms are closely related to each other, despite their fundamentally different representations.

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რეზიუმე. ნაშრომში განხილულია კოშის ამოცანა არაკუმშვადი ნავიე--სტოქსის განტოლებისთვის $\mathbb{R}^n$ სივრცეში, $n \geq 3$, არაკლებადი საწყისი მონაცემების შემთხვევაში. თავდაპირველად, ნაშრომში მოყვანილია არაკუმშვადი ნავიე--სტოქსის განტოლების წნევის არაკლებადი (BMO) წევრის ანალიზი, რომელიც გამოჩნდა H. O. Kreiss-ისა და J. Lorenz-ის ნაშრომში [11]. შემდგომ, ნაშრომში განხილულია გლუვი პერიოდული საწყისი მონაცემები და ფორმალურად მიღებულია წნევის პერიოდული წევრი, რათა გაანალიზდეს წნევის ამ წევრებს შორის კავშირი კოშის ამოცანებში ორი მცირედ განსხვავებული საწყისი მონაცემებით. ეს საერთო ფენომენი საინტერესოა, რადგან წნევის ეს ორი წევრი მჭიდრო კავშირშია ერთმანეთთან, მიუხედავად მათი ფუნდამენტურად განსხვავებული წარმოდგენებისა.

1 Introduction

We consider the Cauchy problem of the incompressible Navier–Stokes equations in $\mathbb{R}^n$, $n \geq 3$:

$$\left.\begin{aligned} u_t + u\cdot\nabla u + \nabla p &= \triangle u && \text{for } x \in \mathbb{R}^n,\ \ t>0,\\ \nabla\cdot u &= 0 && \text{for } x\in\mathbb{R}^n,\ \ t>0,\\ u\big|_{t=0} &= f && \text{for } x\in\mathbb{R}^n, \end{aligned}\right\} \tag{1.1}$$

where $u = u(x,t) = (u_1(x,t),\ldots,u_n(x,t))$ and $p = p(x,t)$ stand for the unknown velocity vector field of the fluid and its pressure, while $f = f(x) = (f_1(x),\ldots,f_n(x))$ is the given initial velocity vector field, with $\nabla\cdot f = 0$. In what follows, we will use the same notations for the space of vector-valued and scalar functions for convenience in writing.

There is a vast literature on the existence and uniqueness of solutions of the Navier–Stokes equations in $\mathbb{R}^n$. For given initial data, solutions of (1.1) have been constructed in various function spaces. For example, if $f \in L^r$ for some r with $3 \leq r < \infty$, then it is well known that there is a unique classical solution in some maximum interval of time: $0 \leq t < T_f$, where $0 < T_f \leq \infty$. But for the uniqueness of the pressure, one requires $|p(x,t)| \to 0$ as $|x| \to \infty$. (See [8, 21] for $r = 3$ and [1] for $3 < r < \infty$.) The solution is C^∞ for $0 < T_f < \infty$.

It is well known that for $f \in L^\infty(\mathbb{R}^n)$, there is a unique, smooth and local-in-time solution u for the Navier–Stokes equations with

$$p = \sum_{i,j} R_i R_j u_i u_j, \tag{1.2}$$

where $R_i = (-\triangle)^{-1/2}\partial_{x_i}$ is the ith Riesz operator. For $f \in L^\infty(\mathbb{R}^n)$, where $n \geq 3$, the existence of a regular solution follows from [2]. The solution is only unique if one puts some growth restrictions on the pressure as $|x| \to \infty$. A simple example of non-uniqueness is demonstrated in [10], where the velocity u is bounded, but $|p(x,t)| \leq C|x|$. In addition, an estimate $|p(x,t)| \leq C(1+|x|^\sigma)$ with $\sigma < 1$ (see [4]) implies the uniqueness. Also, the assumption $p \in L^1_{loc}(0,T;BMO)$ (see [6]) implies the uniqueness. Further, for nondecaying initial data in an exterior domain, the global weak solutions to (1.1) are discussed in [16], and the uniqueness of the very weak solutions for such data in the whole space is presented in [13]. Also, see [12] for the existence and uniqueness of solutions for the nondecaying data in a halfspace. For the existence and uniqueness of global regular solutions to (1.1) in various domains in $\mathbb{R}^2$ for nondecaying initial data, we refer to [14, 15].

Moreover, J. Kato [9] observed that for $f \in L^\infty(\mathbb{R}^n)$, the constructed solution is bounded and may not decay at the space infinity. Even if u solves (1.1), equation (1.2) may not follow. J. Kato provided a simple case for $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, where one can construct a solution of the form $u(x,t) = g(t)$, $p(x,t) = -g'(t)\cdot x$. This pair (u,p) of functions solves (1.1) regardless of the choice of $g(t)$. So, it is clear that if u has a constant initial data, the solution is not unique without assuming (1.2).

Also, in [6], it was discussed that the uniqueness holds if u is bounded and p is of the form

$$p(x,t) = \pi_0 + \sum_{i,j} R_i R_j \pi_{ij}$$

for bounded functions π_0, π_{ij}. More precisely, $\pi_0, \pi_{ij} \in L^\infty(\mathbb{R}^n) \cap L^1_{loc}(\mathbb{R}^n)$ for $t \in (0,T)$, for some maximal time T.

