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L^2-HILBERT–SCHMIDTNESS OF FOURIER INTEGRAL OPERATORS WITH WEIGHTED SYMBOLS

Abstract. In this paper, we define a particular class of Fourier Integral Operators with weighted symbols (FIO, for short). These FIO turn out to be bounded on the spaces $S(\mathbb{R}^n)$ of rapidly decreasing functions (or Schwartz space) and $S'(\mathbb{R}^n)$ of temperate distributions. We also prove that FIO is Hilbert–Schmidt on $L^2(\mathbb{R}^n)$ when the weight of the symbol a belongs to $L^2(\mathbb{R}^{2n})$.

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რეზიუმე. ნაშრომში ჩვენ განვსაზღვრავთ ფურიეს ინტეგრალური ოპერატორების (მოკლედ FIO) კონკრეტულ კლასს შეწონილი სიმბოლოებით. ირკვევა, რომ ეს FIO შემოსაზღვრულია სწრაფად კლებად ფუნქციათა $S(\mathbb{R}^n)$ სივრცეზე (ანუ შვარცის სივრცეზე) და ზომიერი განაწილების $S'(\mathbb{R}^n)$ სივრცეზე. ასევე მტკიცდება, რომ FIO არის ჰილბერტ–შმიდტის ოპერატორი $L^2(\mathbb{R}^n)$-ზე, როცა a სიმბოლოს წონა ეკუთვნის $L^2(\mathbb{R}^{2n})$-ს.

1 Introduction

A Fourier integral operator (FIO, for short) is a singular integral operator of the form

$$I_{a,\phi}u(x) = \iint e^{i\phi(x,y,\xi)}a(x,y,\xi)u(y)\,dy\,d\xi$$

defined under certain assumptions on the regularity and asymptotic properties of the phase function ϕ and the amplitude function (or symbol) a. Here, ξ plays the role of the co-variable. In particular, when $\phi(x,\xi,y) = \langle x-y,\xi\rangle$, $I_{a,\phi} := Op(a)$ is called a pseudodifferential operator.

Several authors have worked hard since 1970 to learn more about this type of operator (see, e.g., [2,6,8,11–14,16]). Local properties are the focus of the first works on Fourier integral operators. We should mention that Hörmander has treated a class of Fourier integral operators for the first time in [14], after they have been initially used by Lax, Maslov, Egorov and others. Duistermaat and Hörmander elaborated the results of [14] in findings in their paper [7], where they studied parametrices of the pseudodifferential operators of principal type as well as the propagation of singularities. In the meantime, FIOs were also used to analyze hyperbolic equations and spectral theory.

The study of FIO was started by a particular class of amplitudes $S^m_{\rho,\delta}$ introduced by Hörmander which consists of functions $a(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^N)$ that satisfy

$$\left|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

with $m\in\mathbb{R}$, $\rho,\delta\in[0,1]$, and the phase functions in $C^\infty(\mathbb{R}^n\times\mathbb{R}^n\setminus 0)$ homogenous of degree 1 in the frequency variable ξ and with non-vanishing determinant of the mixed Hessian matrix (i.e., non-degenerate phase functions).

Furthermore, G. Eskin [9] (in the case $a\in S^0_{1,0}$) and L. Hörmander [14] (in the case $a\in S^0_{\rho,1-\rho}$, $\frac{1}{2}<\rho\le 1$) demonstrated the local L^2 boundedness of FIO with non-degenerate phase functions, R. Beals [3] and A. Greenleaf and G. Uhlmann [10] extended Hörmander's local L^2 result to amplitudes in $S^0_{\frac{1}{2},\frac{1}{2}}$.

Later on, other types of symbols and phase functions have been investigated. In [13,19], D. Robert and B. Helffer treated the symbol class Γ^μ_ρ that consists of smooth functions such that for any multi-indices $(\alpha,\beta,\gamma)\in\mathbb{N}^n\times\mathbb{N}^n\times\mathbb{N}^N$, there exists $C_{\alpha,\beta,\gamma}>0$ such that

$$\left|\partial_x^\alpha\partial_y^\beta\partial_\xi^\gamma a(x,y,\xi)\right| \le C_{\alpha,\beta,\gamma}\langle(x,y,\xi)\rangle^{\mu-\rho(|\alpha|+|\beta|+|\gamma|)},$$

where

$$\langle(x,y,\xi)\rangle := \left(1+|x|^2+|y|^2+|\xi|^2\right)^{\frac{1}{2}}$$

with $\mu\in\mathbb{R}$ and $\rho\in[0,1]$, and they considered phase functions satisfying certain conditions.

