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INFINITE ORDER OF GROWTH OF SOLUTIONS TO LINEAR FRACTIONAL DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

Abstract. In this paper, we investigate the growth of solutions of certain class of linear fractional differential equations with entire coefficients by using the Caputo fractional derivative operator. Under some conditions, we prove that every non-trivial solution is of infinite order.

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1 Introduction

The order of growth of an entire function $f(z)$ is defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log^+ m(r,f)}{\log r},$$

where

$$m(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |f(re^{i\varphi})|\, d\varphi;$$

and we have

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log^+ \log^+ M(r,f)}{\log r},$$

where $M(r,f) = \max\{|f(z)| : |z| = r\}$ (for more details, see [9, 12, 20]). Also, the order of an entire function given by $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ is equal to (see [2])

$$\sigma(f) = \limsup_{n\to+\infty} \frac{n\log n}{-\log|a_n|}.$$

Fractional order differential equations have become a very important tool for modeling phenomena in many diverse fields of science and engineering which traditional differential modeling cannot accomplish (see, e.g., Kilbas et al. [11]). Three kinds of fractional derivatives are often used, the Grünwald Letnikov derivative, the Riemann Liouville derivative and the Caputo derivative. There are many discussions for properties of these derivatives (see [13, 14]). All these studies are limited in real line. In this paper, we use the Caputo derivative which is defined as follows.

Definition ([11, 14, 15])**.** Suppose that $\alpha > 0$, $r \geq 0$. The fractional operator

$$\mathcal{D}^\alpha f(r) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^r \frac{f^{(n)}(t)}{(r-t)^{\alpha+1-n}}\, dt, & n-1<\alpha<n, \\ \dfrac{d^n}{dr^n} f(r), & \alpha = n \in \mathbb{N}\setminus\{0\}, \end{cases}$$

is called the Caputo derivative. It is clear that f should be n times continuously differentiable.

Consider the function $f(z) = \sum_{j=0}^{+\infty} a_j z^j$, where $z = re^{i\theta}$. By using the properties of the Caputo operator derivative, for $n-1<\alpha<n$, we have

$$\mathcal{D}^\alpha f(re^{i\theta}) = \sum_{j=n}^{+\infty} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)}\, a_j r^{j-\alpha} e^{ji\theta}, \tag{1.1}$$

$$r^\alpha \mathcal{D}^\alpha f(re^{i\theta}) = \sum_{j=n}^{+\infty} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)}\, a_j z^j.$$

For $\alpha = n \in \mathbb{N}\setminus\{0\}$,

$$\mathcal{D}^n f(z) = \frac{d^n}{dr^n} f(re^{i\theta}) \neq \frac{d^n}{dz^n} f(z),$$

while

$$\frac{r^n}{z^n}\mathcal{D}^n f(z) = \frac{d^n}{dz^n} f(z).$$

Proposition. *The two functions $f(z) = \sum_{j=0}^{+\infty} a_j z^j$ and $r^\alpha D^\alpha f(z)$ have the same radius of convergence. Consequently, if $f(z)$ is an entire function, then $r^\alpha D^\alpha f(z)$ is equally an entire function.*

Proof. To prove that the two power series

$$f(z)=\sum_{j=0}^{+\infty}a_jz^j,\quad r^\alpha D^\alpha f(z)=\sum_{j=n}^{+\infty}\frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)}\,a_jz^j$$

have the same radius of convergence, we have just to show that

$$\lim_{j\to+\infty}\frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)}\,\frac{\Gamma(j-\alpha+2)}{\Gamma(j+2)}=1.\qquad\square$$

In the study of the growth of solutions of the classical linear differential equation

$$f^{(n)}+A_{n-1}(z)f^{(n-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{1.2}$$

where the coefficients are entire functions, many authors are interested in the following question: what conditions on the coefficients will guarantee that every solution $f(z)\not\equiv 0$ of (1.2) has infinite order? In the literature, there are many papers concerning this question (see, e.g., [1,4,6,7,16]). The main tool used is this investigation is the logarithmic derivative estimates (see [5]). Unfortunately, up to now, there is no similar estimates given in [5] for the fractional derivatives except the Wiman–Valiron theorem in the fractional calculus that is valid only on a neighborhood of the points z, where the function reaches its maximum (see [3]). Despite this obstruction, we will investigate the growth of solutions of certain class of linear fractional differential equations by using the Caputo fractional derivative operator as the following.