In the same paper, one of the authors, J. Kato, improved the result by simply assuming that $p \in L^1_{loc}(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$, Further, Uchiyama [20] indicated that if a function g is BMO, then it is of the form

$$g = \nu_0 + \sum_{i,j} R_i R_j \nu_{ij}$$

with some $\nu_{ij} \in L^\infty(\mathbb{R}^n)$.

Additionally, Sadosky [19], and Fefferman & Stein [3] observed that every $g \in BMO$ can be written as

$$g = \tau_0 + \sum_j R_j \tau_j,$$

where $\tau_j \in L^\infty(\mathbb{R}^n)$.

Going back to Kato's paper [9] which deals with weak solutions. The main result of that paper was that under condition (1.2), if (u,p) solves (1.1) with $f \in L^\infty(\mathbb{R}^n)$ and $p(x,0) \in BMO(\mathbb{R}^n)$, then the solution $(u, \nabla p)$ was unique and

$$\nabla p = \sum_{i,j} \nabla R_i R_j (u_i u_j),$$

where ∇p is understood in the distributional sense. As it can be determined from the discussion about the pressure above, pressure term plays a significant role in the uniqueness of solutions of the Navier–Stokes equations, therefore we turn our attention to the Poisson equation which can be used formally to derive a pressure term that together with the velocity field solves the Navier–Stokes equations.

2 The pressure Poisson equations

Let us rewrite the Navier–Stokes equations as

$$u_t = \triangle u + Q, \;\; \nabla \cdot u = 0, \;\; u = f \;\text{ at }\; t = 0$$

with

$$Q = -\nabla p - u \cdot \nabla u = -\nabla p - \sum_j D_j(u_j u). \tag{2.1}$$

The pressure term is determined by the Poisson equation

$$-\triangle p = \sum_{i,j} D_i D_j (u_i u_j) = \sum_{i,j} (D_i u_j)(D_j u_i).$$

We may formally obtain the Poisson pressure equation

$$p(x,t) = \sum_{i,j} C_0 \int\limits_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} \, (D_i u_j)(D_j u_i)(y,t)\, dy, \tag{2.2}$$

where $C_0 = \frac{1}{n(2-n)\omega_n}$, and ω_n is the surface area of the unit n-sphere.

This solution (2.2) to the Poisson equation (2.1) has an (integrable) singularity at $x = y$, and can (initially) be proven to have a solution if $(D_i u_j D_j u_i)(x,t)$ is a function of compact support in $\mathbb{R}^n$ or has a sufficient decay at space infinity in $\mathbb{R}^n$. Interestingly, since Riesz transforms map L^∞ to BMO, for $u \in L^\infty(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, the pressure equation (2.2) may exist as a BMO-valued function by virtue of the Calderon–Zygmund theory of singular integrals. However, it may fail to exist in the classical sense. For now, we consider that (u,p) is a solution to the Navier–Stokes equations (1.1) with $f \in L^\infty(\mathbb{R}^n)$. Additionally, we also assume $u, Du \in L^\infty(\mathbb{R}^n)$.

Despite the pressure being a critical object in the study of the Navier–Stokes equations, my previous works [17] and [18] on the study of Navier–Stokes equations follow the similar approach as many mathematicians, such as Giga [5–7], as well as Kato [8, 9], where the Leray projector is being used to eliminate the pressure term from the Navier–Stokes equations. While removing the pressure may be a mathematical need, the pressure term is still there. Therefore, this paper focuses on a simple study of the pressure term. Furthermore, interestingly, since only the space derivatives appear in the pressure term of the incompressible Navier–Stokes equations, adding an appropriately chosen time-dependent constant to the pressure term will not matter in terms of solving the Navier–Stokes equations. Therefore, we are interested in constructing a pressure term which has the same underlying structure as (2.2), and solves the Navier–Stokes equations (1.1) with the velocity field u.

Addition of such a properly chosen time-dependent constant to the formal solution of the Poisson equation (2.2) can be used to construct a slightly modified pressure solution of the Poisson pressure equation (2.1). Application of such a modification can be used to properly address the results of the paper due to Kriess and Lorenz [11] in which a significant analysis of the pressure term is being left for the readers. Again, the formal pressure term in (2.2) exists only as a BMO-valued function. The integral in (2.2) may fail to exist, but through a limiting process we can control the growth over the cube or ball so that the integral

$$\sup_Q \frac{1}{|Q|} \int\limits_Q |p(y) - p_Q| \, dy < \infty,$$

over all cubes (or balls) Q, where P_Q is the average of P over Q. The fact that the functions in the space of BMO differ by a constant, and since $p \in L^1_{loc}(0, T; BMO)$, we can add any time-dependent constant $C(t)$ to p without changing the element in the space. Since the pressure term in (2.2) lies in BMO, so $p + C(t)$ will also lie in the same space, and $\|p + C(t)\|_{BMO} = \|p\|_{BMO}$. This fact allows us to modify the formal pressure solution of (2.2) without changing its underlying structure. Next, we introduce the following