In [16], Messirdi and Senoussaoui treated the L^2 boundedness and L^2 compactness of FIO with symbol class just defined. These operators are continuous (respectively, compact) in L^2 if the weight of the symbol is bounded (respectively, tends to 0).

In this work, we use the same technique as in [5] to show the Hilbert–Schmidtness of the operators the type

$$(Fu)(x) = (2\pi)^{-n}\iint e^{i(s(x,\xi)-\langle y,\xi\rangle)}a(x,\xi)u(y)\,dy\,d\xi.$$

We mainly prove that operator F is Hilbert–Schmidt on $L^2(\mathbb{R}^n)$ when the weight of the amplitude a belongs to $L^2(\mathbb{R}^{2n})$.

The article is organized as follows. In Section 2, we define symbol and phase functions used in this paper and recall the continuity of some general class of Fourier integral operators on $\mathcal{S}(\mathbb{R}^n)$ and on $\mathcal{S}'(\mathbb{R}^n)$. In Section 3, we discuss a special case of phase functions of type $\phi(x,y,\xi) = s(x,\xi)-\langle y,\xi\rangle$. The last section is devoted to the proof of the main result.

2 Notations and preliminaries

We assume that $n \in \mathbb{N}$ throughout the paper unless otherwise noted. In particular, $n \neq 0$. For all $x, y, \xi \in \mathbb{R}^n$, we define

$$\langle x, \xi \rangle := \sum_{j=0}^{n} x_j \xi_j \ \text{ and } \ \widehat{d\xi} := (2\pi)^{-n}\, d\xi.$$

Additionally,

$$\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}, \quad \langle (x, y) \rangle := (1 + |x|^2 + |y|^2)^{\frac{1}{2}}$$

and

$$\langle (x, y, \xi) \rangle := \big(1 + |x|^2 + |y|^2 + |\xi|^2\big)^{\frac{1}{2}}.$$

Partial derivatives with respect to a variable $x \in \mathbb{R}^n$ scaled with the factor $-i$ are denoted by

$$D_x^\alpha := (-i)^{|\alpha|}\partial_x^\alpha := (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multi-index and $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$ is the length of α.

Considering two Freshet spaces E and F, the set $\mathcal{L}(E, F)$ contains all linear and bounded operators $A : E \to F$. If $E = F$, we also just write $\mathcal{L}(E)$.

Definition 2.1. The space of tempered weights $\omega(\mathbb{R}^n)$ is the set of all continuous functions $m : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\exists\, C_0 > 0,\ \ \exists\, k_0 \in \mathbb{R}, \quad m(x) \leq C_0 m(y)\big(1 + |x - y|\big)^{k_0},\ \ \forall\, x, y \in \mathbb{R}^n.$$

Example 2.1. The simplest example of tempered weight is given by

$$m(x) = \langle x \rangle^k \ \text{ for } \ x \in \mathbb{R}^n,$$

where $k \in \mathbb{R}$.

Lemma 2.1. *If $k > 2n$, then $\langle x \rangle^{-k} \in L^2(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$.*

Proof. The lemma can easily be proved by using polar coordinates. An alternative approach can be found in [18, Lemma 1.3]. □

Definition 2.2. Let Ω be an open subset of $\mathbb{R}^n$, $m \in \omega(\mathbb{R}^n)$ and $\rho \in [0, 1]$. A smooth function $a : \Omega \to \mathbb{C}$ is called an (m, ρ)-weighted symbol on Ω if

$$\forall\, \alpha \in \mathbb{N}^n,\ \ \exists\, C_\alpha > 0,\ \ |\partial_x^\alpha a(x)| \leq C_\alpha m(x)\big(1 + |x|\big)^{-\rho|\alpha|},\ \ \forall\, x \in \Omega.$$

We note that $\Gamma_\rho^m(\Omega)$ is the space of all (m, ρ)-weighted symbols.

