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# FAST GROWING SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

**Abstract.** In this paper, we use the concept of $\varphi$-order to investigate under suitable conditions the growth and the oscillation of solutions of higher order linear differential equations with meromorphic coefficients on the complex plane. Many existing results due to Li–Cao, Hu–Zheng, Kara–Belaïdi will be revisited and extended for the lower $\varphi$-order and the $\varphi$-convergence exponent.

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**Key words and phrases.** Meromorphic function, $\varphi$-order, $\varphi$-type, $\varphi$-convergence exponent.

**რეზიუმე.** ნაშრომში კომპლექსურ სიბრტყეზე მერომორფულ კოეფიციენტებიანი მაღალი რიგის წრფივი დიფერენციალური განტოლებების ამონახსნების ზრდადობის და რხევის გამოსაკვლევად გამოყენებულია $\varphi$-რიგის კონცეფცია. Li–Cao, Hu–Zheng, Kara–Belaïdi-ს მიერ მიღებული მრავალი შედეგი გადაიხედა და გაფართოვდა უფრო დაბალი $\varphi$-რიგისთვის და $\varphi$-კრებადობის ექსპონენტისთვის.

# 1 Introduction

Throughout this paper, the reader is assumed to be familiar with the fundamental notions and standard notations of Nevanlinna value distribution theory of meromorphic functions such as $M(r,f)$, $T(r,f)$, $m(r,f)$, $N(r,f)$, $\delta(a,f)$ (see [11, 19, 25]). In addition, the term "meromorphic function" will mean "meromorphic function in the whole complex plane $\mathbb{C}$". For all $r \in \mathbb{R}$, we define $\exp_1 r := \exp r = e^r$ and $\exp_{p+1} r := \exp(\exp_p r)$, $p \in \mathbb{N} = \{1, 2, \dots\}$. Inductively, for all $r \in (0, +\infty)$ sufficiently large, we define $\log_1 r := \log r$ and $\log_{p+1} r := \log(\log_p r)$, $p \in \mathbb{N}$. By convention, we denote $\exp_0 r := r = \log_0 r$, $\exp_{-1} r := \log_1 r$ and $\log_{-1} r := \exp_1 r$. We define the linear measure of a set $E \subset (0, +\infty)$ by $m(E) = \int\limits_E dt$ and the logarithmic measure of a set $F \subset (1, +\infty)$ by $lm(F) = \int\limits_F \frac{dt}{t}$.

**Definition 1.1** ([20, 21])**.** Let $p \geq q \geq 1$ be integers. The $[p,q]$-order of a transcendental meromorphic function $f$ is defined by

$$\rho_{[p,q]}(f) := \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

If $f$ is a transcendental entire function, then

$$\rho_{[p,q]}(f) := \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log_q r}.$$

$\rho_{[p,1]}(f)$ is called the iterated $p$-order and simply denoted by $\rho_p(f)$. Moreover, $\rho_{[1,1]}(f) = \rho_1(f)$ coincides with the usual order $\rho(f)$ [11, 19, 25].

Historically, Juneja–Kapoor–Bajpai [15] introduced the concepts of $[p,q]$-order to study some properties of entire functions. In [21], Liu-Tu-Shi made a minor modification on the original definition of $[p,q]$-order to investigate the growth of entire solutions of higher order complex linear differential equations of the form

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0, \tag{1.1}$$

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z), \tag{1.2}$$

where $k \geq 2$ and the coefficients $A_0, \dots, A_{k-1}$, $F \not\equiv 0$ are entire functions of $[p,q]$-order. The case when the coefficients of equations (1.1) and (1.2) are meromorphic of $[p,q]$-order have been discussed by Li-Cao [20], Hu–Zheng [13], Belaïdi [4] and many other authors. We also mention that Belaïdi [3] and Hu–Zheng [14] investigated the growth of solutions of equation (1.1) with analytic coefficients of $[p,q]$-order and lower $[p,q]$-order in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ and obtained similar results to those in the complex plane. However, the iterated $p$-order and the $[p,q]$-order do not cover an arbitrary growth as it is shown in [9, Example 1.4]. A general scale which does not have this disadvantage is called the $\varphi$-order (see [24]), and it is adopted recently by Chyzhykov–Semochko [9], Semochko [22], Belaïdi [5, 6], Kara–Belaïdi [17] in order to study the fast growing of solutions of equations (1.1) and (1.2) in the complex plane and in the unit disc which extend some previous results that considered the iterated $p$-order [2, 7, 12, 18].

**Definition 1.2** ([9])**.** Let $\varphi$ be an increasing unbounded function on $[1, +\infty)$. The $\varphi$-orders of a meromorphic function $f$ are defined by

$$\rho^0_\varphi(f) := \limsup_{r\to+\infty} \frac{\varphi(e^{T(r,f)})}{\log r}, \quad \rho^1_\varphi(f) := \limsup_{r\to+\infty} \frac{\varphi(T(r,f))}{\log r}.$$

If $f$ is an entire function, then the $\varphi$-orders are defined by

$$\widetilde{\rho}^{\,0}_\varphi(f) := \limsup_{r\to+\infty} \frac{\varphi(M(r,f))}{\log r}, \quad \widetilde{\rho}^{\,1}_\varphi(f) := \limsup_{r\to+\infty} \frac{\varphi(\log M(r,f))}{\log r}.$$

We denote by $\Phi$ the class of positive unbounded increasing functions on $[1, +\infty)$ such that $\varphi(e^t)$ grows slowly, i.e.,

$$\forall\, c > 0 : \quad \lim_{t\to+\infty} \frac{\varphi(e^{ct})}{\varphi(e^t)} = 1.$$

**Example.** Let $f$ be a meromorphic function. It is clear that $\varphi(r) = \log_p r$ $(p \geq 2)$ belongs to the class $\Phi$ and $\varphi(r) = \log r \notin \Phi$. For $\varphi(r) = \log_p r$ $(p \in \mathbb{N})$, we have

$$\rho^1_\varphi(f) = \rho^1_{\log_p}(f) = \rho_p(f).$$

In particular, $\rho^0_{\log_2}(f) = \rho_1(f) = \rho(f)$ is the usual order of $f$ and $\rho^1_{\log_2}(f) = \rho_2(f)$ is the hyper-order of $f$.

**Proposition 1.1** ([9])**.** *If $\varphi \in \Phi$, then*

$$\forall\, m > 0,\ \forall\, k \geq 0: \ \frac{\varphi^{-1}(\log x^m)}{x^k} \to +\infty, \ \ x \to +\infty, \tag{1.3}$$

$$\forall\, \delta > 0: \ \frac{\log \varphi^{-1}((1+\delta)x)}{\log \varphi^{-1}(x)} \to +\infty, \ \ x \to +\infty, \tag{1.4}$$

$$\forall\, c > 0: \ \varphi(c\,x) \leq \varphi(x^c) \leq (1 + o(1))\varphi(x), \ \ x \to +\infty. \tag{1.5}$$

**Proposition 1.2** ([9])**.** *Let $\varphi \in \Phi$ and $f$ be an entire function. Then for $j = 0, 1$, we have*

$$\rho^j_\varphi(f) = \widetilde{\rho}^{\,j}_\varphi(f).$$

**Definition 1.3** ([6])**.** Let $\varphi$ be an increasing unbounded function on $[1, +\infty)$. The $\varphi$-types of a meromorphic function $f$ with $0 < \rho^j_\varphi(f) < +\infty$ $(j = 0, 1)$ are defined by

$$\tau^0_\varphi(f) := \limsup_{r \to +\infty} \frac{\exp\{\varphi(e^{T(r,f)})\}}{r^{\rho^0_\varphi(f)}}, \quad \tau^1_\varphi(f) := \limsup_{r \to +\infty} \frac{\exp\{\varphi(T(r,f))\}}{r^{\rho^1_\varphi(f)}}.$$