Definition. Let (u, p) be a solution to the Navier–Stokes equations (1.1) with $f \in L^\infty(\mathbb{R}^n)$ and suppose that $u \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for $0 \le t < T$ for some $T \le \infty$. The **modified Poisson pressure** is given by

$$p^*(x,t) = C_0 \sum_{i,j} \int\limits_{\mathbb{R}^n} \big[G_{ij}(x-y) - G_{ij}(y)\big](u_i u_j)(y,t) \, dy, \tag{2.3}$$

where

$$G_{ij}(x) = D_i D_j(G(x)) \text{ and } G(x) = |x|^{2-n}. \tag{2.4}$$

Next, we will show that the integral in (2.3) exists in the principal value sense, also check its smoothness and the growth at the space infinity. Then we will verify that such a modified pressure term also solves the Poisson equation (2.1); finally, we prove the existence of a smooth solution of (1.1). To reach that goal, we first introduce some propositions without proofs, then state and prove a few lemmas, necessary to prove the main result. In the following, as mentioned earlier, we assume that $u \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $Du \in L^\infty(\mathbb{R}^n)$.

Proposition 2.1. *Let $\mathbb{S}^n$ denote an n-dimensional unit sphere and dS denote the element of surface measure of sphere in $\mathbb{R}^n$, then*

$$\int\limits_{\mathbb{S}^{n-1}} x_i x_j \, dS_x = \delta_{ij}$$

for any $i, j \in \{1, 2, \dots, n\}$.

Proposition 2.2. *For G_{ij}, as defined in (2.4), we have*

$$\int\limits_{\mathbb{S}^{n-1}} G_{ij}(y) \, dS_y = 0.$$

Proposition 2.3. *Let $\Omega = \{y \in \mathbb{R}^n : \epsilon < |x-y| < \delta, \ x \in \mathbb{R}^n\}$ for any $\epsilon, \delta > 0$. Then*

$$\int\limits_{\Omega} G_{ij}(x-y) \, dS_y = 0.$$

Next, we state and prove the following important

Lemma 2.1. *Let G_{ij} be the kernel given in (2.4). Then for some $C > 0$,*

$$|G_{ij}(x-y) - G_{ij}(y)| \le \frac{C|x|}{|y|^{n+1}} \ \text{ for } \ |y| > 2|x|.$$

Proof. Let us define

$$\phi(t) = G_{ij}(y - xt), \ \ 0 \le t \le 1.$$

For $|y| > 2|x|$, the fundamental theorem of calculus applies and gives us

$$G_{ij}(y-x) - G_{ij}(y) = \phi(1) - \phi(0) = \int_0^1 \phi'(t)\,dt$$

and

$$G_{ij}(y-x) - G_{ij}(y) = |\phi(1) - \phi(0)| \le \max_{0\le t\le 1} |\phi'(t)|.$$

We obtain $\phi'(t) = -x \cdot \nabla G_{ij}(y - tx)$, and by the Cauchy–Schwarz inequality,

$$|\phi'(t)| \le |x|\,|\nabla G_{ij}(y-tx)| \le \frac{C|x|}{|y-tx|^{n+1}}\,.$$

For $0 \le t \le 1$ and $|y| > 2|x|$, we have $|y - tx| \ge |y| - t|x| \ge |y| - |x| \ge |y| - \frac{|y|}{2} = \frac{|y|}{2}$. Hence we arrive at

$$\max_{0\le t\le 1} |\phi'(t)| \le \frac{C|x|}{|y|^{n+1}}\,.$$

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Suppose that $G_{ij}(y)$ is the kernel defined in (2.4). Then for any $\delta > 0$ and for fixed $x \in \mathbb{R}^n$,*

$$PV \int_{|x-y|<\delta} G_{ij}(x-y) g(y)\,dy < \infty,$$

where $g(y) = (u_i u_j)(y) \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Proof. For any fixed $x \in \mathbb{R}^n$ and $0 < \epsilon < \delta$, denote

$$I(x) := \int_{\epsilon<|x-y|<\delta} G_{ij}(x-y) g(y)\,dy.$$

Since g is a bounded function, we can use Proposition 2.2, for any $x \in \mathbb{R}^n$, to obtain

$$\int_{\epsilon<|x-y|<\delta} G_{ij}(x-y) g(x)\,dy = 0.$$

Therefore, we are allowed to write

$$I(x) = \int_{\epsilon<|x-y|<\delta} G_{ij}(x-y)[g(y) - g(x)]\,dy.$$

Using $|G_{ij}(x-y)| \le \frac{C}{|x-y|^n}$, for some $C > 0$, we get

$$|I(x)| \le C \int_{\epsilon<|x-y|<\delta} \frac{1}{|x-y|^n} |g(y) - g(x)|\,dy.$$