Theorem 1.1. *Let $A_0(z),A_1(z),\dots,A_{n-1}(z)$ be entire functions and $\rho>0$, $\delta>0$ be constants such that $\max\{\sigma(A_j):\ j=1,\dots,n-1\}<\rho$ and $A_0(0)=0$; let $0<q_1<q_2<\cdots<q_n$. Suppose that for any $\theta\in[0,2\pi)$,*

$$|A_0(re^{i\theta})|\geq\exp\{\delta r^\rho\} \tag{1.3}$$

as $r\to+\infty$. Then every solution $f(z)\not\equiv 0$ of the linear fractional differential equation

$$\frac{r^{q_n}}{z^{[q_n]}}\,\mathcal{D}^{q_n}f(z)+A_{n-1}(z)\,\frac{r^{q_{n-1}}}{z^{[q_{n-1}]}}\,\mathcal{D}^{q_{n-1}}f(z)+\cdots+A_1(z)\,\frac{r^{q_1}}{z^{[q_1]}}\,\mathcal{D}^{q_1}f(z)+zA_0(z)f(z)=0 \tag{1.4}$$

is an entire function of infinite order and, further, if $\sigma(A_0)<\infty$, then $\sigma_2(f)\leq\sigma(A_0)$.

Corollary. *Let $P_1(z),\dots,P_{n-1}(z)$ be polynomials and $0<q_1<q_2<\cdots<q_n$. Then every solution $f(z)\not\equiv 0$ of the linear fractional differential equation*

$$\frac{r^{q_n}}{z^{[q_n]}}\,\mathcal{D}^{q_n}f(z)+P_{n-1}(z)\,\frac{r^{q_{n-1}}}{z^{[q_{n-1}]}}\,\mathcal{D}^{q_{n-1}}f(z)+\cdots+P_1(z)\,\frac{r^{q_1}}{z^{[q_1]}}\,\mathcal{D}^{q_1}f(z)+(\sin z+\sinh z)f(z)=0$$

is an entire function of infinite order with $\sigma_2(f)\leq 1$.

Theorem 1.2. *Let $A_1(z),B(z),A_0(z)\not\equiv 0$, be entire functions and let $\rho>0$, $\delta>0$, $0<\alpha<1$ be constants such that $\max\{\sigma(A_0),\sigma(B)\}<\rho$. Suppose that there exists a set $E\subset[0,2\pi)$ of linear measure zero such that for any $\theta\in[0,2\pi)\setminus E$,*

$$|A_1(re^{i\theta})|\geq\exp\{\delta r^\rho\} \tag{1.5}$$

as $r\to+\infty$. Then every solution $f(z)\not\equiv 0$ of the differential equation

$$e^{-2i\theta}\mathcal{D}^2f(z)+A_1(z)e^{-i\theta}\mathcal{D}^1f(z)+B(z)r^\alpha\mathcal{D}^\alpha f(z)+A_0(z)f(z)=0 \tag{1.6}$$

is an entire function of infinite order and, further, if $\sigma(A_1)<\infty$ then $\sigma_2(f)\leq\sigma(A_1)$.

Example. By Theorem 1.2, every solution $f(z)\not\equiv 0$ of the differential equation

$$e^{-2i\theta}\mathcal{D}^2f(z)+\sin(z^2)e^{-i\theta}\mathcal{D}^1f(z)+e^zr^\alpha\mathcal{D}^\alpha f(z)+zf(z)=0$$

is an entire function of infinite order with $\sigma_2(f)\leq 2$.

