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GENERALIZED DIRECTIONAL DERIVATIVES AND GRADIENT OF MULTIVARIATE FUNCTION ON TIME SCALES

Abstract. In this paper, we introduce a notion of jump operators along the assigned direction $\vec{\omega}$ and the corresponding opposite direction $-\vec{\omega}$ on an n-dimensional time scale Λ^n, due to which the total increment of the multivariate function can be represented accurately. Based on it, we introduce the new notions of directional derivatives and gradient of multivariate function on time scales and demonstrate their nice geometric significance. Moreover, some basic results are established which improve and perfect the previous literature.

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Key words and phrases. Time scale, jump operators along the assigned direction, directional derivative, gradient, partial derivative.

რეზიუმე. სტატიაში შემოტანილია ნახტომის ოპერატორების ცნება მოცემული მიმართულებით $\vec{\omega}$ და შესაბამისი საპირისპირო $-\vec{\omega}$ მიმართულებით n-განზომილებიანი Λ^n დროის მასშტაბით, რომლის საშუალებითაც შესაძლებელია მულტივარიაციული ფუნქციის მთლიანი ზრდის ზუსტად წარმოდგენა. ამის საფუძველზე შემოტანილია მიმართულებითი წარმოებულებისა და მრავალგანზომილებიანი ფუნქციის გრადიენტის ახალი ცნებები დროის მასშტაბებზე და ნაჩვენებია მათი ლამაზი გეომეტრიული მნიშვნელობა. გარდა ამისა, დადგენილია რამდენიმე ძირითადი შედეგი, რომელიც აუმჯობესებს და სრულყოფს წინა ლიტერატურას.

1 Introduction and preliminaries

Time scale theory was proposed by Hilger (see [5]) and used to unify the discrete and continuous analysis in pure and applied mathematics (see [1,4,6,7,9–16]). As a fundamental tool of mathematical analysis and applications, the partial derivatives on time scales were proposed and studied by Bohner and Guseinov (see [3]). In 2009, Aktan and Sarıkaya et al. introduced the directional derivatives and investigated a differential calculus for multivariable functions on n-dimensional time scales (see [2,8]). In [2,8], the authors considered the total increments of two types $f(\sigma_1(t_1), \dots, \sigma_n(t_n)) - f(t_1, \dots, t_n)$ and $f(t_1, \dots, t_n) - f(\rho_1(t_1), \dots, \rho_n(t_n))$ for the multivariate function $f : \Lambda^n \to \mathbb{R}$ and introduced the directional derivatives on n-dimensional time scales $\mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$ with the time scale type $\mathbb{T}_i = \{t_i : t_i^0 + \xi\omega_i, \xi \in \mathbb{T}\}$, $i = 1, 2, \dots, n$. In addition, they established the relations between the partial derivatives and their directional derivatives on time scales. However, it is intractable to study the arbitrary directional derivatives on any arbitrary n-dimensional time scale $\mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, where $\mathbb{T}_i$ is an arbitrary time scale for each $i \in \{1, 2, \dots, n\}$, since the total increment of the multivariate function along the assigned direction is difficult to describe on any closed subsets of an arbitrary time scale space Λ^n. In this paper, the notions of jump operators along the assigned direction $\vec{\omega}$ and the corresponding opposite direction $-\vec{\omega}$ on the n-dimensional time scale Λ^n are introduced, through which the total increment of the multivariate function can be represented precisely. Then new notions of directional derivatives and gradient of multivariate function on time scales are considered and their nice geometric significance and basic results are demonstrated.