Next, we determine a suitable bound for $|g(y) - g(x)|$. Let us start with the following:

$$g(y) - g(x) = (u_i u_j)(y) - (u_i u_j)(x) = u_i(y)u_j(y) - u_i(y)u_j(x) + u_i(y)u_j(x) - u_i(x)u_j(x),$$

so,

$$\begin{aligned} |g(y) - g(x)| &\le |u_i(y)|\,|u_j(y) - u_j(x)| + |u_j(x)|\,|u_i(y) - u_j(x)| \\ &\le |u|_\infty |u(y) - u(x)| + |u|_\infty |u(y) - u(x)| \\ &= 2|u|_\infty |u(y) - u(x)|. \end{aligned}$$

Let

$$\phi(t) = u(x + (y - x)t), \quad 0 \le t \le 1.$$

Then

$$\phi(1) - \phi(0) = \int_0^1 \phi'(t)\, dt = \int_0^1 \nabla u(x + (y - x)t) \cdot (y - x)\, dt.$$

Therefore,

$$|u(y) - u(x)| = |\phi(1) - \phi(0)| \le \max_{0 \le t \le 1} |\nabla u(x + (y - x)t)|\, |y - x| \le |\nabla u|_\infty |y - x|.$$

Since $u \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $Du \in L^\infty(\mathbb{R}^n)$, we obtain

$$|g(y) - g(x)| \le 2|u|_\infty |\nabla u|_\infty |y - x| \le C|y - x|.$$

Hence we arrive at

$$|I(x)| \le C \int_{\epsilon < |x-y| < \delta} \frac{1}{|x - y|^n} |g(y) - g(x)|\, dy \le C \int_{\epsilon < |x-y| < \delta} \frac{1}{|x - y|^{n-1}}\, dy.$$

Changing to polar coordinates gives

$$|I(x)| \le C \int_\epsilon^\delta \frac{1}{r^{n-1}}\, r^{n-1}\, dr = C(\delta - \epsilon).$$

Finally,

$$\begin{aligned} PV \int_{|x-y|<\delta} G_{ij}(x - y) g(y)\, dy &= \lim_{\epsilon \to 0} \int_{\epsilon < |x-y| < \delta} G_{ij}(x - y) g(y)\, dy \\ &= \lim_{\epsilon \to 0} I(x) \le C \lim_{\epsilon \to 0} (\delta - \epsilon) \le C\delta < \infty. \end{aligned}$$

This completes the proof of Lemma 2.2. □

Next, we prove an important theorem of this paper.

Theorem 2.1. *Suppose that (u, p) is a solution to the Navier–Stokes equations (1.1) with $f \in L^\infty(\mathbb{R}^n)$, and also suppose that $u \in C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ for any $t \in [0, T)$ for some $T \le \infty$. Then for any fixed $x \in \mathbb{R}^n$, the* ***modified Poisson pressure*** *given by (2.3) exists, i.e., $p* < \infty$, and it does not grow faster than a logarithmic function of $|x|$ as $x \to \infty$.*

Proof. In what follows, we suppress the t-dependence in our notations and denote $g(y) = (u_i u_j)(y)$. In addition, we allow the constant to change line by line. First, if $x = 0$, then $p^*(x) = 0$, therefore there is nothing to prove. Let $x \ne 0$ be fixed and write the integral given by (2.3) as

$$\begin{aligned} p^*(x) &= PV \sum_{i,j} C \int_{|y|<2|x|} \big[G_{ij}(x - y) - G_{ij}(y)\big] g(y)\, dy + PV \sum_{i,j} C \int_{|y|>2|x|} \big[G_{ij}(x - y) - G_{ij}(y)\big] g(y)\, dy \\ &= I_1 + I_2, \end{aligned}$$

where

$$\begin{aligned} I_1(x) &:= PV \sum_{i,j} C \int_{|y|<2|x|} \big[G_{ij}(x - y) - G_{ij}(y)\big] g(y)\, dy, \\ I_2(x) &:= PV \sum_{i,j} C \int_{|y|>2|x|} \big[G_{ij}(x - y) - G_{ij}(y)\big] g(y)\, dy. \end{aligned}$$

Let us proceed with I_1 as below:

$$\begin{aligned} I_1(x) &= PV \sum_{i,j} C \int\limits_{|y|<2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y)\, dy \\ &= PV \sum_{i,j} C \int\limits_{|y|<2|x|} G_{ij}(x-y) g(y)\, dy - PV \sum_{i,j} C \int\limits_{|y|<2|x|} G_{ij}(y) g(y)\, dy \\ &=: J_1 + J_2. \end{aligned}$$

With the use of Lemma 2.2 for fixed x, we obtain

$$|J_2(x)| < \infty.$$

Denote the ball of radius r and centered at $x \in \mathbb{R}^n$ by $B_r(x)$. For some $\epsilon > 0$, write

$$\begin{aligned} J_1(x) &= PV \sum_{i,j} \int\limits_{|y|<2|x|} G_{ij}(x-y) g(y)\, dy \\ &= \sum_{i,j} C \int\limits_{|y|<2|x| \setminus B_\epsilon(x)} G_{ij}(x-y) g(y)\, dy + PV \sum_{i,j} C \int\limits_{B_\epsilon(x)} G_{ij}(x-y) g(y)\, dy \\ &= J_1^* + J_1^{**}. \end{aligned}$$