Theorem 1.3. *Let $A_1(z), B(z), A_0(z) \not\equiv 0$, $F(z)$ be entire functions and let $\rho > 0$, $\delta > 0$, $0 < \alpha < 1$ be constants such that $\max\{\sigma(A_0), \sigma(B), \sigma(F)\} < \rho$. Suppose that there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for any $\theta \in [0, 2\pi) \setminus E$,*

$$|A_1(re^{i\theta})| \geq \exp\{\delta r^\rho\} \tag{1.7}$$

as $r \to +\infty$. Then every solution $f(z) \not\equiv 0$ of the differential equation

$$e^{-2i\theta}\mathcal{D}^2 f(z) + A_1(z)e^{-i\theta}\mathcal{D}^1 f(z) + B(z)\,\frac{r^\alpha}{z^{[\alpha]}}\,\mathcal{D}^\alpha f(z) + A_0(z)f(z) = F(z) \tag{1.8}$$

is an entire function of infinite order.

2 Preliminary lemmas

To prove these results we need the following lemmas.

Lemma 2.1. *Let f be a non-constant entire function and suppose that $|\mathcal{D}^1 f(z)|$ is unbounded on some ray $\arg z = \theta \in [0, 2\pi) \setminus E$, where E is of linear measure zero. Then there exists an infinite sequence of points r_m $(m \geq 1)$, $r_m \to +\infty$, such that $|\mathcal{D}^1 f(r_m e^{i\theta})| \to +\infty$ and*

$$\Big|\frac{\mathcal{D}^\alpha f(r_m e^{i\theta})}{\mathcal{D}^1 f(r_m e^{i\theta})}\Big| \leq \frac{r_m^{1-\alpha}}{\Gamma(2-\alpha)}\,, \quad 0 < \alpha < 1, \tag{2.1}$$

$$\Big|\frac{f(r_m e^{i\theta})}{\mathcal{D}^1 f(r_m e^{i\theta})}\Big| \leq r_m + o(1) \tag{2.2}$$

as $m \to +\infty$.

Proof. By definition, we have

$$\mathcal{D}^\alpha f(re^{i\theta}) = \frac{1}{\Gamma(1-\alpha)} \int_0^r \frac{\mathcal{D}^1 f(te^{i\theta})}{(r-t)^\alpha}\, dt. \tag{2.3}$$

Since $|\mathcal{D}^1(r_m e^{i\theta})|$ is unbounded, we can construct a sequence r_m $(m \geq 1)$, $r_m \to +\infty$, such that $|\mathcal{D}^1(r_m e^{i\theta})| \to +\infty$ and $|\mathcal{D}^1(r_m e^{i\theta})| = \max\{|\mathcal{D}^1(te^{i\theta})| : \ t \in [0, r_m]\}$. By (2.3), we have

$$|\mathcal{D}^\alpha f(r_m e^{i\theta})| \leq \frac{1}{\Gamma(1-\alpha)} \int_0^{r_m} \frac{|\mathcal{D}^1 f(te^{i\theta})|}{(r_m - t)^\alpha}\, dt;$$

and then

$$|\mathcal{D}^\alpha f(r_m e^{i\theta})| \leq \frac{|\mathcal{D}^1 f(r_m e^{i\theta})|}{\Gamma(1-\alpha)} \int_0^{r_m} \frac{1}{(r_m - t)^\alpha}\, dt,$$

so, we obtain

$$\Big|\frac{\mathcal{D}^\alpha f(r_m e^{i\theta})}{\mathcal{D}^1 f(r_m e^{i\theta})}\Big| \leq \frac{r_m^{1-\alpha}}{\Gamma(2-\alpha)}\,.$$