Now, we introduce some basic results which will be utilized in our discussion. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real line $\mathbb{R}$, on which the intervals are denoted by

$$[a,b]_{\mathbb{T}} := \big\{t \in \mathbb{T} : \ a \leqslant t \leqslant b\big\}, \quad [a,b)_{\mathbb{T}} := \big\{t \in \mathbb{T} : \ a \leqslant t < b\big\},$$
$$(a,b]_{\mathbb{T}} := \big\{t \in \mathbb{T} : \ a < t \leqslant b\big\}, \quad (a,b)_{\mathbb{T}} := \big\{t \in \mathbb{T} : \ a < t < b\big\}.$$

The forward and backward jump operators on time scales are defined by $\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\}$, $\rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}$, respectively, and the graininess functions are given by $\mu(t) := \sigma(t) - t$, $\nu(t) := t - \rho(t)$. We call t a right-scattered point if $\sigma(t) > t$; t is called a left-scattered point if $\rho(t) < t$. Also, if $\sigma(t) = t$, then t is called a right-dense point; if $\rho(t) = t$, then t is called a left-dense point. We set

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}, & \sup \mathbb{T} \text{ is left-dense}, \\ \mathbb{T} \setminus \sup \mathbb{T}, & \sup \mathbb{T} \text{ is left-scattered}, \end{cases} \qquad \mathbb{T}_\kappa = \begin{cases} \mathbb{T}, & \inf \mathbb{T} \text{ is right-dense}, \\ \mathbb{T} \setminus \inf \mathbb{T}, & \inf \mathbb{T} \text{ is right-scattered}. \end{cases}$$

2 Improvement of Directional Derivatives on Time Scales

For the convenience of our discussion, we will use the following notations:

$$A_{\mathbb{T}} := \big\{t \in \mathbb{T} : \ t \text{ is a left-dense and right-dense point}\big\},$$
$$B_{\mathbb{T}} := \big\{t \in \mathbb{T} : \ t \text{ is a left-dense and right-scattered point}\big\},$$
$$C_{\mathbb{T}} := \big\{t \in \mathbb{T} : \ t \text{ is a left-scattered and right-dense point}\big\},$$
$$D_{\mathbb{T}} := \big\{t \in \mathbb{T} : \ t \text{ is an isolated point}\big\}.$$

To discuss the directional derivatives of multivariate function on time scales, we need the following n-dimensional time scale Λ^n $(n \in \mathbb{N})$.

Definition 2.1 ([3]). Let $\mathbb{T}_i$ $(i = 1, 2, \dots, n)$ be time scales, an n-dimensional time scale is defined by the Cartesian product $\Lambda^n := \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$. The forward and backward jump operators on each time scale $\mathbb{T}_i$ are denoted by σ_i and ρ_i, respectively.

Remark 2.1. Note that since the n-dimensional time scale Λ^n is a countable union of the closed hypercuboids, then $(t_1, t_2, \dots, t_n) \in \Lambda^n$ belongs to the interior of each hypercuboid if $t_i \in A_{\mathbb{T}_i}$ and $(t_1, t_2, \dots, t_n) \in \Lambda^n$ belongs to the surface of each hypercuboid if $t_i \notin A_{\mathbb{T}_i}$ or $t_i \in B_{\mathbb{T}_i} \cup C_{\mathbb{T}_i} \cup D_{\mathbb{T}_i}$ for each $i \in \{1, 2, \dots, n\}$.

Based on the notion of n-dimensional time scale Λ^n, for any fixed directional vector $\vec{\omega} = (\omega_1, \omega_2, \dots, \omega_n)$, we introduce two directional jump operators along $\vec{\omega}$ and $-\vec{\omega}$ on Λ^n as follows.

Definition 2.2. Let Λ^n be an n-dimensional time scale, $t = (t_1, t_2, \dots, t_n) \in \Lambda^n$, $\vec{\omega} = (\omega_1, \omega_2, \dots, \omega_n)$ be any fixed directional vector. The directional jump operator $\Pi^{\vec{\omega}}(t)$ along $\vec{\omega}$ on Λ^n is defined as follows:

$$\Pi^{\vec{\omega}}(t) = \left(\Pi_1^{\vec{\omega}}(t_1), \dots, \Pi_n^{\vec{\omega}}(t_n)\right) = \begin{cases} (t_1, t_2, \dots, t_n) & \text{if } t_i \in A_{\mathbb{T}_i}, \\ (s_1, s_2, \dots, s_n) & \text{otherwise,} \end{cases} \tag{2.1}$$

where $(s_1, \dots, s_n)$ satisfies

$$\Big[\sum_{i=1}^{n}(s_i - t_i)^2\Big]^{1/2} = \inf\left\{\Big[\sum_{i=1}^{n}(q_i - t_i)^2\Big]^{1/2} : \ \frac{q_1 - t_1}{\omega_1} = \dots = \frac{q_n - t_n}{\omega_n}, \ q = (q_1, \dots, q_n) \in \Lambda^n\right\}. \tag{2.2}$$