Again, from Lemma 2.2, we immediately get $|J_1^{**}(x)| \leq C\epsilon < \infty$. Let us notice that $\{y \in \mathbb{R}^n : |y| < 2|x| \setminus B_\epsilon(x)\} \subset \{y \in \mathbb{R}^n : \epsilon < |x-y| < 3|x|\}$ to get

$$|J_1^*(x)| \leq \sum_{i,j} C \int\limits_{\epsilon<|x-y|<3|x|} |G_{ij}(x-y)|\, |g(y)|\, dy.$$

Using Lemma 2.2 one more time, we get

$$|J_1^*(x)| \leq \sum_{i,j} C \int\limits_{\epsilon<|x-y|<3|x|} |G_{ij}(x-y)|\, |g(y)|\, dy \leq C(3|x| - \epsilon) < \infty.$$

Therefore, for J_1, we have the following estimate:

$$|J_1(x)| \leq |J_1^*(x)| + |J_2^{**}(x)| \leq C + C\epsilon \leq C < \infty \quad \text{for a fixed} \quad x \in \mathbb{R}^n.$$

Thus we deduce that for any $x \in \mathbb{R}^n$,

$$|I_1(x)| \leq |J_1(x)| + |J_2(x)| < \infty.$$

Next, we prove $|I_2(x)| < \infty$ for a fixed x. To do this, let us begin with

$$I_2(x) = \lim_{R\to\infty} \sum_{i,j} C \int\limits_{2|x|<|y|<R} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y)\, dy.$$

Use Lemma 2.1 to obtain

$$|I_2(x)| \leq \lim_{R\to\infty} \sum_{i,j} C \int\limits_{2|x|<|y|<R} \frac{|x|}{|y|^4}\, dy = \lim_{R\to\infty} \sum_{i,j} C|x| \int\limits_{2|x|<|y|<R} \frac{1}{|y|^{n+1}}\, dy.$$

Evaluating the integral using polar coordinates gives

$$|I_2(x)| \leq \lim_{R\to\infty} \sum_{i,j} C|x| \Big[\frac{1}{2|x|} - \frac{1}{R}\Big] \leq \lim_{R\to\infty} \sum_{i,j} C\Big[\frac{1}{2} - \frac{|x|}{R}\Big] < \infty \quad \text{for any fixed } x \in \mathbb{R}^n.$$

Therefore, for any fixed $x \in \mathbb{R}^n$, we prove

$$|I(x)| \leq |I_1(x)| + |I_2(x)| < \infty.$$

We finally have

$$p^*(x,t) = C_0 \sum_{i,j} \int_{\mathbb{R}^n} [G_{ij}(x-y) - G_{ij}(y)](u_i u_j)(y,t)\, dy < \infty.$$

Next, it remains to prove that p^* does not grow faster than a *logarithmic* function of $|x|$ for large x. For that, choose $|x| > 1$. Let us recall p^* as follows:

$$\begin{aligned} p^*(x) &= PV \sum_{i,j} C \int_{|y|<2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y)\, dy + PV \sum_{i,j} C \int_{|y|>2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y)\, dy \\ &=: p_1^* + p_2^*. \end{aligned}$$

With the same argument as for I_2 in the previous part of the proof of this theorem, we can conclude that $p_2^* < \infty$.

For p_1^*, we proceed as below:

$$\begin{aligned} p_1^*(x) &= PV \sum_{i,j} C \int_{|y|<2|x|} \big[G_{ij}(x-y) - G_{ij}(y)\big] g(y)\, dy \\ &= PV \sum_{i,j} C \int_{|y|<2|x|} G_{ij}(x-y) g(y)\, dy - PV \sum_{i,j} C \int_{|y|<2|x|} G_{ij}(y) g(y)\, dy. \end{aligned}$$

We can write the above integral as

$$\begin{aligned} p_1^*(x) = \sum_{i,j} C \int_{|y|<2|x| \setminus B_1(x)} & G_{ij}(x-y) g(y)\, dy + PV \sum_{i,j} C \int_{B_1(x)} G_{ij}(x-y) g(y)\, dy \\ & - PV \sum_{i,j} C \int_{|y|<1} G_{ij}(y) g(y)\, dy - \sum_{i,j} C \int_{1<|y|<2|x|} G_{ij}(y) g(y)\, dy \\ =:& \; T_1 + T_2 + T_3 + T_4. \end{aligned}$$

Again, Lemma 2.2 can be applied, for a fixed x, on T_2 and T_3 to verify

$$|T_2(x)| \leq C < \infty \;\text{ and }\; |T_3(x)| \leq C < \infty.$$

To estimate T_1, we observe that $\{y \in \mathbb{R}^n : \; |y| < 2|x| \setminus B_1(x)\} \subset \{y \in \mathbb{R}^n : \; 1 < |x-y| < 3|x|\}$ and $|G_{ij}(x-y)| \leq \frac{C}{|x-y|^n}$ for some $C > 0$. Therefore,

$$|T_1(x)| \leq \sum_{i,j} C \int_{1<|x-y|<3|x|} |G_{ij}(x-y)|\, dy \leq \sum_{i,j} C \int_{1<|x-y|<3|x|} \frac{1}{|x-y|^n}\, dy.$$