If $f$ is an entire function with $0 < \widetilde{\rho}^{\,j}_\varphi(f) < +\infty$ $(j = 0, 1)$, then the $\varphi$-types are defined by

$$\widetilde{\tau}^{\,0}_\varphi(f) := \limsup_{r \to +\infty} \frac{\exp\{\varphi(M(r,f))\}}{r^{\widetilde{\rho}^{\,0}_\varphi(f)}}, \quad \widetilde{\tau}^{\,1}_\varphi(f) := \limsup_{r \to +\infty} \frac{\exp\{\varphi(\log M(r,f))\}}{r^{\widetilde{\rho}^{\,1}_\varphi(f)}}.$$

**Definition 1.4** ([17])**.** Let $\varphi$ be an increasing unbounded function on $[1, +\infty)$. The $\varphi$-convergence exponents of the sequence of zeros of a meromorphic function $f$ are defined by

$$\lambda^0_\varphi(f) := \limsup_{r \to +\infty} \frac{\varphi(e^{N(r,\frac{1}{f})})}{\log r}, \quad \lambda^1_\varphi(f) := \limsup_{r \to +\infty} \frac{\varphi(N(r,\frac{1}{f}))}{\log r}.$$

Similarly, the notations $\overline{\lambda}^{\,0}_\varphi(f)$ and $\overline{\lambda}^{\,1}_\varphi(f)$ can be used to denote the $\varphi$-convergence exponents of the sequence of distinct zeros of $f$

$$\overline{\lambda}^{\,0}_\varphi(f) := \limsup_{r \to +\infty} \frac{\varphi(e^{\overline{N}(r,\frac{1}{f})})}{\log r}, \quad \overline{\lambda}^{\,1}_\varphi(f) := \limsup_{r \to +\infty} \frac{\varphi(\overline{N}(r,\frac{1}{f}))}{\log r}.$$

Now, we can introduce by analogous manner the following quantities.

**Definition 1.5** ([5,6])**.** Let $\varphi$ be an increasing unbounded function on $[1, +\infty)$. The lower $\varphi$-orders of a meromorphic function $f$ are defined by

$$\mu^0_\varphi(f) := \liminf_{r \to +\infty} \frac{\varphi(e^{T(r,f)})}{\log r}, \quad \mu^1_\varphi(f) := \liminf_{r \to +\infty} \frac{\varphi(T(r,f))}{\log r}.$$

If $f$ is an entire function, then the lower $\varphi$-orders are defined by

$$\widetilde{\mu}^{\,0}_\varphi(f) := \liminf_{r \to +\infty} \frac{\varphi(M(r,f))}{\log r}, \quad \widetilde{\mu}^{\,1}_\varphi(f) := \liminf_{r \to +\infty} \frac{\varphi(\log M(r,f))}{\log r}.$$

**Proposition 1.3** ([5])**.** *Let $\varphi \in \Phi$ and $f$ be an entire function. Then for $j = 0, 1$, we have*

$$\mu^j_\varphi(f) = \widetilde{\mu}^{\,j}_\varphi(f).$$

**Definition 1.6** ([6, 17]). Let $\varphi$ be an increasing unbounded function on $[1,+\infty)$. The lower $\varphi$-types of a meromorphic function $f$ with $0<\mu_\varphi^j(f)<+\infty$ $(j=0,1)$ are defined by

$$\underline{\tau}_\varphi^0(f)=\liminf_{r\to+\infty}\frac{\exp\{\varphi(e^{T(r,f)})\}}{r^{\mu_\varphi^0(f)}},\quad \underline{\tau}_\varphi^1(f)=\liminf_{r\to+\infty}\frac{\exp\{\varphi(T(r,f))\}}{r^{\mu_\varphi^1(f)}}.$$

If $f$ is an entire function with $0<\widetilde{\mu}_\varphi^j(f)<+\infty$ $(j=0,1)$, then the lower $\varphi$-types are defined by

$$\underline{\widetilde{\tau}}_\varphi^0(f)=\liminf_{r\to+\infty}\frac{\exp\{\varphi(M(r,f))\}}{r^{\widetilde{\mu}_\varphi^0(f)}},\quad \underline{\widetilde{\tau}}_\varphi^1(f)=\liminf_{r\to+\infty}\frac{\exp\{\varphi(\log M(r,f))\}}{r^{\widetilde{\mu}_\varphi^1(f)}}.$$

**Definition 1.7.** Let $\varphi$ be an increasing unbounded function on $[1,+\infty)$. We define the lower $\varphi$-convergence exponents of the sequence of zeros of a meromorphic function $f$ by

$$\underline{\lambda}_\varphi^0(f)=\liminf_{r\to+\infty}\frac{\varphi(e^{N(r,\frac{1}{f})})}{\log r},\quad \underline{\lambda}_\varphi^1(f)=\liminf_{r\to+\infty}\frac{\varphi(N(r,\frac{1}{f}))}{\log r}.$$

Similarly, the notations $\underline{\overline{\lambda}}_\varphi^0(f)$ and $\underline{\overline{\lambda}}_\varphi^1(f)$ can be used to denote the $\varphi$-convergence exponents of the sequence of distinct zeros of $f$:

$$\underline{\overline{\lambda}}_\varphi^0(f)=\liminf_{r\to+\infty}\frac{\varphi(e^{\overline{N}(r,\frac{1}{f})})}{\log r},\quad \underline{\overline{\lambda}}_\varphi^1(f)=\liminf_{r\to+\infty}\frac{\varphi(\overline{N}(r,\frac{1}{f}))}{\log r}.$$

## 2 Main results

Before we state our main results, it is essential to recall some existing results. The first work that considered the concept of $\varphi$-order to study the growth of entire solutions of equation (1.1) was made by Chyzhykov–Semochko in [9]. They gave the precise estimate of $\widetilde{\rho}_\varphi^1(f)$ when $A_0$ dominates the growth of the other coefficients.

**Theorem 2.1** ([9]). *Let $\varphi\in\Phi$ and $A_0,A_1,\dots,A_{k-1}$ be entire functions such that*

$$\max\left\{\widetilde{\rho}_\varphi^0(A_j):\ \ j=1,\dots,k-1\right\}<\widetilde{\rho}_\varphi^0(A_0)<+\infty.$$

*Then every solution $f\not\equiv 0$ of (1.1) satisfies $\widetilde{\rho}_\varphi^1(f)=\widetilde{\rho}_\varphi^0(A_0)$.*

After that, the second author [5, 6] extended Theorem 2.1 by considering the lower $\varphi$-order and the lower $\varphi$-type. He also obtained similar results when there is more than one dominant coefficient in equation (1.1).

**Theorem 2.2** ([6]). *Let $\varphi\in\Phi$ and $A_0,A_1,\dots,A_{k-1}$ be entire functions. Assume that*

$$\max\left\{\widetilde{\rho}_\varphi^0(A_j):\ \ j=1,\dots,k-1\right\}\le\widetilde{\mu}_\varphi^0(A_0)\le\widetilde{\rho}_\varphi^0(A_0)<+\infty\ \ (\widetilde{\mu}_\varphi^0(A_0)>0),$$
$$\max\left\{\widetilde{\tau}_\varphi^0(A_j):\ \ \widetilde{\rho}_\varphi^0(A_j)=\widetilde{\mu}_\varphi^0(A_0)\right\}<\underline{\widetilde{\tau}}_\varphi^0(A_0)=\underline{\tau}_0\ \ (0<\underline{\tau}_0<+\infty).$$

*Then every solution $f\not\equiv 0$ of (1.1) satisfies*

$$\widetilde{\mu}_\varphi^1(f)=\widetilde{\mu}_\varphi^0(A_0)\le\widetilde{\rho}_\varphi^1(f)=\widetilde{\rho}_\varphi^0(A_0).$$

Very recently, the authors in [17] investigated the growth of solutions of equations (1.1) and (1.2) when the coefficients are meromorphic with $\varphi$-order which improve and generalise under some conditions on the poles of coefficients some results in [5, 6, 9, 13, 17].