Remark 2.1. Instead of $\Gamma_\rho^m(\mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n)$ we simply write Γ_ρ^m. Furthermore, if $\rho = 0$, we write Γ^m.

Now, we are interested in giving a sense to the following integral transformations

$$I_{a,\phi}u(x) = \iint e^{i\phi(x,\xi,y)} a(x, \xi, y) u(y)\, dy\, \widehat{d\xi},\ \ u \in \mathcal{S}(\mathbb{R}^n), \tag{2.1}$$

where $a \in \Gamma_\rho^m$, $m \in \omega$ and ϕ is a phase function which satisfies the following conditions:

(C_1) $\phi : \mathbb{R}_x^n \times \mathbb{R}_\xi^N \times \mathbb{R}_y^n \to \mathbb{R}$ is a C^∞ application.

(C_2) $\forall\, (\alpha, \beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n$, $\exists\, C_{\alpha,\beta,\gamma} \geq 0$ such that

$$\big|\partial_x^\alpha \partial_\xi^\beta \partial_y^\gamma \phi(x, \xi, y)\big| \leq C_{\alpha,\beta,\gamma} \langle (x, \xi, y) \rangle^{(2-|\alpha|-|\beta|-|\gamma|)}.$$

(C_3) There exists $K > 0$ such that

$$\frac{\langle(\partial_y\phi, \partial_\xi\phi, y)\rangle}{\langle(x,\xi,y)\rangle} \leq K, \ \ \forall\,(x,\xi,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y.$$

(C_3^*) There exists $K^* > 0$ such that

$$\frac{\langle(x, \partial_\xi\phi, \partial_x\phi)\rangle}{\langle(x,\xi,y)\rangle} \leq K^*, \ \ \forall\,(x,\xi,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\xi \times \mathbb{R}^n_y.$$

In the next theorem, we give a sense to the right-hand side of (2.1), by using the oscillatory integral technique, and prove the boundedness on the Schwartz space and on its dual. So, we consider $g \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}^N_\xi)$ such that $g(0,0,0) = 1$.

If $a \in \Gamma^m$, we define

$$a_p(x,\xi,y) := g\Big(\frac{x}{p}, \frac{\xi}{p}, \frac{y}{p}\Big)a(x,\xi,y), \ \ p > 0.$$

Theorem 2.1. *If ϕ satisfies (C_1), (C_2), (C_3) and (C_3*), then*

(1) *For all $u \in \mathcal{S}(\mathbb{R}^n)$, $\lim\limits_{p\to\infty} I_{a_p,\phi}u(x)$ exists for every $x \in \mathbb{R}^n$ and is independent of the choice of the function g. Then we set*

$$I_{a,\phi}u(x) := \lim_{p\to\infty} I_{a_p,\phi}u(x), \ \ \forall\, x \in \mathbb{R}^n.$$

(2) $I_{a,\phi} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ *and* $I_{a,\phi} \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$.

Proof. Let $\delta \in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\delta \subseteq [-2,2]$ and $\delta \equiv 1$ on $[-1,1]$.

For all $\varepsilon > 0$, we set

$$\omega_\varepsilon(x,\xi,y) = \delta\Big(\frac{|\partial_y\phi|^2 + |\partial_\xi\phi|^2}{\varepsilon\langle(x,\xi,y)\rangle^2}\Big).$$

The condition (C_3) implies that there exists $\gamma > 0$ such that on the support of ω_ε we have

$$\langle(x,\xi,y)\rangle \leq \gamma\big[(1+|y|^2)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}\langle(x,\xi,y)\rangle\big].$$

Therefore, there exist ε_0 and a constant γ_0, such that for all $\varepsilon \leq \varepsilon_0$, we have the inequality

$$\langle(x,\xi,y)\rangle \leq \gamma_0\big(1+|y|^2\big)^{\frac{1}{2}}$$

in the support of ω_ε.