Changing to a polar form gives

$$|T_1(x)| \leq \sum_{i,j} C \int_1^{3|x|} \frac{1}{r}\, dr \leq C \ln 3|x|.$$

Similarly as for T_1, we obtain

$$|T_4(x)| \leq C \ln 2|x| \leq C \ln 3|x|.$$

Therefore, for p_1^* we obtain the following bound

$$|p_1^*(x)| \leq |T_1(x)| + |T_2(x)| + |T_3(x)| + |T_4(x)| \leq C\big(1 + \ln 3|x|\big).$$

Combining the results from the above, we get a bound for p^* as

$$|p^*(x)| \le |p_1^*(x)| + |p_2^*(x)| \le C\big(1 + \ln 3|x|\big).$$

Hence we have proved that the modified Poisson pressure p^* exists in the principal value sense and does not grow faster than a *logarithmic* function of $|x|$ for large x. $\square$

Theorem 2.2. *The modified pressure p^* defined in (2.3) is a solution to the Poisson equation*

$$-\triangle p(x,t) = \sum_{i,j} (D_i u_j)(D_j u_i)(x,t).$$

Proof. Once again, we will suppress the t-dependence in our notations. Suppose there is $R > 0$ such that $2|x| < R$. Let us introduce a C^∞ cut-off function $\phi(r)$ with $\phi(r) = 1$ if $0 \le r \le R$ and $\phi(r) = 0$ for $r \ge R+1$. We write p^* using this cut-off function as follows:

$$\begin{aligned}
p^*(x) &= PV \sum_{i,j} \frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \big[G_{ij}(x-y) - G_{ij}(y)\big] \big(\phi(|y|) - (1 - \phi(|y|))\big)(u_i u_j)(y)\, dy \\
&= PV \sum_{i,j} \frac{1}{4\pi} \int\limits_{|y|<R+1} \big[G_{ij}(x-y) - G_{ij}(y)\big]\phi(|y|)(u_i u_j)(y)\, dy \\
&\qquad + \sum_{i,j} \frac{1}{4\pi} \int\limits_{|y|>R} \big[G_{ij}(x-y) - G_{ij}(y)\big]\big(1 - \phi(|y|)\big)(u_i u_j)(y)\, dy \\
&= p^*_{loc}(x) + p^*_{glb}(x).
\end{aligned}$$

Using integration by parts and the fact that ϕ vanishes on the boundary, we obtain

$$\begin{aligned}
p^*_{loc}(x) &= \sum_{i,j} \frac{1}{4\pi} \int\limits_{|y|<R+1} \big[G(x-y) - G(y)\big] D_i D_j \big(\phi(|y|) u_i u_j\big)(y)\, dy \\
&= \sum_{i,j} \frac{1}{4\pi} \int\limits_{|y|<R} \big[G(x-y) - G(y)\big] D_i D_j (u_i u_j)(y)\, dy \\
&\qquad + \sum_{i,j} \frac{1}{4\pi} \int\limits_{R<|y|<R+1} \big[G(x-y) - G(y)\big] D_i D_j (\phi(|y|) u_i u_j)(y)\, dy \\
&= I_1(x) + I_2(x).
\end{aligned}$$

Then

$$\triangle_x p^*_{loc} = \triangle_x I_1(x) + \triangle_x I_2(x).$$

It is known that for of $x \ne y$, we have $\triangle_x G(x-y) = 0$ and $\triangle_x G_{ij}(x-y) = 0$. This clearly implies $\triangle_x p^*_{glb}(x) = 0$ and $\triangle_x I_2(x) = 0$. Therefore, we arrive at

$$\begin{aligned}
\triangle_x p^*(x) = \triangle_x I_1(x) &= PV \sum_{i,j} \frac{1}{4\pi} \int\limits_{|y|<R} \triangle_x G(x-y) D_i D_j (u_i u_j)(y)\, dy \\
&= PV \frac{1}{4\pi} \int\limits_{|y|<R} \triangle_x G(x-y) \sum_{i,j} D_i D_j (u_i u_j)(y)\, dy \\
&= -\sum_{i,j} D_i D_j (u_i u_j)(x) = -\sum_{i,j} (D_i u_j)(D_j u_i)(x).
\end{aligned}$$

This completes the proof of Theorem 2.2. $\square$

Finally, Theorem 2.2 has proved that the *modified pressure term* p^* solves the pressure Poisson equations (2.1), whereas Theorem 2.1 proves that such *modified pressure* does not grow faster than a *logarithmic* function of $|x|$.