**Theorem 2.3** ([17]). *Let $\varphi\in\Phi$ and $A_0,A_1,\dots,A_{k-1}$ be meromorphic functions such that*

$$\lambda_\varphi^0\Big(\frac{1}{A_0}\Big)<\rho_\varphi^0(A_0),$$
$$\max\left\{\rho_\varphi^0(A_j):\ \ j=1,\dots,k-1\right\}\le\rho_\varphi^0(A_0)<+\infty,$$
$$\max\left\{\tau_\varphi^0(A_j):\ \ \rho_\varphi^0(A_j)=\rho_\varphi^0(A_0)>0,\ \ j=1,\dots,k-1\right\}<\tau_\varphi^0(A_0)=\tau_0\ \ (0<\tau_0<+\infty).$$

*Then any non-zero meromorphic solution $f$ whose poles are of uniformly bounded multiplicities of (1.1) satisfies $\rho_\varphi^1(f)=\rho_\varphi^0(A_0)$.*

Thus, the following questions may arise:

**Question 1.** How about the growth of solutions of equation (1.1) with meromorphic coefficients of lower $\varphi$-order?

**Question 2.** Which conditions can be added to extend Theorem 2.2 from entire coefficients to meromorphic coefficients?

**Question 3.** Can we replace the dominant coefficient $A_0$ by an arbitrary coefficient $A_s$ ($s \in \{0, 1, \dots, k-1\}$)?

In this paper, we are going to give answers to the above questions. We also obtain the results for the $\varphi$-convergence exponent and the lower $\varphi$-convergence exponent.

**Theorem 2.4.** *Let $\varphi \in \Phi$ and $A_0, A_1, \dots, A_{k-1}$ be meromorphic functions satisfying*

$$\lambda_\varphi^0\left(\frac{1}{A_0}\right) < \mu_\varphi^0(A_0) = \mu_0,$$
$$\max\left\{\rho_\varphi^0(A_j): \ \ j = 1, \dots, k-1\right\} \le \mu_\varphi^0(A_0) \le \rho_\varphi^0(A_0) < +\infty,$$
$$\max\left\{\tau_\varphi^0(A_j): \ \ \rho_\varphi^0(A_j) = \mu_\varphi^0(A_0), \ \ j = 1, \dots, k-1\right\} < \underline{\tau}_\varphi^0(A_0) = \underline{\tau}_0 \ \ (0 < \underline{\tau}_0 < +\infty).$$

*Then for any non-zero meromorphic solution $f$ of (1.1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} < \varphi^{-1}(\alpha \log r)$ such that $0 < \alpha \le \mu_0$, we have*

$$\underline{\overline{\lambda}}_\varphi^1(f-g) = \mu_\varphi^1(f) = \mu_\varphi^0(A_0) \le \rho_\varphi^0(A_0) = \rho_\varphi^1(f) = \overline{\lambda}_\varphi^1(f-g),$$

*where $g \not\equiv 0$ is a meromorphic function satisfying $\rho_\varphi^1(g) < \mu_\varphi^0(A_0)$.*

**Remark 2.1.** Theorem 2.4 is an improvement and an extension of Theorem 2.2 from entire solution to meromorphic solution. Furthermore, by setting $\varphi(r) = \log_{p+1} r$ ($p \ge 1$) in Theorem 2.4, we obtain Theorem 3 of Hu–Zheng [13] in the case where $p > q = 1$.

**Theorem 2.5.** *Let $\varphi \in \Phi$ and $A_0, A_1, \dots, A_{k-1}$ be meromorphic functions satisfying*

$$\lambda_\varphi^0\left(\frac{1}{A_0}\right) < \mu_\varphi^0(A_0) = \mu_0, \quad \limsup_{r \to +\infty} \frac{\sum_{j=1}^{k-1} m(r, A_j)}{m(r, A_0)} < 1$$

*and*

$$\max\left\{\rho_\varphi^0(A_j): \ \ j = 1, \dots, k-1\right\} \le \mu_\varphi^0(A_0) \le \rho_\varphi^0(A_0) < +\infty.$$

*Then for any non-zero meromorphic solution $f$ of (1.1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} < \varphi^{-1}(\alpha \log r)$ such that $0 < \alpha \le \mu_0$, we have*

$$\underline{\overline{\lambda}}_\varphi^1(f-g) = \mu_\varphi^1(f) = \mu_\varphi^0(A_0) \le \rho_\varphi^0(A_0) = \rho_\varphi^1(f) = \overline{\lambda}_\varphi^1(f-g),$$

*where $g \not\equiv 0$ is a meromorphic function satisfying $\rho_\varphi^1(g) < \mu_\varphi^0(A_0)$.*

**Remark 2.2.** Theorem 2.5 is an improvement and an extension of Theorem 1.13 of Belaïdi [5] from entire solution to meromorphic solution. Furthermore, by setting $\varphi(r) = \log_{p+1} r$ ($p \ge 1$) in Theorem 2.5, we obtain Theorem 4 of Hu–Zheng [13] in the case where $p > q = 1$.

**Theorem 2.6.** *Let $\varphi \in \Phi$ and $A_0, A_1, \dots, A_{k-1}$ be meromorphic functions. Assume there exists one coefficient $A_s(z)$ ($0 \le s \le k-1$) satisfying $\lambda_\varphi^0(\frac{1}{A_s}) < \mu_\varphi^0(A_s)$ and*

$$\max\left\{\rho_\varphi^0(A_j): \ \ j \ne s\right\} \le \mu_\varphi^0(A_s) \le \rho_\varphi^0(A_s) < +\infty,$$
$$\max\left\{\tau_\varphi^0(A_j): \ \ \rho_\varphi^0(A_j) = \mu_\varphi^0(A_s), \ \ j \ne s\right\} < \underline{\tau}_\varphi^0(A_s) = \underline{\tau}_s (0 < \underline{\tau}_s < +\infty).$$

*Then for any transcendental meromorphic solution $f$ of (1.1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} < \varphi^{-1}(\alpha \log r)$ such that $0 < \alpha \le \mu_\varphi^0(A_s)$, we have $\mu_\varphi^1(f) \le \mu_\varphi^0(A_s) \le \mu_\varphi^0(f)$ and $\rho_\varphi^1(f) \le \rho_\varphi^0(A_s) \le \rho_\varphi^0(f)$. Moreover, every non-transcendental meromorphic solution $f$ of (1.1) is a polynomial of degree $\deg(f) \le s-1$.*

**Remark 2.3.** Putting $\varphi(r) = \log_{p+1} r$ $(p \geq 1)$ in Theorem 2.6, we obtain Theorem 5 of Hu–Zheng [13] in the case where $p > q = 1$.

**Remark 2.4.** The condition $\lambda_\varphi^0(\frac{1}{A_0}) < \mu_\varphi^0(A_0)$ in the above theorems can be replaced by

$$N(r, A_0) = o(m(r, A_0)) \text{ as } r \to +\infty$$

or

$$\delta(\infty, A_0) := \liminf_{r\to+\infty} \frac{m(r, A_0)}{T(r, A_0)} > 0.$$

# 3 Preliminary lemmas

**Lemma 3.1** ([8]). *Let $f$ be a meromorphic solution of equation (1.1). Suppose that not all coefficients $A_j$ are constants. Given a real number $\gamma > 1$ and denoting $T(r) = \sum_{j=0}^{k-1} T(r, A_j)$, the inequalities*

$$\log m(r, f) < T(r)\big\{(\log r) \log T(r)\big\}^\gamma \text{ if } s = 0,$$
$$\log m(r, f) < r^{2s+\gamma-1} T(r)\big\{\log T(r)\big\}^\gamma \text{ if } s > 0$$

*take place outside of an exceptional set $E_s$ with $\int_{E_s} t^{s-1}\, dt < +\infty$.*

**Lemma 3.2** ([1, 10]). *Let $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions. If*

(i) *$g(r) \leq h(r)$ outside of an exceptional set $E_1 \subset (0, +\infty)$ of finite linear measure*

*or*

(ii) *$g(r) \leq h(r)$ for all $r \notin E_2 \cup [0, 1]$, where $E_2 \subset (1, +\infty)$ is a set of finite logarithmic measure, then for any $\alpha > 1$, there exists $r_0 = r_0(\alpha) > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

**Lemma 3.3** ([11, 25]). *Let $f$ be a non-constant meromorphic function and $k \in \mathbb{N}$. Then for sufficiently large $r$, we have*