On the other hand, we have

$$f(re^{i\theta}) = f(0) + e^{i\theta} \int_0^r \mathcal{D}^1 f(te^{i\theta})\, dt,$$

and then

$$|f(r_m e^{i\theta})| \leq |f(0)| + |\mathcal{D}^1 f(r_m e^{i\theta})| r_m,$$

which implies

$$\Big|\frac{f(r_m e^{i\theta})}{\mathcal{D}^1 f(r_m e^{i\theta})}\Big| \leq o(1) + r_m, \quad m \to +\infty. \qquad \square$$

Lemma 2.2. *Let $f(z)$ be an entire function and suppose that*

$$G(re^{i\theta}) := \frac{\log^+ |\mathcal{D}^1 f(re^{i\theta})|}{r^\rho}$$

is unbounded on some ray $\arg z = \theta \in [0, 2\pi)$, where $\alpha > 0$ and $\rho > 0$. Then there exists an infinite sequence of points r_m $(m \geq 1)$, $r_m \to +\infty$, such that $G(r_m e^{i\theta}) \to +\infty$ and inequalities (2.1), (2.2) hold.

Proof. If $G(re^{i\theta})$ is unbounded on some ray $\arg z = \theta \in [0, 2\pi)$, then we can immediately see that $|\mathcal{D}^1 f(re^{i\theta})|$ is unbounded on the ray $\arg z = \theta$. So, by Lemma 2.1, (2.1), (2.2) hold. □

By [5] and by taking into account that $e^{-ni\theta}\mathcal{D}^n f(z) = \frac{d^n}{dz^n} f(z)$, we obtain the following

Lemma 2.3. *Let f be a non-constant entire function of finite order $\sigma(f) = \sigma < \infty$; let $\varepsilon > 0$ be a given constant. Then the following two statements hold.*

(i) *There exists a set $F \subset (1, +\infty)$ of a finite logarithmic measure such that for all $r \in (1, +\infty) \setminus F$ and for integers k, j $(0 \leq k \leq j)$, we have*

$$\left| \frac{\mathcal{D}^j f(z)}{\mathcal{D}^k f(z)} \right| \leq r^{(j-k)(\sigma - 1 + \varepsilon)}. \tag{2.4}$$

(ii) *There exists a set $E \subset [0, 2\pi)$ of a linear measure zero such that for all $\theta \in [0, 2\pi) \setminus E$, there exists a constant $r_0 = r_0(\theta) > 0$ such that for all z satisfying $\arg(z) \in [0, 2\pi) \setminus E$ and $r = |z| \geq r_0$, inequality (2.4) holds.*

Lemma 2.4 ([3])**.** *Let $f(z)$ be an entire function, $\gamma > 0$, $0 < \delta < \frac{1}{4}$ and z be such that $|z| = r$ and $|f(z)| > M(r, f)\nu(r)^{-\frac{1}{4}+\delta}$ holds, where $\nu(r)$ is the central index of f. Then there exists a set $E \subset (0, +\infty)$ of finite logarithmic measure, that is, $\int\limits_E \frac{dt}{t} < +\infty$ such that*

$$\frac{r^\gamma \mathcal{D}^\gamma f(z)}{f(z)} = (\nu(r))^\gamma (1 + o(1))$$

holds for $r \to +\infty$ and $r \notin E$.

Remark. We signal here that the fractional derivative used in the proof of Lemma 2.4 is the Riemann-Liouville operator and for an entire function $f(z) = \sum\limits_{j=0}^{+\infty} a_j z^j$, we have

$$\mathcal{D}^\alpha_{RL} f(z) = \sum_{j=0}^{+\infty} \frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)} a_j r^{j-\alpha} e^{ji\theta}. \tag{2.5}$$

By (1.1) and (2.5), we immediately conclude that the proof of Lemma 2.1 is valid also for the Caputo fractional derivative operator.