3 Periodic pressure term

In this section, we turn out our attention to somehow a simpler, however, an interesting phenomenon in various pressure terms of the NS equations. We begin by assuming that the initial data $f \in C^\infty(\mathbb{T}^n)$, where $\mathbb{T}^n = [0, 2\pi)^n$. It is well known that the unique smooth periodic solution to (1.1) exists for some maximal interval of time, where a pressure term can be determined by

$$-\triangle p(x,t) = \sum_{i,j} (D_i u_j)(D_j u_i)(x,t) =: g(x,t) \tag{3.1}$$

with the periodic boundary conditions. The integral

$$p(x,t) = \int_{\mathbb{T}^n} K(x-y) g(y,t)\, dy, \tag{3.2}$$

where $g \in C^\infty(\mathbb{T}^n)$, solves the Poisson equations (3.1), $K(x)$ in equation (3.2) is the periodic Green's function of the Poisson equation (3.1) which can formally be derived, with the use of the Fourier expansion on (3.1), as follows:

$$K(x) = \frac{1}{(2\pi)^n} \sum_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} \frac{e^{ik\cdot x}}{|k|^2}\,. \tag{3.3}$$

Apparently, the solution of (3.1) exists, and it can be given by

$$p(x) = \sum_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} \widehat{p}(k) e^{ik\cdot x}, \tag{3.4}$$

where $\widehat{p}(k) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{T}^n} \frac{e^{ik\cdot y}}{|k|^2}\, g(y)\, dy$. Note that we have suppressed the t-dependence in our notation. Interestingly, the series in (3.3) does not converge absolutely; however, because of some oscillations in the summand, the series still manages to converge for $x \neq 0$. How about finding an alternative form of the periodic Green's function whose convergence is easy to observe, unlike to (3.3)? Proposition below answers that question.

Proposition 3.1. *The Green's function in (3.3) can be rewritten as*

$$K(x) = \sum_{\substack{k \in \mathbb{Z}^n \\ k \neq 0}} \frac{e^{ik\cdot x}}{|k|^2}\, e^{-|k|^2\alpha} + \frac{1}{8\pi\sqrt{\pi}} \sum_{k\in\mathbb{Z}^n} \int_0^\alpha e^{-|x+2\pi k|^2/4\tau} \tau^{-3/2}\, d\tau \quad \text{for } \alpha > 0.$$

Proof. See Appendix. □

Note that $f \in C^\infty(\mathbb{T}^n) \subset L^\infty(\mathbb{R}^n)$ can be extended periodically to the whole space so that $(u, \nabla p^*)$, where p^* is the modified pressure term in (2.3), uniquely solves (1.1). Also, for $f \in C^\infty(\mathbb{T}^n)$, a smooth periodic pressure term p can be obtained as in (3.2) or (3.4) such that $(u, \nabla p)$ also uniquely solves the same Navier–Stokes equations (1.1) as $(u, \nabla p^*)$ does. Thus a natural observation in this discussion suggests to claim that the ∇p^* and ∇p must coincide in $\mathbb{T}^n$. This means that p^* is also a smooth periodic function of x, even if, structurally, it does not suggest so. Next, we state and prove a theorem that proves our claim.

Theorem 3.1. *If p^* given by (2.3) and p given by (3.4) solve the same Poisson pressure equation (3.1), then $p(x,t) = p^*(x,t) + C(t)$ for some constant C that depends only on t.*

Proof. Let $p_1 = p - p^*$. Then p_1 is harmonic in $\mathbb{R}^n$. Suppose, for any $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \neq x_2$, we apply a volume version of the mean value property of a harmonic function to obtain

$$p_1(x_1) = \frac{1}{vol(B_r(x_1))} \int_{B_r(x_1)} p_1(y)\, dy,$$

$$p_1(x_2) = \frac{1}{vol(B_r(x_2))} \int_{B_r(x_2)} p_1(y)\, dy,$$

where dy is an element of volume measure of the sphere and $r > 0$ is arbitrarily large.

Let $A = B_r(x_1) \setminus B_r(x_2) \cup B_r(x_2) \setminus B_r(x_1)$. Then

$$\begin{aligned} |p_1(x_1) - p_1(x_2)| &\le \frac{1}{vol(B_r(0))} \int\limits_A |p_1(y)|\, dy \\ &\le \frac{1}{vol(B_r(0))} \int\limits_A |p(y) - p^*(y)|\, dy \\ &\le \frac{1}{vol(B_r(0))} \int\limits_A \big(|p(y)| + |p^*(y)|\big)\, dy. \end{aligned}$$

Since $p(y,t)$ in (3.4) is bounded in $\mathbb{R}^n \times [0,T]$, for some $T > 0$ and from Theorem 2.1, we have $|p^*(y,t)| \le C(1 + \ln 3|y|)$, where C depends on t which we have suppressed in our notations. Therefore, we get

$$|p_1(x_1) - p_1(x_2)| \le \frac{1}{vol(B_r(0))} \int\limits_A C\big(1 + \ln 3|y|\big)\, dy \le \frac{C}{vol(B_r(0))} (1 + \ln 3r) vol(A).$$