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O(\log r + \log T(r, f))$$

*possibly outside of an exceptional set $E_3 \subset (0, +\infty)$ of finite linear measure. Moreover, if $f$ is of finite order (i.e. $\rho(f) < +\infty$), then*

$$m\Big(r, \frac{f^{(k)}}{f}\Big) = O(\log r), \quad r \to +\infty.$$

**Lemma 3.4** ([9, 16]). *Let $\varphi \in \Phi$ and $f_1$, $f_2$ be two meromorphic functions. Then for $j = 0, 1$, we have*

$$\rho_\varphi^j(f_1 + f_2) \leq \max\big\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\big\},$$
$$\rho_\varphi^j(f_1 f_2) \leq \max\big\{\rho_\varphi^j(f_1), \rho_\varphi^j(f_2)\big\}.$$

*Moreover, if $\rho_\varphi^j(f_1) < \rho_\varphi^j(f_2)$, then $\rho_\varphi^j(f_1 + f_2) = \rho_\varphi^j(f_1 f_2) = \rho_\varphi^j(f_2)$.*

**Lemma 3.5.** *Let $\varphi \in \Phi$ and $f_1, f_2$ be two meromorphic functions. Then for $j = 0, 1$, we have*

$$\mu_\varphi^j(f_1 + f_2) \leq \max\big\{\rho_\varphi^j(f_1), \mu_\varphi^j(f_2)\big\},$$
$$\mu_\varphi^j(f_1\, f_2) \leq \max\big\{\rho_\varphi^j(f_1), \mu_\varphi^j(f_2)\big\}.$$

*Moreover, if $\rho_\varphi^j(f_1) < \mu_\varphi^j(f_2)$, then $\mu_\varphi^j(f_1 + f_2) = \mu_\varphi^j(f_1 f_2) = \mu_\varphi^j(f_2)$.*

*Proof.* We prove the lemma only for $j=0$, the proof for $j=1$ is similar. Without loss of generality, we assume that $\rho_\varphi^0(f_1)<+\infty$ and $\mu_\varphi^0(f_2)<+\infty$. From the definition of the lower $\varphi-$order, there exists a sequence $r_n\to+\infty$ $(n\to+\infty)$ such that

$$\lim_{n\to+\infty}\frac{\varphi(e^{T(r_n,f_2)})}{\log r_n}=\mu_\varphi^0(f_2).$$

Then for any given $\varepsilon>0$, there exists a positive integer $N_1$ such that

$$T(r_n,f_2)\le\log\Big(\varphi^{-1}\big\{(\mu_\varphi^0(f_2)+\varepsilon)\log r_n\big\}\Big)$$

holds for $n>N_1$. From the definition of the $\varphi-$order, for any given $\varepsilon>0$, there exists a positive number $R$ such that

$$T(r,f_1)\le\log\Big(\varphi^{-1}\big\{(\rho_\varphi^0(f_1)+\varepsilon)\log r\big\}\Big)$$

holds for $r\ge R$. Since $r_n\to+\infty$ $(n\to+\infty)$, there exists a positive integer $N_2$ such that $r_n>R$, and thus

$$T(r_n,f_1)\le\log\Big(\varphi^{-1}\big\{(\rho_\varphi^0(f_1)+\varepsilon)\log r_n\big\}\Big)$$

holds for $n>N_2$. Note that

$$T(r,f_1+f_2)\le T(r,f_1)+T(r,f_2)+\log 2$$

and

$$T(r,f_1f_2)\le T(r,f_1)+T(r,f_2).$$

Then for any given $\varepsilon>0$, for $n>\max\{N_1,N_2\}$ we have

$$\begin{aligned}T(r_n,f_1+f_2)&\le T(r_n,f_1)+T(r_n,f_2)+\log 2\\&\le\log\Big(\varphi^{-1}\big\{(\rho_\varphi^0(f_1)+\varepsilon)\log r_n\big\}\Big)+\log\Big(\varphi^{-1}\big\{(\mu_\varphi^0(f_2)+\varepsilon)\log r_n\big\}\Big)+\log 2\\&\le 2\log\Big(\varphi^{-1}\Big\{\big(\max\big\{\rho_\varphi^0(f_1),\mu_\varphi^0(f_2)\big\}+\varepsilon\big)\log r_n\Big\}\Big)+\log 2\end{aligned}\tag{3.1}$$

and

$$T(r_n,f_1f_2)\le T(r_n,f_1)+T(r_n,f_2)\le 2\log\Big(\varphi^{-1}\{(\max\{\rho_\varphi^0(f_1),\mu_\varphi^0(f_2)\}+\varepsilon)\log r_n\}\Big).\tag{3.2}$$

Since $\varepsilon>0$ is arbitrary, from (3.1) and (3.2) we easily obtain

$$\mu_\varphi^0(f_1+f_2)\le\max\big\{\rho_\varphi^0(f_1),\mu_\varphi^0(f_2)\big\}\tag{3.3}$$

and

$$\mu_\varphi^0(f_1f_2)\le\max\big\{\rho_\varphi^0(f_1),\mu_\varphi^0(f_2)\big\}.\tag{3.4}$$

Suppose now that $\mu_\varphi^0(f_2)>\rho_\varphi^0(f_1)$. Considering that

$$T(r,f_2)=T(r,f_1+f_2-f_1)\le T(r,f_1+f_2)+T(r,f_1)+\log 2\tag{3.5}$$

and

$$T(r,f_2)=T\Big(r,\frac{f_1f_2}{f_1}\Big)\le T(r,f_1f_2)+T\Big(r,\frac{1}{f_1}\Big)=T(r,f_1f_2)+T(r,f_1)+O(1),\tag{3.6}$$

by (3.5), (3.6) and the same method as above, we obtain

$$\mu_\varphi^0(f_2)\le\max\big\{\mu_\varphi^0(f_1+f_2),\rho_\varphi^0(f_1)\big\}=\mu_\varphi^0(f_1+f_2),\tag{3.7}$$

$$\mu_\varphi^0(f_2)\le\max\big\{\mu_\varphi^0(f_1f_2),\rho_\varphi^0(f_1)\big\}=\mu_\varphi^0(f_1f_2).\tag{3.8}$$

By using (3.3) and (3.7), we obtain $\mu_\varphi^0(f_1+f_2)=\mu_\varphi^0(f_2)$, and by (3.4) and (3.8), we obtain $\mu_\varphi^0(f_1f_2)=\mu_\varphi^0(f_2)$. □

**Lemma 3.6** ([9,16])**.** *Let $f$ be a meromorphic function and $\varphi \in \Phi$. Then for $j = 0, 1$, we have*

$$\rho^j_\varphi(f') = \rho^j_\varphi(f).$$

**Lemma 3.7.** *Let $\varphi \in \Phi$ and $f$ be a meromorphic function with $0 < \rho^0_\varphi(f) < +\infty$. Then there exists a set $E_4 \subset (1, +\infty)$ of infinite logarithmic measure such that*

$$\tau^0_\varphi(f) = \lim_{\substack{r\to+\infty\\ r\in E_4}} \frac{\exp\{\varphi(e^{T(r,f)})\}}{r^{\rho^0_\varphi(f)}}.$$

*Therefore, for any given $\varepsilon > 0$ and sufficiently large $r \in E_4$, we have*

$$T(r, f) > \log \varphi^{-1}\Big(\log\big[(\tau^0_\varphi(f) - \varepsilon) r^{\rho^0_\varphi(f)}\big]\Big).$$

*Proof.* The definition of $\tau^0_\varphi(f)$ implies that there exists a sequence $\{r_n, n \geq 1\}$ tending to $+\infty$ satisfying $(1 + \frac{1}{n})r_n < r_{n+1}$ and

$$\tau^0_\varphi(f) = \lim_{r_n\to+\infty} \frac{\exp\{\varphi(e^{T(r_n,f)})\}}{r_n^{\rho^0_\varphi(f)}}.$$