Lemma 2.5 ([12])**.** *Let $f(z)$ be an entire function of finite order $\sigma(f) < +\infty$. Then*

$$\limsup_{r \to +\infty} \frac{\log^+ \nu(r)}{\log r} = \sigma(f),$$
$$\limsup_{r \to +\infty} \frac{\log^+ \log^+ \nu(r)}{\log r} = \sigma_2(f),$$

where $\nu(r)$ is the central index of f.

Lemma 2.6. *Let $A_0(z), A_1(z), \ldots, A_{n-1}(z)$ be entire functions such that* $\max\{\sigma(A_j) : j = 0, 1, \ldots, n-1\} = \rho < +\infty$*; let $0 < q_1 < q_2 < \cdots < q_n$. Then every solution $f(z) \not\equiv 0$ of the linear fractional differential equation*

$$\frac{r^{q_n}}{z^{[q_n]}} \mathcal{D}^{q_n} f(z) + A_{n-1}(z) \frac{r^{q_{n-1}}}{z^{[q_{n-1}]}} \mathcal{D}^{q_{n-1}} f(z) + \cdots + A_1(z) \frac{r^{q_1}}{z^{[q_1]}} \mathcal{D}^{q_1} f(z) + z A_0(z) f(z) = 0 \tag{2.6}$$

is an entire function satisfying $\sigma_2(f) \leq \rho$.

Proof. By definition and the assumption that $\max\{\sigma(A_j) : \ j = 0, 1, \dots, n-1\} = \rho$, for any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $r \geq r_0$, we have

$$|A_j(z)| \leq \exp\{r^{\rho+\varepsilon}\}, \ \ j = 0, 1, \dots, n-1. \tag{2.7}$$

From (2.6), we can write

$$\begin{aligned}\Big|\frac{r^{q_n}\mathcal{D}^{q_n}f(z)}{f(z)}\Big| \leq |A_{n-1}(z)|r^{[q_n]-[q_{n-1}]}\Big|\frac{r^{q_{n-1}}\mathcal{D}^{q_{n-1}}f(z)}{f(z)}\Big| + \cdots \\ + |A_1(z)|r^{[q_n]-[q_1]}\Big|\frac{r^{q_1}\mathcal{D}^{q_1}f(z)}{f(z)}\Big| + r^{1+[q_n]}|A_0(z)|.\end{aligned} \tag{2.8}$$

By Lemma 2.4, (2.7) and (2.8), we obtain

$$(\nu(r))^{q_n}(1 + o(1)) \leq cr^{[q_n]-[q_{n-1}]}(\nu(r))^{q_{n-1}} \exp\{r^{\rho+\varepsilon}\},$$

where $c > 0$ is some constant; and then

$$(\nu(r))^{q_n-q_{n-1}}(1 + o(1)) \leq cr^{[q_n]-[q_{n-1}]} \exp\{r^{\rho+\varepsilon}\}.$$

From (2.7) and by Lemma 2.5, we obtain $\sigma_2(f) \leq \rho$. □

Lemma 2.7 ([18])**.** *Let f be an entire function of finite order $\sigma(f)$. Suppose that there exists a set $E \subset [0, 2\pi)$ of a linear measure zero such that*

$$\log^+|f(re^{i\theta})| \leq Mr^{\rho}$$

for any $\theta \in [0, 2\pi) \setminus E$, where M is a positive constant depending on θ, while ρ is a positive constant independent of θ. Then $\sigma(f) \leq \rho$.