In the sequel, we fix $\varepsilon = \varepsilon_0$. Then it is immediate that $I(\omega_{\varepsilon_0}a_p,\phi)f$ is an absolutely convergent integral, and we have

$$\lim_{p\to\infty} I(\omega_{\varepsilon_0}a_p,\phi)f = I(\omega_{\varepsilon_0}a,\phi)f. \tag{2.2}$$

Using (C_2), we also prove that $I(\omega_{\varepsilon_0}a,\phi)f$ is a continuous operator from $S(\mathbb{R}^n)$ into itself. To study $\lim\limits_{p\to\infty} I((1-\omega_{\varepsilon_0})a_p,\phi)f$, we introduce the operator

$$L = \frac{1}{i}\,\frac{\Big(\sum\limits_{j=1}^{n}(\partial_{y_j}\phi)\,\frac{\partial}{\partial y_j} + \sum\limits_{j=1}^{N}(\partial_{\xi_j}\phi)\frac{\partial}{\partial\xi_j}\Big)}{|\partial_y\phi|^2 + |\partial_\xi\phi|^2}\,.$$

Clearly, we have

$$L(e^{i\phi}) = e^{i\phi}. \tag{2.3}$$

Let Ω_0 be the open subset of $\mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$ defined by

$$\Omega_0 = \Big\{(x,\xi,y),\ |\partial_y\phi|^2 + |\partial_\xi\phi|^2 > \frac{\varepsilon_0}{2}\,\langle(x,\xi,y)\rangle^2\Big\}.$$

We need the following

Lemma 2.2. *For all integer $q \geq 0$ and $b \in C^\infty(\mathbb{R}^n_y \times \mathbb{R}^N_\xi)$, we have*

$$({}^tL)^q((1-\omega_{\varepsilon_0})b) = \sum_{|\alpha|+|\beta|\leq q} g^q_{\alpha,\beta}\partial^\beta_y \partial^\beta_\xi((1-\omega_{\varepsilon_0})b),$$

where the $g^q_{\alpha,\beta}$ are in $\Gamma_0^{-q}(\Omega_0)$ and depend only on ϕ. Recall that ${}^tL$ designates the transpose of L.

Proof. We prove the lemma by the recurrence. It is obvious for $q = 0$. Now we can see easily that

$$ {}^tL = \sum_j F_j \frac{\partial}{\partial y_j} + \sum_j G_j \frac{\partial}{\partial \xi_j} + H, \tag{2.4}$$

where $F_j \in \Gamma_0^{-1}(\Omega_0)$, $G_j \in \Gamma_0^{-1}(\Omega_0)$ and $H \in \Gamma_0^{-2}(\Omega_0)$ (which results from the hypothesis (C_2)). So, the recurrence is immediately proved. □

For all integer $q \geq 0$, from (2.3) we have

$$I((1-\omega_{\varepsilon_0})a_p,\phi)f(x) = \iint e^{i\phi(x,\xi,y)}({}^tL)^q((1-\omega_{\varepsilon_0})a_p f)\, dy\, d\xi. \tag{2.5}$$

Now $({}^tL)^q((1-\omega_{\varepsilon_0})a_p f)$ describes (when p varies) a bound of $\Gamma_0^{\mu-q}$, and

$$\lim_{p\to\infty} ({}^tL)^q((1-\omega_{\varepsilon_0})a_p f)(x,\xi,y) = ({}^tL)^q((1-\omega_{\varepsilon_0})af)(x,\xi,y) \tag{2.6}$$

for all $(x,\xi,y) \in \mathbb{R}^n \times \mathbb{R}^N \times \mathbb{R}^n$.

Finally, for all $s > n + N$, we have

$$\iint \langle (x,\xi,y)\rangle^{-s}\, d\xi\, dy \leq \gamma_s \langle x\rangle^{n+N-s}. \tag{2.7}$$

From (2.5)–(2.7) and by using the Lebesgue's theorem, we get

$$\lim_{p\to\infty} I((1-\omega_{\varepsilon_0})a_p,\phi)f(x) = \iint e^{i\phi(x,\xi,y)}({}^tL)^q((1-\omega_{\varepsilon_0})af)\, dy\, d\xi, \tag{2.8}$$

where q satisfies $q > n + N + \mu$. The first part of the theorem can be proven by using (2.2) and (2.8).