Then for any given $\varepsilon > 0$, there exists an integer number $n_1$ such that for all $n \geq n_1$ and $r \in [r_n, (1 + \frac{1}{n})r_n]$, we have

$$\frac{\exp\{\varphi(e^{T(r_n,f)})\}}{r_n^{\rho^0_\varphi(f)}} \Big(\frac{1}{1+\frac{1}{n}}\Big)^{\rho^0_\varphi(f)} \leq \frac{\exp\{\varphi(e^{T(r,f)})\}}{r^{\rho^0_\varphi(f)}}.$$

Set

$$E_4 = \bigcup_{n=n_1}^{+\infty} \Big[r_n, \Big(1 + \frac{1}{n}\Big)r_n\Big].$$

By the fact that

$$\Big(\frac{1}{1+\frac{1}{n}}\Big)^{\rho^0_\varphi(f)} \to 1 \text{ as } n \to +\infty,$$

we get

$$\lim_{\substack{r\to+\infty\\ r\in E_4}} \frac{\exp\{\varphi(e^{T(r,f)})\}}{r^{\rho^0_\varphi(f)}} \geq \lim_{r_n\to+\infty} \frac{\exp\{\varphi(e^{T(r_n,f)})\}}{r_n^{\rho^0_\varphi(f)}} = \tau^0_\varphi(f)$$

and the logarithmic measure of $E_4$ satisfies

$$lm(E_4) = \int_{E_4} \frac{dr}{r} = \sum_{n=n_1}^{+\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum_{n=n_1}^{+\infty} \log\Big(1 + \frac{1}{n}\Big) = +\infty.$$

It is obvious that

$$\lim_{\substack{r\to+\infty\\ r\in E_4}} \frac{\exp\{\varphi(e^{T(r,f)})\}}{r^{\rho^0_\varphi(f)}} \leq \limsup_{r\to+\infty} \frac{\exp\{\varphi(e^{T(r,f)})\}}{r^{\rho^0_\varphi(f)}} = \tau^0_\varphi(f).$$

Therefore,

$$\tau^0_\varphi(f) = \lim_{\substack{r\to+\infty\\ r\in E_4}} \frac{\exp\{\varphi(e^{T(r,f)})\}}{r^{\rho^0_\varphi(f)}}$$

and for any given $\varepsilon > 0$ and sufficiently large $r \in E_4$, we have

$$T(r, f) > \log \varphi^{-1}\Big(\log\big[(\tau^0_\varphi(f) - \varepsilon) r^{\rho^0_\varphi(f)}\big]\Big).$$

□

Using analogous arguments as in [5, Lemma 2.4, p. 31] and the above proof, we can easily obtain the following lemma.

**Lemma 3.8.** *Let $\varphi \in \Phi$ and $f$ be a meromorphic function with $\mu_{\varphi}^{0}(f) < +\infty$. Then there exists a set $E_5 \subset (1, +\infty)$ of infinite logarithmic measure such that*

$$\mu_{\varphi}^{0}(f) = \lim_{\substack{r \to +\infty \\ r \in E_5}} \frac{\varphi(e^{T(r,f)})}{\log r}.$$

*Therefore, for any given $\varepsilon > 0$ and sufficiently large $r \in E_5$, we have*

$$T(r,f) < \log \varphi^{-1}\big((\mu_{\varphi}^{0}(f) + \varepsilon) \log r\big).$$

**Lemma 3.9** ([17])**.** *Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be meromorphic functions. If $f$ is a meromorphic solution of equation (1.2) such that*

$$\max\big\{\rho_{\varphi}^{1}(F), \rho_{\varphi}^{1}(A_j): \ \ j = 0, 1, \ldots, k-1\big\} < \rho_{\varphi}^{1}(f) < +\infty,$$

*then we have $\overline{\lambda}_{\varphi}^{1}(f) = \lambda_{\varphi}^{1}(f) = \rho_{\varphi}^{1}(f)$.*

**Lemma 3.10.** *Let $\varphi \in \Phi$ and let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be meromorphic functions. If $f$ is a meromorphic solution of equation (1.2) such that*

$$\max\big\{\rho_{\varphi}^{1}(F), \rho_{\varphi}^{1}(A_j): \ \ j = 0, 1, \ldots, k-1\big\} < \mu_{\varphi}^{1}(f),$$

*then we have $\underline{\overline{\lambda}}_{\varphi}^{1}(f) = \underline{\lambda}_{\varphi}^{1}(f) = \mu_{\varphi}^{1}(f)$.*

*Proof.* Equation (1.2) can be written as

$$\frac{1}{f} = \frac{1}{F}\Big(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\Big). \tag{3.9}$$

If $f$ has a zero at $z_0$ of order $l > k$ and if the coefficients $A_0, \ldots, A_{k-1}$ are all analytic at $z_0$, then $F$ should have a zero at $z_0$ of order at least $l - k$. Hence

$$N\Big(r, \frac{1}{f}\Big) \le k\overline{N}\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{F}\Big) + \sum_{j=0}^{k-1} N(r, A_j). \tag{3.10}$$

By Lemma 3.3 and (3.9), we find that

$$m\Big(r, \frac{1}{f}\Big) \le m\Big(r, \frac{1}{F}\Big) + \sum_{j=0}^{k-1} m(r, A_j) + O\big(\log r + \log T(r,f)\big) \tag{3.11}$$

holds for all $|z| = r \notin E_3$, where $E_3$ is a set of finite linear measure. It follows from (3.10) and (3.11) that

$$\begin{aligned} T(r,f) = T\Big(r, \frac{1}{f}\Big) + O(1) = m\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{f}\Big) + O(1) \\ \le k\overline{N}\Big(r, \frac{1}{f}\Big) + T(r,F) + \sum_{j=0}^{k-1} T(r, A_j) + O\big(\log r + \log T(r,f)\big). \end{aligned} \tag{3.12}$$

We denote $\rho = \max\{\rho_{\varphi}^{1}(F), \rho_{\varphi}^{1}(A_j): \ \ j = 0, 1, \ldots, k-1\} < \mu_{\varphi}^{1}(f) = \mu$. Then for any given $\varepsilon$ $(0 < 2\varepsilon < \mu - \rho)$ and sufficiently large $r$, we get

$$T(r,f) \ge \varphi^{-1}((\mu - \varepsilon) \log r). \tag{3.13}$$

We have

$$\max_{j=0,1,\dots,k-1}\big\{T(r,F),T(r,A_j)\big\}\le\varphi^{-1}((\rho+\varepsilon)\log r),\tag{3.14}$$

$$O\big(\log r+\log T(r,f)\big)=o(T(r,f)).\tag{3.15}$$

Since $\varepsilon$ satisfies $0<2\varepsilon<\mu-\rho$, by (3.13), (3.14) and Proposition 1.1, we obtain

$$\begin{aligned}\max_{j=0,1,\dots,k-1}&\Big\{\frac{T(r,F)}{T(r,f)},\frac{T(r,A_j)}{T(r,f)}\Big\}\\ &\le\frac{\exp\{\log\varphi^{-1}(((\rho+\varepsilon))\log r)\}}{\exp\{\log\varphi^{-1}((\mu-\varepsilon)\log r)\}}=\exp\big\{\log\varphi^{-1}((\rho+\varepsilon)\log r)-\log\varphi^{-1}((\mu-\varepsilon)\log r)\big\}\\ &=\exp\Big\{\Big(1-\frac{\log\varphi^{-1}((\mu-\varepsilon)\log r)}{\log\varphi^{-1}((\rho+\varepsilon)\log r)}\Big)\log\varphi^{-1}((\rho+\varepsilon)\log r)\Big\}\to0\ \text{ as }\ r\to+\infty.\end{aligned}\tag{3.16}$$

Then, by substituting (3.15) and (3.16) into (3.12), it follows that

$$(1-o(1))T(r,f)\le k\overline{N}\Big(r,\frac{1}{f}\Big),\quad r\notin E_3,\quad r\to+\infty.$$

By Lemma 3.2, the monotonicity of $\varphi$ and (1.5), we get $\mu_\varphi^1(f)\le\overline{\lambda}_\varphi^1(f)$. Since $\mu_\varphi^1(f)\ge\lambda_\varphi^1(f)\ge\overline{\lambda}_\varphi^1(f)$, we deduce that $\overline{\lambda}_\varphi^1(f)=\lambda_\varphi^1(f)=\mu_\varphi^1(f)$. □