Lemma 2.8 ([8])**.** *Let $A_0(z) \not\equiv 0$, $A_1(z), \dots, A_{n-1}(z)$ be entire functions such that $A_0(0) = 0$; let $0 < q_1 < q_2 < \cdots < q_n$ be real constants. Then all solutions of (1.4) are entire functions.*

3 Proof of theorems

Proof of Theorem 1.1. First, by Lemma 2.8, all solutions of (1.4) are entire functions. If we suppose that $f(z) \not\equiv 0$ is a solution of (1.4) with $\sigma(f) < \rho$, then by the assumptions of Theorem 1.1 and by taking into account that $\sigma(r^{q_i}\mathcal{D}^{q_i}f) = \sigma(f) < \rho$, we can immediately see that the term $zA_0(z)f(z)$ is the only dominant term in (1.4) which leads to a contradiction. So, $\sigma(f) \geq \rho$. Now, suppose that $\sigma(f) = \sigma < \infty$. By Lemma 2.5, for any given $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $r \geq r_0$, we have $\nu(r) \leq r^{\sigma+\varepsilon}$; and by Lemma 2.4, for $r \notin E$ and $\arg z = \theta_0$, we have

$$\Big|\frac{r^{\gamma}\mathcal{D}^{\gamma}f(z)}{f(z)}\Big| \leq r^{\gamma(\sigma+\varepsilon)}. \tag{3.1}$$

Set $\max\{\sigma(A_j) : \ j = 1, \dots, n-1\} = \rho_1$. For any given ϵ such that $0 < \varepsilon < \rho - \rho_1$, we have

$$|A_j(z)| \leq \exp\{r^{\rho_1+\varepsilon}\}, \ \ j = 1, \dots, n-1. \tag{3.2}$$

From (1.4), we have

$$\begin{aligned}|A_0(z)| \leq r^{q_n-[q_n]-1}\Big|\frac{\mathcal{D}^{q_n}f(z)}{f(z)}\Big| \\ + |A_{n-1}(z)|r^{q_{n-1}-[q_{n-1}]-1}\Big|\frac{\mathcal{D}^{q_{n-1}}f(z)}{f(z)}\Big| + \cdots + |A_1(z)|r^{q_1-[q_1]-1}\Big|\frac{\mathcal{D}^{q_1}f(z)}{f(z)}\Big|.\end{aligned} \tag{3.3}$$

Combining (3.1), (3.2)), (1.3) with (3.3), we obtain

$$\exp\{\delta r^{\rho}\} \leq r^{\gamma(\sigma+\varepsilon)} \exp\{r^{\rho_1+\varepsilon}\}. \tag{3.4}$$

Since $\varepsilon < \rho - \rho_1$, (3.4) leads to a contradiction as $r \to +\infty$. So, $\sigma(f) = +\infty$. Now, if $\sigma(A_0) < \infty$, then by the assumptions, we have $\max\{\sigma(A_j) : \ j = 1, \dots, n-1\} = \sigma(A_0)$ and by Lemma 2.6, we get $\sigma_2(f) \leq \sigma(A_0)$. □

Proof of Theorem 1.2. By the same method of proof of Lemma 2.8, we can prove that all solutions of (1.6) are entire functions. If we suppose that $f(z) \not\equiv 0$ is a solution of (1.6) with $\sigma(f) = \sigma < \rho$, then by the assumptions, we can immediately see that the term $A_1(z)e^{-i\theta}\mathcal{D}^1(z)$ is the only dominant term in (1.6) which leads to a contradiction. So, $\sigma(f) \geq \rho$. Now, we suppose that $\sigma(f) < \infty$. From (1.6), we can write

$$|A_1(z)| \leq \Big|\frac{\mathcal{D}^2 f(z)}{\mathcal{D}^1 f(z)}\Big| + |B(z)|\Big|\frac{r^\alpha \mathcal{D}^\alpha f(z)}{\mathcal{D}^1 f(z)}\Big| + |A_0(z)|\Big|\frac{f(z)}{\mathcal{D}^1 f(z)}\Big|. \tag{3.5}$$

Set $\max\{\sigma(A_0), \sigma(B)\} = \rho_1$. For any given ϵ $(0 < \varepsilon < \rho - \rho_1)$, we have

$$\max\big\{|A_0(z)|, |B(z)|\big\} \leq \exp\{r^{\rho_1+\varepsilon}\}. \tag{3.6}$$