Now let us show that $I((1-\omega_{\varepsilon_0})a,\phi)$ is continuous. Taking into account (2.4) and (2.8), we get

$$I((1-\omega_{\varepsilon_0})a,\phi)f(x) = \sum_{|\gamma|\leq q} \iint e^{i\phi(x,\xi,y)} b^{(q)}_\gamma(x,\xi,y)\partial^\gamma_y f(y)\, dy\, d\xi, \tag{2.9}$$

with $b^{(q)}_\gamma \in \Gamma_0^{\mu-q}$. On the other hand, we have

$$x^\alpha \partial^\beta_x(e^{i\phi}b^{(q)}_\gamma(x,\xi,y)) \in \Gamma_0^{\mu-q+|\alpha|+|\beta|}. \tag{2.10}$$

We deduce from (2.9) and (2.10) that, for all $q > n + N + \mu + |\alpha| + |\beta|$, there exists a constant $C_{\alpha,\beta,q}$ such that

$$\left| x^\alpha \partial^\beta_x I((1-\omega_{\varepsilon_0})a,\phi)f(x)\right| \leq C_{\alpha,\beta,q} \sup_{\substack{x\in\mathbb{R}^n \\ |\gamma|\leq q}} |\partial^\gamma_x f(x)|,$$

which proves the continuity of $I((1-\omega_{\varepsilon_0})a,\phi)$. □

3 About the special class of FIO

From now on, we will focus, in a particular case, on the phase function ϕ, which is extremely useful in applications for solving Cauchy problems [15, 17]. Let

$$\phi(x, \xi, y) = s(x, \xi) - \langle y, \xi \rangle,$$

where s satisfies the conditions:

(H_1) $s \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$.

(H_2) For all $(\alpha, \beta) \in \mathbb{N}^{2n}$, there exist $C_{\alpha,\beta} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta s(x, \xi)\right| \leq C_{\alpha,\beta} \langle (x, \xi) \rangle^{(2-|\alpha|-|\beta|)}.$$

(H_3) There exists $\delta_0 > 0$ such that

$$\inf_{x,\xi \in \mathbb{R}^n} \left| \det \frac{\partial^2 s}{\partial x \partial \xi} (x, \xi) \right| \geq \delta_0.$$

Proposition 3.1. *If s satisfies $(H_1), (H_2)$ and (H_3), then the function $\phi(x, \xi, y) = s(x, \xi) - \langle y, \xi \rangle$ satisfies (C_1), (C_2), (C_3) and (C_3^*).*

Proof. (C_1) and (C_2) are trivially satisfied.

To prove (C_3) and (C_3^*) we use the following

Lemma 3.1. *Assume that s satisfies (H_1), (H_2) and (H_3), then s satisfies the following inequalities: There exist $c_1, c_2 > 0$ such that*

$$\begin{cases} |x| \leq c_1 \langle (\xi, \partial_\xi s) \rangle, & \forall\, (x, \xi) \in \mathbb{R}^{2n}, \\ |\xi| \leq c_2 \langle (x, \partial_x s) \rangle, & \forall\, (x, \xi) \in \mathbb{R}^{2n}. \end{cases} \tag{3.1}$$

Also, there exists $c_3 > 0$ such that for all $(x, \xi), (x', \xi') \in \mathbb{R}^{2n}$,

$$|x - x'| + |\xi - \xi'| \leq c_3 \Big[\left|(\partial_\xi s)(x, \xi) - (\partial_\xi s)(x', \xi')\right| + |\xi - \xi'| \Big].$$