**Lemma 3.11.** *Let $\varphi\in\Phi$ and let $A_0,\dots,A_{k-1}$ be meromorphic functions such that*

$$\max\{\rho_\varphi^0(A_j):j\ne s\}\le\mu_\varphi^0(A_s)<+\infty.$$

*If $f\not\equiv0$ is a solution of (1.1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)}<\varphi^{-1}(\alpha\log r)$, where $0<\alpha\le\mu_\varphi^0(A_s)$, then we have $\mu_\varphi^1(f)\le\mu_\varphi^0(A_s)$.*

*Proof.* From equation (1.1), we see that the poles of $f$ can only occur at the poles of the coefficients $A_0,A_1,\dots,A_{k-1}$. Since $\frac{N(r,f)}{\overline{N}(r,f)}<\varphi^{-1}(\alpha\log r)$ $(\alpha\le\mu_\varphi^0(A_s))$, we get

$$N(r,f)\le\varphi^{-1}(\alpha\log r)\overline{N}(r,f)\le\varphi^{-1}(\alpha\log r)\sum_{j=0}^{k-1}\overline{N}(r,A_j)\le\varphi^{-1}(\alpha\log r)\sum_{j=0}^{k-1}T(r,A_j),$$

and therefore

$$T(r,f)\le m(r,f)+\varphi^{-1}(\alpha\log r)\sum_{j=0}^{k-1}T(r,A_j).\tag{3.17}$$

Since $\max\{\rho_\varphi^0(A_j):\ j\ne s\}\le\mu_\varphi^0(A_s)<+\infty$, for any given $\varepsilon>0$ and sufficiently large $r$, we have

$$T(r,A_j)\le\log\varphi^{-1}((\mu_\varphi^0(A_s)+\varepsilon)\log r),\quad j\ne s.\tag{3.18}$$

By applying Lemma 3.8 to the coefficient $A_s$ and for the above $\varepsilon$, we have

$$T(r,A_s)<\log\varphi^{-1}\big((\mu_\varphi^0(A_s)+\varepsilon)\log r\big),\quad r\in E_5,\quad r\to+\infty,\tag{3.19}$$

where $E_5$ is a set of infinite logarithmic measure. By Lemma 3.1, (3.18) and (3.19), there exists a set $E_0$ of finite logarithmic measure such that for sufficiently large $r\in E_5\setminus E_0$, we have

$$m(r,f)\le\exp\Big\{\Big(\sum_{j=0}^{k-1}T(r,A_j)\Big)\Big(\log r\ \log\Big(\sum_{j=0}^{k-1}T(r,A_j)\Big)\Big)^{\gamma}\Big\}\le\varphi^{-1}\big((\mu_\varphi^0(A_s)+2\varepsilon)\log r\big).\tag{3.20}$$

It follows from (3.17)–(3.20) that

$$\begin{aligned}T(r,f)&\le\varphi^{-1}\big((\mu_\varphi^0(A_s)+2\varepsilon)\log r\big)+k\varphi^{-1}(\alpha\log r)\log\varphi^{-1}\big((\mu_\varphi^0(A_s)+\varepsilon)\log r\big)\\ &\le\varphi^{-1}\big((\mu_\varphi^0(A_s)+3\varepsilon)\log r\big),\quad r\in E_5\setminus E_0,\quad r\to+\infty.\end{aligned}$$

Thus, by Lemma 3.2, arbitrariness of $\varepsilon>0$ and the monotonicity of $\varphi$, we obtain $\mu_\varphi^1(f)\le\mu_\varphi^0(A_s)$. □

**Lemma 3.12.** *Let $\varphi \in \Phi$ and $A_0, A_1, \ldots, A_{k-1}$ be meromorphic functions satisfying*

$$\lambda_\varphi^0\Big(\frac{1}{A_0}\Big) < \mu_\varphi^0(A_0) = \mu_0,$$
$$\max\big\{\rho_\varphi^0(A_j) : \ \ j = 1, \ldots, k-1\big\} \le \mu_\varphi^0(A_0) < +\infty,$$
$$\max\big\{\tau_\varphi^0(A_j) : \ \ \rho_\varphi^0(A_j) = \mu_\varphi^0(A_0), \ \ j = 1, \ldots, k-1\big\} < \underline{\tau}_\varphi^0(A_0) = \underline{\tau}_0 \ \ (0 < \underline{\tau}_0 < +\infty).$$

*Then, for any non-zero meromorphic solution $f$ of (1.1) we have $\mu_\varphi^1(f) \ge \mu_\varphi^0(A_0)$.*

*Proof.* Assume that $f$ is a non-zero meromorphic solution of equation (1.1). By (1.1), we have

$$-A_0 = \frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_1\frac{f'}{f}.$$

Then, by Lemma 3.3, we find that

$$m(r, A_0) \le \sum_{j=1}^{k-1} m(r, A_j) + \sum_{j=1}^{k} m\Big(r, \frac{f^{(j)}}{f}\Big) + O(1) \le \sum_{j=1}^{k-1} m(r, A_j) + O(\log r + \log T(r, f)) \qquad (3.21)$$

holds possibly outside of an exceptional set $E_3 \subset (0, +\infty)$ of finite linear measure. Since $\lambda_\varphi^0(\frac{1}{A_0}) < \mu_\varphi^0(A_0)$, we get

$$N(r, A_0) = o(T(r, A_0)), \ \ r \to +\infty. \qquad (3.22)$$

Since $T(r, A_0) = m(r, A_0) + N(r, A_0)$, then by (3.21) and (3.22), we obtain

$$(1 - o(1))T(r, A_0) \le \sum_{j=1}^{k-1} m(r, A_j) + O(\log r + \log T(r, f)), \ \ r \notin E_3, \ \ r \to +\infty. \qquad (3.23)$$

Set

$$a = \max\big\{\rho_\varphi^0(A_j) : \ \ \rho_\varphi^0(A_j) < \mu_\varphi^0(A_0) = \mu_0, \ \ j = 1, \ldots, k-1\big\}.$$

If $\rho_\varphi^0(A_j) < \mu_\varphi^0(A_0) = \mu_0$, then for any given $\varepsilon$ $(0 < 2\varepsilon < \mu_0 - a)$ and sufficiently large $r$, we have

$$m(r, A_j) \le T(r, A_j) \le \log \varphi^{-1}(\log r^{a+\varepsilon}) < \log \varphi^{-1}(\log r^{\mu_0 - \varepsilon}), \ \ j \ne 0. \qquad (3.24)$$

Set

$$\tau = \max\big\{\tau_\varphi^0(A_j) : \ \ \rho_\varphi^0(A_j) = \mu_\varphi^0(A_0) = \mu_0, \ \ j = 1, \ldots, k-1\big\}.$$

Then $\tau < \underline{\tau}_0 = \underline{\tau}_\varphi^0(A_0)$. If $\rho_\varphi^0(A_j) = \mu_\varphi^0(A_0) = \mu_0$, $\tau_\varphi^0(A_j) \le \tau < \underline{\tau}_0$, then for any given $\varepsilon$ $(0 < 2\varepsilon < \underline{\tau}_0 - \tau)$ and sufficiently large $r$, we have

$$m(r, A_j) \le T(r, A_j) \le \log \varphi^{-1}\Big(\log\big[(\tau + \varepsilon) r^{\mu_0}\big]\Big), \ \ j \ne 0. \qquad (3.25)$$

The definition of the lower $\varphi$-type $\underline{\tau}_\varphi^0(A_0) = \underline{\tau}_0$ implies that for any given $\varepsilon > 0$ and sufficiently large $r$, we have

$$T(r, A_0) > \log \varphi^{-1}\Big(\log\big[(\underline{\tau}_0 - \varepsilon) r^{\mu_0}\big]\Big). \qquad (3.26)$$