By Lemma 2.3, there exists a set $E \subset [0, 2\pi)$ of a linear measure zero such that for all $\theta \in [0, 2\pi) \setminus E$, there exists a constant $r_0 = r_0(\theta) > 0$ such that for all z satisfying $\arg(z) \in [0, 2\pi)\setminus E$ and $r = |z| \geq r_0$, we have

$$\Big|\frac{\mathcal{D}^2 f(z)}{\mathcal{D}^1 f(z)}\Big| \leq r^{\sigma-1+\varepsilon}. \tag{3.7}$$

We will prove that $|\mathcal{D}^1 f(z)|$ is bounded in $[0, 2\pi) \setminus E$; toward this end, we suppose to the contrary that $\mathcal{D}^1 f(z)$ is unbounded on some ray $\arg z = \theta \in [0, 2\pi) \setminus E$. Then there exists an infinite sequence of points r_m $(m \geq 1)$, $r_m \to +\infty$, such that $|\mathcal{D}^1 f(r_m e^{i\theta})| \to +\infty$ and

$$\Big|\frac{\mathcal{D}^\alpha f(r_m e^{i\theta})}{\mathcal{D}^1 f(r_m e^{i\theta})}\Big| \leq \frac{r_m^{1-\alpha}}{\Gamma(2-\alpha)}, \quad 0 < \alpha < 1, \tag{3.8}$$

$$\Big|\frac{f(r_m e^{i\theta})}{\mathcal{D}^1 f(r_m e^{i\theta})}\Big| \leq r_m + o(1). \tag{3.9}$$

Using (3.6)–(3.9) and (1.5) in (3.5), we get

$$\exp\{\delta r_m^\rho\} \leq c r_m^d \exp\{r_m^{\rho_1+\varepsilon}\}, \tag{3.10}$$

where $c > 0$, $d > 0$. Since $\varepsilon < \rho - \rho_1$, (3.10) leads to a contradiction as $m \to +\infty$. So, $|e^{i\theta}\mathcal{D}^1 f(z)|$ is bounded in $[0, 2\pi) \setminus E$. By the Phragmen–Lindelöf theorem, $e^{i\theta}\mathcal{D}^1 f(z)$ has to be constant in the whole complex plane which implies that $f(z)$ is a polynomial of degree one; but this is impossible. So, $\sigma(f) = +\infty$. Now, if $\sigma(A_1) < \infty$, then by the assumptions, we have $\max\{\sigma(A_0), \sigma(A_1), \sigma(B)\} = \sigma(A_1)$ and by Lemma 2.6, we have $\sigma_2(f) \leq \sigma(A_1)$. □

Proof of Theorem 1.3. As above, all solutions of (1.8) are entire functions. Suppose first that $f(z) \not\equiv 0$ is a solution of (1.8) with $\sigma(f) = \sigma < \rho$. From (1.8), we can write

$$A_1(z)e^{-i\theta}\mathcal{D}^1(z) = F(z) - e^{-2i\theta}\mathcal{D}^2 f(z) - B(z)\,\frac{r^\alpha}{z^{[\alpha]}}\,\mathcal{D}^\alpha f(z) - A_0(z)f, \tag{3.11}$$

then by the assumptions, the left-hand side of (3.11) is of order greather than or equal to ρ, while the right-hand side of (3.11) is of order, strictly smaller than ρ, which is a contradiction. So, $\sigma(f) \geq \rho$. Now, to prove that $\sigma(f) = \infty$, we suppose to the contrary that $\sigma(f) = \sigma < \infty$. Set $\max\{\sigma(A_0), \sigma(B), \sigma(F)\} = \rho_1$. For any given ε $(0 < \varepsilon < \rho - \rho_1)$, we have

$$\max\big\{|A_0(z)|, |B(z)|, |F(z)|\big\} \leq \exp\{r^{\rho_1+\varepsilon}\}. \tag{3.12}$$