Similarly, we define $\Pi^{-\vec{\omega}}(t)$ as follows:

$$\Pi^{-\vec{\omega}}(t) = \left(\Pi_1^{-\vec{\omega}}(t_1), \dots, \Pi_n^{-\vec{\omega}}(t_n)\right) = \begin{cases} (t_1, t_2, \dots, t_n), & \text{if } t_i \in A_{\mathbb{T}_i}, \\ (s_1, s_2, \dots, s_n), & \text{otherwise,} \end{cases} \tag{2.3}$$

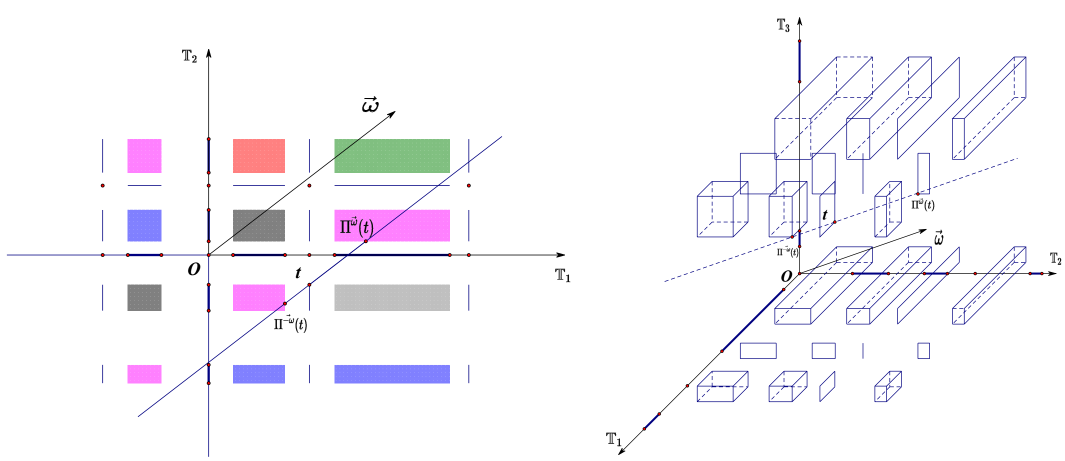
where $(s_1, \dots, s_n)$ satisfies

$$\Big[\sum_{i=1}^{n}(s_i - t_i)^2\Big]^{1/2} = \inf\left\{\Big[\sum_{i=1}^{n}(q_i - t_i)^2\Big]^{1/2} : \ \frac{q_1 - t_1}{-\omega_1} = \dots = \frac{q_n - t_n}{-\omega_n}, \ q = (q_1, \dots, q_n) \in \Lambda^n\right\}. \tag{2.4}$$

For $t = (t_1, t_2, \dots, t_n)$, if there exists some $i \in \{1, \dots, n\}$ such that $t_i = \max \mathbb{T}_i$ and $\omega_i > 0$, then we define $\Pi^{\vec{\omega}}(t) = t$. Similarly, if there exists some $i \in \{1, \dots, n\}$ such that $t_i = \min \mathbb{T}_i$ and $\omega_i < 0$, then we define $\Pi^{-\vec{\omega}}(t) = t$.

Remark 2.2. It follows from Definition 2.2 that the set of points $q = (q_1, q_2, \dots, q_n) \in \Lambda^n$ along the direction $\vec{\omega}$ given by (2.2) and (2.4) is nonempty, which indicates that $s = (s_1, s_2, \dots, s_n)$ in (2.1) and (2.3) are well-defined.

Remark 2.3. Note that if $\Pi_i^{\vec{\omega}}(t) = \sigma_i(t)$ and $\Pi_i^{-\vec{\omega}}(t) = \rho_i(t)$, Definition 2.2 includes the notions of directional derivatives discussed in [2, 3, 8].