Substituting (3.24)–(3.26) into (3.23), for any given $\varepsilon$ $(0 < 2\varepsilon < \min\{\mu_0 - a; \underline{\tau}_0 - \tau\})$ we obtain

$$(1 - o(1)) \log \varphi^{-1}\Big(\log\big[(\underline{\tau}_0 - \varepsilon) r^{\mu_0}\big]\Big) \le O(\log r + \log T(r, f)), \ \ r \notin E_3, \ \ r \to +\infty. \qquad (3.27)$$

From Lemma 3.2, (3.27), the monotonicity of $\varphi^{-1}$ and (1.5), we deduce that $\mu_0 = \mu_\varphi^0(A_0) \le \mu_\varphi^1(f)$. $\square$

**Lemma 3.13** ([17])**.** *Let $f$ be a meromorphic function. If $\rho_\varphi^0(f) < +\infty$, then $\rho_\varphi^1(f) = 0$.*

**Lemma 3.14.** *Let $f$ be a rational function, then $\rho_\varphi^0(f) = 0$.*

*Proof.* Since $f$ is a rational function, we have $T(r, f) = O(\log r)$. By Karamata's theorem (see [23]), it follows that $\varphi(e^t) = t^{o(1)}$ as $t \to +\infty$. Hence

$$\rho_\varphi^0(f) := \limsup_{r\to+\infty} \frac{\varphi(e^{T(r,f)})}{\log r} = \limsup_{r\to+\infty} \frac{(O(\log r))^{o(1)}}{\log r} = 0. \qquad \square$$

## 4 Proofs of the main results

*Proof of Theorem 2.4.* First, we prove that $\rho_\varphi^1(f) = \rho_\varphi^0(A_0)$ and $\mu_\varphi^1(f) = \mu_\varphi^0(A_0)$. Since

$$\max\left\{\rho_\varphi^0(A_j): \quad j \neq 0\right\} \leq \mu_\varphi^0(A_0) \leq \rho_\varphi^0(A_0) < +\infty,$$

by Lemma 3.1 and (3.17), we have that

$$T(r,f) \leq \varphi^{-1}\big((\rho_\varphi^0(A_0) + 3\varepsilon)\log r\big) \tag{4.1}$$

holds for any given $\varepsilon > 0$ and $r \notin E_0$, $r \to +\infty$, where $E_0$ is a set of finite logarithmic measure. From Lemma 3.2, the monotonicity of $\varphi$ and (4.1), we obtain that $\rho_\varphi^1(f) \leq \rho_\varphi^0(A_0)$. Set

$$b = \max\left\{\rho_\varphi^0(A_j): \quad \rho_\varphi^0(A_j) < \rho_\varphi^0(A_0) = \rho_0, \quad j = 1, \ldots, k-1\right\}.$$

If $\rho_\varphi^0(A_j) < \mu_\varphi^0(A_0) \leq \rho_\varphi^0(A_0) = \rho_0$ or $\rho_\varphi^0(A_j) \leq \mu_\varphi^0(A_0) < \rho_\varphi^0(A_0) = \rho_0$, then for any given $\varepsilon$ $(0 < 2\varepsilon < \rho_0 - b)$ and sufficiently large $r$, we have

$$T(r, A_j) \leq \log \varphi^{-1}(\log r^{b+\varepsilon}) < \log \varphi^{-1}(\log r^{\rho_0 - \varepsilon}), \quad j \neq 0. \tag{4.2}$$

Set

$$\tau = \max\left\{\tau_\varphi^0(A_j): \quad \rho_\varphi^0(A_j) = \mu_\varphi^0(A_0) = \mu_0, \quad j = 1, \ldots, k-1\right\}.$$

Then $\tau < \underline{\tau}_0 = \underline{\tau}_\varphi^0(A_0)$. If $\rho_\varphi^0(A_j) = \mu_\varphi^0(A_0) = \rho_\varphi^0(A_0) = \rho_0$, $\tau < \underline{\tau}_0 \leq \tau_0 = \tau_\varphi^0(A_0)$, then for any given $\varepsilon$ $(0 < 2\varepsilon < \tau_0 - \tau)$ and sufficiently large $r$, we have

$$T(r, A_j) \leq \log \varphi^{-1}\big(\log\left[(\tau + \varepsilon) r^{\rho_0}\right]\big), \quad j \neq 0. \tag{4.3}$$

Applying Lemma 3.7 to the coefficient $A_0$, we find that

$$T(r, A_0) > \log \varphi^{-1}\big(\log\left[(\tau_0 - \varepsilon) r^{\rho_0}\right]\big) \tag{4.4}$$

holds for any given $\varepsilon > 0$ and $r \in E_4$, $r \to +\infty$, where $E_4$ is a set of infinite logarithmic measure. Substituting (4.2)–(4.4) into (3.23), it follows for any given $\varepsilon (0 < 2\varepsilon < \min\{\rho_0 - b; \tau_0 - \tau\})$, $r \in E_4 \setminus E_3$ $(r \to +\infty)$ that

$$(1 - o(1)) \log \varphi^{-1}\big(\log\left[(\tau_0 - \varepsilon) r^{\rho_0}\right]\big) \leq O(\log r + \log T(r,f)), \tag{4.5}$$

where $E_3$ is a set of finite linear measure. By Lemma 3.2, the monotonicity of $\varphi^{-1}$ and (1.5), from (4.5) we obtain $\rho_0 = \rho_\varphi^0(A_0) \leq \rho_\varphi^1(f)$. Therefore, $\rho_\varphi^1(f) = \rho_\varphi^0(A_0)$. On the other hand, by Lemma 3.11 and Lemma 3.12, we deduce that $\mu_\varphi^1(f) = \mu_\varphi^0(A_0)$.

Secondly, we prove that $\underline{\overline{\lambda}}_\varphi^1(f-g) = \mu_\varphi^1(f)$ and $\overline{\lambda}_\varphi^1(f-g) = \rho_\varphi^1(f)$. Let $h = f - g$. Since $\rho_\varphi^1(g) < \mu_\varphi^0(A_0) = \mu_\varphi^1(f) \leq \rho_\varphi^1(f)$, it follows from Lemma 3.4 and Lemma 3.5 that $\rho_\varphi^1(h) = \rho_\varphi^1(f) = \rho_\varphi^0(A_0)$ and $\mu_\varphi^1(h) = \mu_\varphi^1(f) = \mu_\varphi^0(A_0)$. By substituting $f = g + h$, $f' = g' + h', \ldots, f^{(k)} = g^{(k)} + h^{(k)}$ into (1.1), we obtain

$$h^{(k)} + A_{k-1}(z) h^{(k-1)} + \cdots + A_0(z) h = -\big(g^{(k)} + A_{k-1}(z) g^{(k-1)} + \cdots + A_0(z) g\big). \tag{4.6}$$

If $g^{(k)} + A_{k-1}(z) g^{(k-1)} + \cdots + A_0(z) g = G \equiv 0$, then by Lemma 3.12, we have $\mu_\varphi^1(g) \geq \mu_\varphi^0(A_0)$ which contradicts the assumption $\rho_\varphi^1(g) < \mu_\varphi^0(A_0)$. Hence $G \not\equiv 0$. By Lemma 3.6 and Lemma 3.13, we get

$$\begin{aligned}\rho_\varphi^1(G) &\leq \max\left\{\rho_\varphi^1(g), \ \rho_\varphi^1(A_j) \ \ (j = 0, 1, \ldots, k-1)\right\} \\ &= \rho_\varphi^1(g) < \mu_\varphi^0(A_0) = \mu_\varphi^1(f) = \mu_\varphi^1(h) \leq \rho_\varphi^1(h) = \rho_\varphi^1(f) = \rho_\varphi^0(A_0).\end{aligned}$$

Then it follows from Lemma 3.10, Lemma 3.9 and (4.6) that $\overline{\lambda}_\varphi^1(h) = \lambda_\varphi^1(h) = \rho_\varphi^1(h) = \rho_\varphi^1(f)$ and $\underline{\overline{\lambda}}_\varphi^1(h) = \underline{\lambda}_\varphi^1(h) = \mu_\varphi^1(h) = \mu_\varphi^1(f)$. Therefore, $\underline{\overline{\lambda}}_\varphi^1(f-g) = \mu_\varphi^1(f)$ and $\overline{\lambda}_\varphi^1(f-g) = \rho_\varphi^1(f)$ which completes the proof of Theorem 2.4. $\square$