Notice, if $B = B_{r+|x_1-x_2|}(x_1) \setminus B_{r-|x_1-x_2|}(x_2)$, then $vol(A) \le vol(B)$. Also, it is not difficult to see that $vol(B) = O(r^{n-1})$. To this end, we have

$$|p_1(x_1) - p_1(x_2)| \le \frac{C(1 + \ln 3r)O(r^{n-1})}{r^n}.$$

Therefore, as $r \to \infty$, we obtain $p_1(x_1) = p_1(x_2)$ for any x_1, x_2. Hence we have proved that p_1 is a constant function of x which further proves $p(x,t) = p^*(x,t) + C(t)$. Finally, we have proved that p^* is also a smooth periodic solution of the Poisson pressure equation (3.1), as a result, ∇p^* and ∇p coincide in $\mathbb{T}^n$. $\square$

4 Appendix: proof of Proposition 3.1

Proof. Using the Fourier expansion in (3.1), we get

$$\begin{aligned} \widehat{p}(k) = \frac{1}{|k|^2} \widehat{g}(k) &= \widehat{g}(k) \int\limits_0^\infty e^{-|k|^2\tau}\, d\tau \\ &= \widehat{g}(k) \int\limits_0^\alpha e^{-|k|^2\tau}\, d\tau + \widehat{g}(k) \int\limits_\alpha^\infty e^{-|k|^2\tau}\, d\tau, \quad \text{for } \alpha > 0 \\ &= \widehat{g}(k) \int\limits_0^\alpha e^{-|k|^2\tau}\, d\tau + \widehat{g}(k) \frac{e^{-|k|^2\alpha}}{|k|^2} \\ &= \widehat{p_1}(k) + \widehat{p_2}(k). \end{aligned}$$

Let us rewrite (3.4) as

$$p(x) = \sum_{\substack{k\in\mathbb{Z}^3 \\ k\neq 0}} \widehat{p}(k) e^{ik\cdot x} = \sum_{\substack{k\in\mathbb{Z}^3 \\ k\neq 0}} \big(\widehat{p_1}(k) + \widehat{p_2}(k)\big) e^{ik\cdot x} = \sum_{\substack{k\in\mathbb{Z}^3 \\ k\neq 0}} \widehat{p_1}(k) e^{ik\cdot x} + \sum_{\substack{k\in\mathbb{Z}^3 \\ k\neq 0}} \widehat{p_2}(k) e^{ik\cdot x} = p_1(x) + p_2(x).$$

Take

$$p_1(x) = \sum_{k\in\mathbb{Z}^3} \widehat{p_1}(k) e^{ik\cdot x} = \sum_{k\in\mathbb{Z}^3} \widehat{g}(k) \int\limits_0^\alpha e^{-|k|^2\tau}\, d\tau e^{ik\cdot x}.$$

The Lebesgue dominated convergence theorem allows us to write

$$p_1(x) = \int_0^\alpha \Big(\sum_{k\in\mathbb{Z}^3} \widehat{g}(k) e^{-|k|^2\tau} e^{ik\cdot x} \Big)\, d\tau = \int_0^\alpha \sum_{k\in\mathbb{Z}^3} (\widehat{g * \theta_\tau})(k) e^{ik\cdot x}\, d\tau.$$

Note that θ_t is the n-dimensional heat kernel. Since

$$\mathcal{F}^{-1}(e^{-|\xi|^2\tau})(x) = \frac{e^{-|x|^2/4\tau}}{(4\pi\tau)^{3/2}} \ \text{ for } \ \tau > 0,$$

where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^n$, and using the *Poisson Summation Formula*, we obtain

$$p_1(x) = \sum_{k\in\mathbb{Z}^3} \int_0^\alpha \Big(\int_{\mathbb{T}^3} \frac{e^{-|x-y+2\pi k|^2/4\tau}}{(4\pi\tau)^{3/2}}\, g(y)\, dy \Big)\, d\tau = \int_{\mathbb{T}^3} \sum_{k\in Z^3} \int_0^\alpha \frac{e^{-|x-y+2\pi k|^2/4\tau}}{(4\pi\tau)^{3/2}}\, d\tau g(y)\, dy.$$

Also,

$$\begin{aligned} p_2(x) &= \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \widehat{p_2}(k) e^{ik\cdot x} = \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \widehat{g}(k)\, \frac{e^{-|k|^2\tau}}{|k|^2}\, e^{ik\cdot x} \\ &= \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{e^{-|k|^2\tau}}{|k|^2} \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} g(y) e^{-ik\cdot y}\, dy\, e^{ik\cdot x} = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{e^{ik\cdot(x-y)}}{|k|^2}\, e^{-|k|^2\tau} g(y)\, dy. \end{aligned}$$

Finally, we arrive at the following expression:

$$\begin{aligned} p(x) &= p_1(x) + p_2(x) \\ &= \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \Big(\sum_{\substack{k\in\mathbb{Z}^3\\ k\neq 0}} \frac{e^{ik\cdot(x-y)}}{|k|^2}\, e^{-|k|^2\alpha} + \frac{1}{8\pi\sqrt{\pi}} \sum_{k\in\mathbb{Z}^3} \int_0^\alpha e^{-|x-y+2\pi k|^2/4\tau} \tau^{-3/2}\, d\tau \Big) g(y)\, dy. \end{aligned}$$

Hence the proposition is proved. □

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