(a) The case for $n = 2$ (b) The case for $n = 3$

Figure 2.1: Geometric diagrams of $\Pi^{\vec{\omega}}(t)$ and $\Pi^{-\vec{\omega}}(t)$

It is clear that Λ^n consists of the mutually disjoint pairwise closed hypercuboids, i.e., $\Lambda^n = \bigcup\limits_{i=1}^{n} V_i$, where $V_i \in \mathbb{R}^n$ and $V_i \cap V_j = \varnothing$ as $i \neq j$. By observing Figure 2.1, for $t \in \Lambda^n$, $\Pi^{\vec{\omega}}(t)$ moves t on a straight line to the direction that is parallel to the directional vector $\vec{\omega}$ and, similarly, $\Pi^{-\vec{\omega}}(t)$ moves t to the opposite direction. Note that $s = (s_1, s_2, \dots, s_n)$ is the point, nearest to t along the direction $\vec{\omega}$.

Definition 2.3. Let $f : \Lambda^n \to \mathbb{R}$ be a function, $t \in \Lambda^n$. If $s \in \Lambda^n \setminus \{\Pi^{\vec{\omega}}(t)\}$ belongs to an arbitrary neighbourhood of t and $\Delta^\omega f(t) = f(\Pi^{\vec{\omega}}(t)) - f(s)$ can be represented as

$$\Delta^\omega f(t) = f(\Pi^{\vec{\omega}}(t)) - f(s) = \sum_{i=1}^{n} M_i \Delta_i^\omega t_i + o(\theta_{\Delta^\omega}),$$

where $\Delta_i^\omega t_i = \Pi_i^{\vec{\omega}}(t_i) - s_i$, $\theta_{\Delta^\omega} = \sqrt{\sum\limits_{i=1}^{n} (\Delta_i^\omega t_i)^2}$, M_i is a constant related only to the point t, and $o(\theta_{\Delta^\omega})$ is a higher-order infinitesimal of θ_{Δ^ω}, then we call f is Δ^ω-differentiable at point t. $\sum\limits_{i=1}^{n} M_i \Delta_i^\omega t_i$ is called the total Δ^ω-differential of f at t and denoted by $d_{\Delta^\omega} z = d_{\Delta^\omega} f(t) = \sum\limits_{i=1}^{n} M_i \Delta_i^\omega t_i$.

Similarly, if $s \in \Lambda^n \setminus \{\Pi^{-\vec{\omega}}(t)\}$ belongs to an arbitrary neighbourhood of t and $\nabla^\omega f(t) = f(s) - f(\Pi^{-\vec{\omega}}(t))$ can be represented as

$$\nabla^\omega f(t) = f(s) - f(\Pi^{-\vec{\omega}}(t)) = \sum_{i=1}^{n} M_i \nabla_i^\omega t_i + o(\theta_{\nabla^\omega}),$$

where $\nabla_i^\omega t_i = s_i - \Pi_i^{-\vec{\omega}}(t_i)$, $\theta_{\nabla^\omega} = \sqrt{\sum\limits_{i=1}^{n} (\nabla_i^\omega t_i)^2}$, M_i is a constant related only to the point t, and $o(\theta_{\nabla^\omega})$ is a higher-order infinitesimal of θ_{∇^ω}, then we call f is ∇^ω-differentiable at point t. $\sum\limits_{i=1}^{n} M_i \nabla_i^\omega t_i$ is called the total ∇^ω-differential of f at t and denoted by $d_{\nabla^\omega} z = d_{\nabla^\omega} f(t) = \sum\limits_{i=1}^{n} M_i \nabla_i^\omega t_i$.

Next, we give the notions of partial delta and nabla derivative of $f : \Lambda^n \to \mathbb{R}$.