We will prove that $\frac{\log^+ |\mathcal{D}^1 f(re^{i\theta})|}{r^{\rho_1+\varepsilon}}$ is bounded in $[0, 2\pi) \setminus E$; to this end, we suppose to the contrary that $\frac{\log^+ |\mathcal{D}^1 f(re^{i\theta})|}{r^{\rho_1+\varepsilon}}$ is unbounded on some ray $\arg z = \theta \in [0, 2\pi) \setminus E$. Then by Lemma 2.2, there exists an infinite sequence of points r_m $(m \geq 1)$, $r_m \to +\infty$, such that

$$\frac{\log^+ |\mathcal{D}^1 f(r_m e^{i\theta})|}{r_m^{\rho_1+\varepsilon}} \to +\infty, \tag{3.13}$$

$$\Big|\frac{\mathcal{D}^\alpha f(r_m e^{i\theta})}{\mathcal{D}^1 f(r_m e^{i\theta})}\Big| \leq \frac{r_m^{1-\alpha}}{\Gamma(2-\alpha)}\,, \tag{3.14}$$

and

$$\Big|\frac{f(r_m e^{i\theta})}{\mathcal{D}^1 f(r_m e^{i\theta})}\Big| \le r_m + o(1). \tag{3.15}$$

From (3.13), for any sufficiently large number $c > 1$ and for $m \ge m_0$, we have

$$|\mathcal{D}^1 f(r_m e^{i\theta})| \ge \exp\{c r_m^{\rho_1+\varepsilon}\}. \tag{3.16}$$

From (3.12) and (3.16), we get

$$\Big|\frac{F(r_m e^{i\theta})}{\mathcal{D}^1 f(r_m e^{i\theta})}\Big| \le \exp\big\{(1-c) r_m^{\rho_1+\varepsilon}\big\} \to 0, \quad m \to +\infty. \tag{3.17}$$

From (1.8), we can write

$$|A_1(z)| \le \Big|\frac{\mathcal{D}^2 f(z)}{\mathcal{D}^1 f(z)}\Big| + |B(z)| \Big|\frac{r^\alpha \mathcal{D}^\alpha f(z)}{\mathcal{D}^1 f(z)}\Big| + |A_0(z)| \Big|\frac{f(z)}{\mathcal{D}^1 f(z)}\Big| + \Big|\frac{F(z)}{\mathcal{D}^1 f(z)}\Big|. \tag{3.18}$$

Substituting (3.7), (3.12), (3.14), (3.15), (3.17) and (1.7) into (3.18), we obtain

$$\exp\{\delta r_m^\rho\} \le c' r_m^{d'} \exp\{r_m^{\rho_1+\varepsilon}\},$$

where $c' > 0$, $d' > 0$. Since $\varepsilon < \rho - \rho_1$, (3.18) leads to a contradiction as $m \to +\infty$. So, $\frac{\log^+ |\mathcal{D}^1 f(re^{i\theta})|}{r^{\rho_1+\varepsilon}}$ is bounded in $[0, 2\pi) \setminus E$; so $|\mathcal{D}^1 f(re^{i\theta})| \le \exp\{M r_m^{\rho_1+\varepsilon}\}$, where $M > 0$ is a constant and by the inequality

$$|f(re^{i\theta})| \le |f(0)| + \int_0^r |\mathcal{D}^1 f(te^{i\theta})|\, dt,$$

we get

$$|f(re^{i\theta})| \le \exp\{r^{\rho_1+2\varepsilon}\};$$

and by Lemma 2.7, we get $\sigma(f) \le \rho_1 + 2\varepsilon$ for any $\varepsilon > 0$; so, $\sigma(f) \le \rho_1$ which is a contradiction with $\sigma(f) \ge \rho > \rho_1$. So, we conclude that $\sigma(f) = +\infty$. □

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