*Proof of Theorem 2.5.* Since

$$\limsup_{r\to+\infty}\frac{\sum\limits_{j=1}^{k-1} m(r,A_j)}{m(r,A_0)}<1,$$

there exists $\eta$ $(0<\eta<1)$ such that

$$\sum_{j=1}^{k-1} m(r,A_j)\le \eta\, m(r,A_0),\quad r\to+\infty. \tag{4.7}$$

By (4.7) and (3.23), for $r\notin E_3$, $r\to+\infty$ we have

$$(1-o(1)-\eta)T(r,A_0)\le O(\log r+\log T(r,f)), \tag{4.8}$$

where $E_3$ is a set of finite linear measure. By Lemma 3.2, (4.8), the monotonicity of $\varphi$ and (1.5) we obtain $\rho^1_\varphi(f)\ge\rho^0_\varphi(A_0)$ and $\mu^1_\varphi(f)\ge\mu^0_\varphi(A_0)$. From the first part of the proof of Theorem 2.4, we have $\rho^1_\varphi(f)\le\rho^0_\varphi(A_0)$ and by applying Lemma 3.11, we have $\mu^1_\varphi(f)\le\mu^0_\varphi(A_0)$. Then we deduce $\mu^1_\varphi(f)=\mu^0_\varphi(A_0)\le\rho^1_\varphi(f)=\rho^0_\varphi(A_0)$. The second part of the proof of Theorem 2.4 completes the proof of Theorem 2.5. □

*Proof of Theorem 2.6.* First, we suppose that $f$ is a rational function. If the function $f$ is either rational with a pole at $z_0$ of multiplicity $k\ge1$ or polynomial of degree $\deg(f)\ge s$, then $f^{(s)}(z)\not\equiv0$. If

$$\max\left\{\rho^0_\varphi(A_j):\quad j\neq s\right\}<\mu^0_\varphi(A_s)=\mu_s,$$

then by Lemma 3.5, Lemma 3.6, Lemma 3.14 and (1.1), we get

$$0=\mu^0_\varphi(0)=\mu^0_\varphi(f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_0(z)f)=\mu^0_\varphi(A_s)=\mu_s>0,$$

which is a contradiction. Set

$$c=\max\left\{\rho^0_\varphi(A_j):\quad \rho^0_\varphi(A_j)<\mu^0_\varphi(A_s)=\mu_s,\quad j\neq s\right\}.$$

If $\rho^0_\varphi(A_j)<\mu^0_\varphi(A_s)=\mu_s$, then for any given $\varepsilon(0<2\varepsilon<\mu_s-c)$ and sufficiently large $r$, we have

$$m(r,A_j)\le T(r,A_j)\le\log\varphi^{-1}(\log r^{c+\varepsilon}),\quad j\neq s. \tag{4.9}$$

Set $\tau=\max\{\tau^0_\varphi(A_j):\ \rho^0_\varphi(A_j)=\mu^0_\varphi(A_s),\ j\neq s\}$. Then there exist two constants $\beta_1$ and $\beta_2$ such that $\tau<\beta_1<\beta_2<\underline{\tau}_s=\underline{\tau}^0_\varphi(A_s)$. If $\rho^0_\varphi(A_j)=\mu^0_\varphi(A_s)$, $\tau^0_\varphi(A_j)\le\tau<\underline{\tau}_s$, then for sufficiently large $r$, we have

$$m(r,A_j)\le T(r,A_j)\le\log\varphi^{-1}\big(\log\left[\beta_1 r^{\mu_s}\right]\big),\quad j\neq s. \tag{4.10}$$

Since $\lambda^0_\varphi(\frac{1}{A_s})<\mu^0_\varphi(A_s)$, we have

$$N(r,A_s)=o(T(r,A_s)),\quad r\to+\infty. \tag{4.11}$$

By the definition of the lower $\varphi$-type $\underline{\tau}^0_\varphi(A_s)=\underline{\tau}_s$, we have

$$T(r,A_s)>\log\varphi^{-1}\big(\log\left[\beta_2 r^{\mu_s}\right]\big),\quad r\to+\infty. \tag{4.12}$$

Equation (1.1) can be written as

$$-A_s=\frac{f}{f^{(s)}}\Big(\frac{f^{(k)}}{f}+A_{k-1}\frac{f^{(k-1)}}{f}+\cdots+A_{s+1}\frac{f^{(s+1)}}{f}+A_{s-1}\frac{f^{(s-1)}}{f}+\cdots+A_0\Big). \tag{4.13}$$

By Lemma 3.3 and (4.13), for sufficiently large $r$, we get

$$T(r,A_s)=N(r,A_s)+m(r,A_s)\le N(r,A_s)+\sum_{j\neq s} m(r,A_j)+O(\log r). \tag{4.14}$$

Substituting (4.9)–(4.12) into (4.14) and using (1.3), we obtain

$$(1-o(1))\log\varphi^{-1}\big(\log[\beta_2 r^{\mu_s}]\big) \\ \le O\Big(\log\varphi^{-1}\big(\log\big[\beta_1 r^{\mu_s}\big]\big)\Big) + O(\log r) = O\Big(\log\varphi^{-1}\big(\log\big[\beta_1 r^{\mu_s}\big]\big)\Big), \quad r\to+\infty.$$

By Lemma 3.2, the monotonicity of $\varphi$ and (1.5), we obtain $\beta_2 \le \beta_1$ which is a contradiction. Hence we deduce that $f$ should be a polynomial of degree $\deg(f) \le s-1$ if $f$ is a non-transcendental meromorphic solution of (1.1).

Secondly, we suppose that $f$ is a transcendental meromorphic solution of (1.1). From Lemma 3.3, (4.13) and the fact that

$$m\Big(r, \frac{f}{f^{(s)}}\Big) \le T(r,f) + T\Big(r, \frac{1}{f^{(s)}}\Big) = T(r,f) + T(r,f^{(s)}) + O(1) = O(T(r,f)),$$

it follows that

$$T(r,A_s) \le N(r,A_s) + \sum_{j\neq s} m(r,A_j) + O(T(r,f)) \tag{4.15}$$

holds for sufficiently large $r$, $r\notin E_3$, where $E_3$ is a set of finite linear measure. By substituting (4.9)–(4.12) into (4.15), we obtain

$$(1-o(1))\log\varphi^{-1}\big(\log(\beta_2 r^{\mu_s})\big) \le O(T(r,f)), \quad r\notin E_3, \quad r\to+\infty. \tag{4.16}$$

By Lemma 3.2, the monotonicity of $\varphi$, (1.5) and (4.16), we can deduce that $\mu^0_\varphi(A_s) \le \mu^0_\varphi(f)$ and also, by using the same arguments as in the proof of the first part of Theorem 2.4 and (4.15), we can obtain $\rho^0_\varphi(A_s) \le \rho^0_\varphi(f)$. Since $\max\{\rho^0_\varphi(A_j):\ j\neq s\} \le \mu^0_\varphi(A_s) \le \rho^0_\varphi(A_s) < +\infty$, by Lemma 3.1 and (3.17), for any given $\varepsilon>0$, we have

$$T(r,f) \le \varphi^{-1}\big((\rho^0_\varphi(A_s)+3\varepsilon)\log r\big), \quad r\to+\infty, \quad r\notin E_0, \tag{4.17}$$

where $E_0$ is a set of finite logarithmic measure. From Lemma 3.2, the monotonicity of $\varphi$ and (4.17), we obtain that $\rho^1_\varphi(f) \le \rho^0_\varphi(A_s)$. By Lemma 3.11, we have $\mu^1_\varphi(f) \le \mu^0_\varphi(A_s)$. Therefore, $\rho^1_\varphi(f) \le \rho^0_\varphi(A_s) \le \rho^0_\varphi(f)$ and $\mu^1_\varphi(f) \le \mu^0_\varphi(A_s) \le \mu^0_\varphi(f)$ which completes the proof of Theorem 2.6. □

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