Definition 2.4 ([3])**.** Let $f : \Lambda^n \to \mathbb{R}$ be a function, the partial delta derivative of f with respect to $t_i \in \mathbb{T}_i^\kappa$ $(i = 1, \dots, n)$ is defined as the limit

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \sigma_i(t_i)}} \frac{f(t_1, \dots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, s_i, t_{i+1}, \dots, t_n)}{\sigma_i(t_i) - s_i},$$

provided that this limit exists as a finite number, and is denoted by any of the following symbols:

$$\frac{\partial f(t_1, \dots, t_n)}{\Delta_i t_i}, \quad \frac{\partial f(t)}{\Delta_i t_i}, \quad \frac{\partial f}{\Delta_i t_i}(t), \quad f_{t_i}^{\Delta_i}(t).$$

The partial nabla derivative of f with respect to $t_i \in (\mathbb{T}_i)_\kappa$ $(i = 1, \dots, n)$ is defined as the limit

$$\lim_{\substack{s_i \to t_i \\ s_i \neq \rho_i(t_i)}} \frac{f(t_1, \dots, t_{i-1}, \rho_i(t_i), t_{i+1}, \dots, t_n) - f(t_1, \dots, t_{i-1}, s_i, t_{i+1}, \dots, t_n)}{\rho_i(t_i) - s_i}$$

and denoted by $\frac{\partial f(t)}{\nabla_i t_i}$, provided that this limit exists as a finite number.

According to the concepts of $\Pi^{\vec{\omega}}(t)$ and $\Pi^{-\vec{\omega}}(t)$ which are given in Definition 2.2, we introduce the definitions of delta and nabla directional derivatives of $f : \Lambda^n \to \mathbb{R}$, which will reduce to Definition 2.4 when the partial derivatives are considered.

Definition 2.5. Let $f : \Lambda^n \to \mathbb{R}$ be a function, $t = (t_1, \dots, t_n) \in \Lambda^n$, $\vec{\omega} = (\omega_1, \dots, \omega_n)$ be any fixed directional vector. If $s \in \Lambda^n \setminus \{\Pi^{\vec{\omega}}(t)\}$ belongs to an arbitrary neighbourhood of t and the limit

$$\eta = \lim_{s \to t} \frac{f(\Pi_1^{\vec{\omega}}(t_1), \dots, \Pi_n^{\vec{\omega}}(t_n)) - f(s_1, \dots, s_n)}{\sqrt{(\Pi_1^{\vec{\omega}}(t_1) - s_1)^2 + \cdots + (\Pi_n^{\vec{\omega}}(t_n) - s_n)^2}}$$

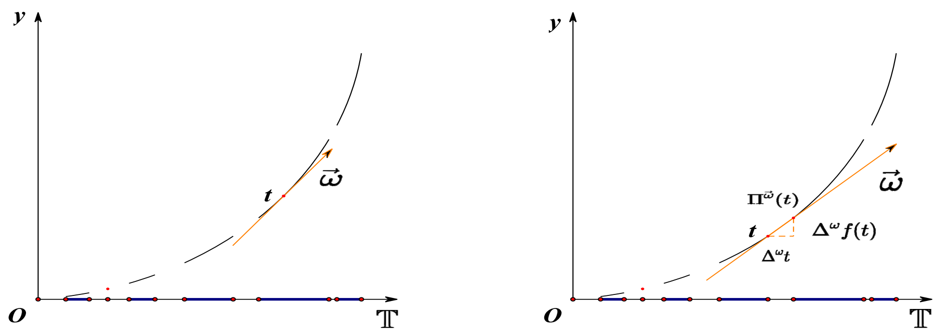


Figure 2.2: Schematic diagram of directional derivatives of f on Λ^1

exists, then we call $\frac{\partial f_{\Delta}}{\partial \omega}\big|_t = \eta$ the delta directional derivative of f along the direction $\vec{\omega}$ at the point t.

If $s \in \Lambda^n \setminus \{\Pi^{-\vec{\omega}}(t)\}$ belongs to an arbitrary neighbourhood of t and the limit

$$\eta = \lim_{s \to t} \frac{f(s_1, \ldots, s_n) - f(\Pi_1^{-\vec{\omega}}(t_1), \ldots, \Pi_n^{-\vec{\omega}}(t_n))}{\sqrt{(\Pi_1^{-\vec{\omega}}(t_1) - s_1)^2 + \cdots + (\Pi_n^{-\vec{\omega}}(t_n) - s_n)^2}}$$

exists, then we call $\frac{\partial f_{\nabla}}{\partial \omega}\big|_t = \eta$ the nabla directional derivative of f along the direction $\vec{\omega}$ at the point t.

Figure 2.2 shows the geometric meaning of one-dimensional delta (nabla) directional derivative.

Theorem 2.1. *Let $f : \Lambda^n \to \mathbb{R}$ be a function, $t = (t_1, t_2, \ldots, t_n) \in \Lambda^n$ and $\vec{\omega} = (\omega_1, \ldots, \omega_n)$ be any fixed directional vector. If f is Δ^{ω}-differentiable (resp. ∇^{ω}-differentiable) at the point t, then the partial delta (resp. nabla) derivative of f with respect to t_i exists and*

$$d_{\Delta^{\omega}} z = \sum_{i=1}^{n} \frac{\partial f(t)}{\Delta_i t_i} \Delta_i^{\omega} t_i \quad \left(resp.\ d_{\nabla^{\omega}} z = \sum_{i=1}^{n} \frac{\partial f(t)}{\nabla_i t_i} \nabla_i^{\omega} t_i \right), \tag{2.5}$$

where $i = 1, 2, \ldots, n$.

Proof. According to Definition 2.3, for $s \in \Lambda^n \setminus \{\Pi^{\vec{\omega}}(t)\}$ that belongs to an arbitrary neighbourhood of t, $\Delta^{\omega} f(t) = f\big(\Pi^{\vec{\omega}}(t)\big) - f(s)$ can be represented as

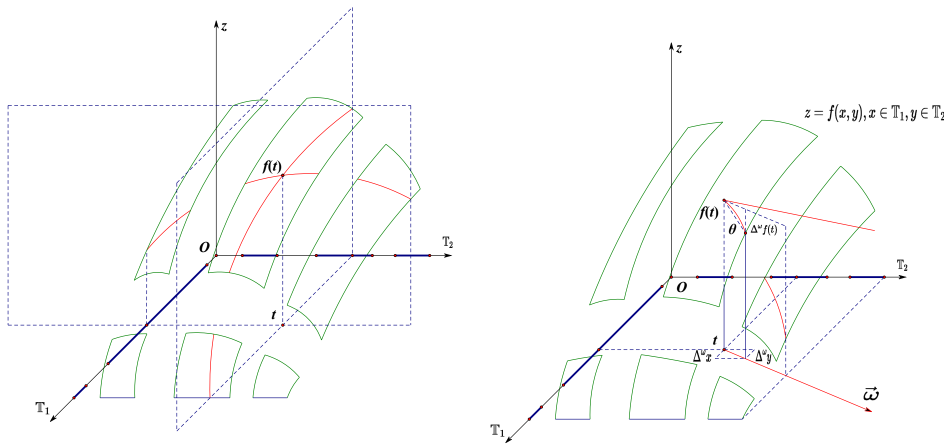
$$\Delta^{\omega} f(t) = f(\Pi^{\vec{\omega}}(t)) - f(s) = \sum_{i=1}^{n} M_i \Delta_i^{\omega} t_i + o(\theta_{\Delta^{\omega}}),$$

where $\Delta_i^{\omega} t_i = \Pi_i^{\vec{\omega}}(t_i) - s_i$, $\theta_{\Delta^{\omega}} = \sqrt{\sum_{i=1}^{n} (\Delta_i^{\omega} t_i)^2}$, M_i is a constant related only to the point t, and $o(\theta_{\Delta^{\omega}})$ is the higher-order infinitesimal of $\theta_{\Delta^{\omega}}$. Then for a fixed i, let $\Delta_i^{\omega} t_i \neq 0$, $\Delta_j^{\omega} t_j = 0$, $j \neq i$, this implies that

$$\Delta^{\omega} f(t) = f\big(\Pi_i^{\vec{\omega}}(t_1), \ldots, \Pi_i^{\vec{\omega}}(t_i), \Pi_n^{\vec{\omega}}(t_n)\big) - f(s_1, \ldots, s_i, \ldots, s_n) = \sum_{i=1}^{n} M_i \Delta_i^{\omega} t_i + o(\theta_{\Delta^{\omega}})$$

with $\Pi_i^{\vec{\omega}}(t_i) \neq s_i$, $\Pi_j^{\vec{\omega}}(t_j) = t_j$, $j \neq i$. Thus we have $\Delta^{\omega} f(t) = M_i \Delta_i^{\omega} t_i + o(\theta)$, which implies $M_i = \lim\limits_{s_i \to t_i} \frac{\Delta^{\omega} f(t)}{\Delta_i^{\omega} t_i} - \frac{o(\theta)}{\Delta_i^{\omega} t_i}$, from Definition 2.4, the partial delta (nabla) derivative of f with respect to t_i exists and is equal to M_i. Hence (2.5) holds. This completes the proof. □

To observe the relationship between the delta (nabla) directional derivative and the partial delta (nabla) derivative, we give their geometric diagram in two-dimensional cases (see Figure 2.3).



(a) Partial derivative of f on Λ^2 (b) Directional direvative of f on Λ^2

Figure 2.3: Comparison of partial derivative and directional derivative

Theorem 2.2. *Let $f : \Lambda^n \to \mathbb{R}$ be a function, $\vec{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$ be any fixed vector, where $\omega_i \in \mathbb{R}$, $i = 1, \ldots, n$. If f is differentiable at the point t, then*

$$\frac{\partial f_\Delta}{\partial \omega}\Big|_t = \sum_{i=1}^{n} \frac{\partial f(t)}{\Delta_i t_i} \cos\alpha_i \quad \left(resp.\ \frac{\partial f_\nabla}{\partial \omega}\Big|_t = \sum_{i=1}^{n} \frac{\partial f(t)}{\nabla_i t_i} \cos\alpha_i \right),$$

where $\cos\alpha_i = \frac{\omega_i}{|\omega|} = \frac{\omega_i}{\sqrt{\omega_1^2 + \omega_2^2 + \cdots + \omega_n^2}}$, $i = 1, 2, \ldots, n$.

Proof. According to Definition 2.2, for $s \in \Lambda^n \setminus \{\Pi^{\vec{\omega}}(t)\}$ that belongs to an arbitrary neighbourhood of t, we have

$$\Pi_i^{\vec{\omega}}(t_i) - s_i = \theta_{\Delta^\omega} \cos\alpha_i, \quad i = 1, 2, \ldots,$$

where $\theta_{\Delta^\omega} = \sqrt{\sum_{i=1}^{n} \left(\Pi_i^{\vec{\omega}}(t_i) - s_i\right)^2}$. By the assumption, $f(\Pi^{\vec{\omega}}(t)) - f(s) = \sum_{i=1}^{n} M_i \Delta_i^\omega t_i + o(\theta_{\Delta^\omega})$, thus

$$\frac{f(\Pi^{\vec{\omega}}(t)) - f(t)}{\theta_{\Delta^\omega}} = \sum_{i=1}^{n} M_i \cos\alpha_i + \frac{o(\theta_{\Delta^\omega})}{\theta_{\Delta^\omega}}.$$

From Theorem 2.1, we have $\frac{\partial f_\Delta}{\partial \omega}\big|_t = \sum_{i=1}^{n} \frac{\partial f(t)}{\Delta_i t_i} \cos\alpha_i$. Similarly, we can obtain $\frac{\partial f_\nabla}{\partial \omega}\big|_t = \sum_{i=1}^{n} \frac{\partial f(t)}{\nabla_i t_i} \cos\alpha_i$. This completes the proof. □

Definition 2.6. Let $f : \Lambda^n \to \mathbb{R}$ be a function, for $t = \big(t_1, t_2, \ldots, t_n\big) \in \Lambda^n$, the delta gradient of f at the point t is defined as follows:

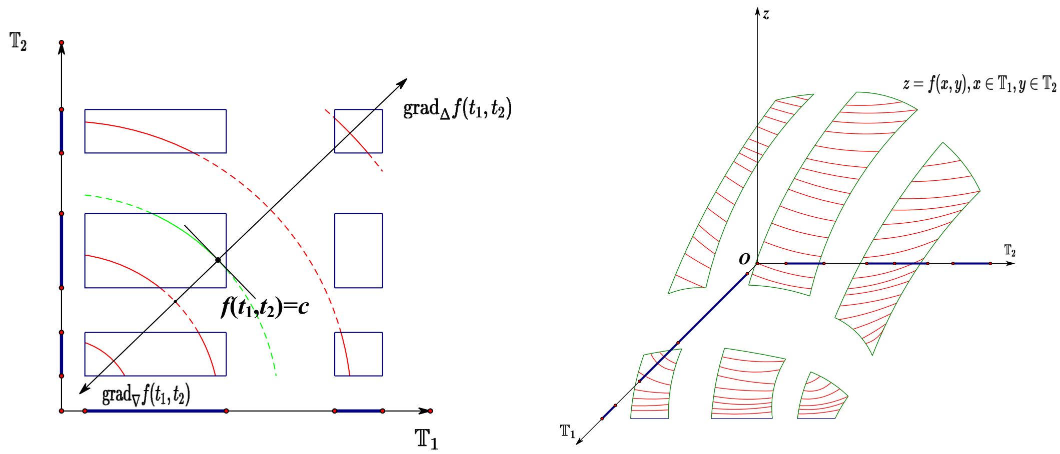
$$\mathrm{grad}_\Delta f = \Big(\frac{\partial f(t)}{\Delta_1 t_1}, \frac{\partial f(t)}{\Delta_2 t_2}, \ldots, \frac{\partial f(t)}{\Delta_n t_n} \Big).$$

Similarly, we define the nabla gradient of f at the point t as

$$\mathrm{grad}_\nabla f = \Big(\frac{\partial f(t)}{\nabla_1 t_1}, \frac{\partial f(t)}{\nabla_2 t_2}, \ldots, \frac{\partial f(t)}{\nabla_n t_n} \Big).$$

We give the geometric diagrams of the contour curve and surface as Figure 2.4, and this shows the geometric significance of the gradient.

Now we establish the following



(a) Contour curve and gradient of f in Λ^2. (b) Contour curve and gradient of $z = f(x, y)$.

Figure 2.4: Contour curve and gradient of f.

Theorem 2.3. *Let $f : \Lambda^2 \to \mathbb{R}$ be a function and there exist a continuous differentiable function $t_2 : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{T}_2 \subset R(t_2)$, where $R(t_2)$ is the range of t_2, $t = (t_1, t_2) \in \Lambda^2$. If $t_i \in A_{\mathbb{T}_i}$, then the gradient at t is perpendicular to the tangent of contour curve at t.*

Proof. Since $t_i \in A_{\mathbb{T}_i}$, $\mathrm{grad}_\Delta f = \mathrm{grad}_\nabla f = \mathrm{grad} f$. Assume that the space surface satisfies the equation $z = f(t_1, t_2)$, then the contour curve equation is

$$\begin{cases} z = f(t_1, t_2), \\ z = c. \end{cases}$$

For $c = f(t_1, t_2)$ with $t_i \in A_{\mathbb{T}_i}$, according to the Theorem of Implicit Function (see Section 9 in [3]), the slope at this point of the contour curve is $\frac{dt_2}{dt_1} = -\frac{f_{t_1}^{\Delta_1}(t)}{f_{t_2}^{\Delta_1}(t)}$, which implies that the tangent vector of the contour curve at the point t is $(f_{t_2}^{\Delta_1}(t), -f_{t_1}^{\Delta_1}(t))$. Since the gradient of f at t is the vector $(\frac{\partial f(t)}{\Delta_1 t_1}, \frac{\partial f(t)}{\Delta_2 t_2})$, from Theorem 2.2, we have $\frac{\partial f(t)}{\Delta_1 t_1} = f_{t_1}^{\Delta_1}(t)$, $\frac{\partial f(t)}{\Delta_2 t_2} = f_{t_2}^{\Delta_2}(t)$, thus $(f_{t_2}^{\Delta_1}(t), -f_{t_1}^{\Delta_1}(t)) \cdot (\frac{\partial f(t)}{\Delta_1 t_1}, \frac{\partial f(t)}{\Delta_2 t_2}) = 0$, the desired result follows. This completes the proof. □

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