Proof. The mappings

$$\xi \to f_x(\xi) = \partial_x s(x, \xi), \quad x \to g_\xi(x) = \partial_\xi s(x, \xi)$$

are diffeomorphisms of $\mathbb{R}^n$. From (H_2) and (H_3) it follows that $\|(f_x^{-1})'\|$, $\|(g_\xi^{-1})'\|$ are uniformly bounded on $\mathbb{R}^n$ and $\|(\psi_2^{-1})'\|$ is uniformly bounded on $\mathbb{R}^{2n}$, where

$$\psi_2(x, \xi) = (\xi, \partial_\xi s(x, \xi)).$$

Thus (H_2) and Taylor's theorem lead to the following estimate: there exist $M, N > 0$ such that for all $(x, \xi), (x', \xi') \in \mathbb{R}^{2n}$,

$$|\xi| = \left|f_x^{-1}(f_x(\xi)) - f_x^{-1}(f_x(0))\right| \leq M \left|\partial_x s(x, \xi) - \partial_x s(x, 0)\right| \leq c_4 \langle (x, \partial_x s) \rangle$$

with $c_4 > 0$;

$$|x| = \left|g_\xi^{-1}(g_\xi(\xi)) - g_\xi^{-1}(g_\xi(0))\right| \leq N \left|\partial_\xi s(x, \xi) - \partial_\xi s(0, \xi)\right| \leq c_5 \langle (\partial_\xi s, \xi) \rangle$$

with $c_5 > 0$;

$$|(x, \xi) - (x', \xi')| = \left|h_2^{-1}(h_2(x, \xi)) - h_2^{-1}(h_2(x', \xi'))\right| \leq c_5 \left|(\xi, \partial_\xi s(x, \xi)) - (\xi', \partial_\xi s(x', \xi'))\right|. \qquad \square$$

From (3.1), we have

$$\langle(x,y,\xi)\rangle \le \langle(x,\xi)\rangle + \langle y\rangle \le c_6\big[\langle(\xi,\partial_\xi s)\rangle + \langle y\rangle\big]$$

with $c_6 > 0$.

Also, we have $\partial_{y_j}\phi = -\xi_j$ and $\partial_{\xi_j}\phi = \partial_{\xi_j}s - y_j$, so,

$$\langle(\xi,\partial_\xi s)\rangle = \langle(\partial_y\phi, \partial_\xi\phi + y)\rangle \le 2\langle(\partial_y\phi,\partial_\xi\phi,y)\rangle,$$

which for some $c_7 > 0$ finally gives

$$\langle(x,\xi,y)\rangle \le 2c_6\langle(\partial_y\phi,\partial_\xi\phi,y)\rangle \le \frac{1}{c_7}\,\langle(\partial_y\phi,\partial_\xi\phi,y)\rangle.$$

The second inequality in (C_3) is a consequence of (3.1). We can demonstrate condition (C_3^*) using a similar argument. □

Example 3.1. Consider the following function:

$$s(x,\xi) = k_1x^2 + k_2\xi^2 + k_3x\xi,$$

where $k_1, k_2, k_3 \in \mathbb{R}$, $s(x,\xi)$ satisfies (H_1), (H_2) and (H_3).

4 The boundedness of FIO on $H^s(\mathbb{R}^n)$

This section goes over a different collection of bounded operators, or to be more specific, the Hilbert–Schmidt operators. The class of Hilbert–Schmidt operators has a natural Hilbert space structure. Let us start with an elementary proposition.

Proposition 4.1. *Let H be a separable Hilbert space. If $\{e_n\}$ and $\{f_m\}$ are orthonormal bases for H and $A \in \mathcal{L}(H)$, then*

$$\sum_n \|Ae_n\|^2 = \sum_m \|A^*f_m\|^2 = \sum_n\sum_m \big|\langle Ae_n, f_m\rangle\big|^2.$$

Remark 4.1. This result can be taken to mean that one of these infinite sums converges if and only if they all do, in which case the three sums are equal.

Proof. It follows from Parseval's Identity that for each n,

$$\|Ae_n\|^2 = \sum_m |\langle Ae_n, f_m\rangle|^2.$$

Also, for every m,

$$\|A^*f_m\|^2 = \sum_n |\langle e_n, A^*f_m\rangle|^2.$$

□

Definition 4.1. An operator A on H is called a Hilbert–Schmidt operator if

$$\sum_{n=0}^{\infty} \|Ae_n\|_H^2 < +\infty. \tag{4.1}$$

The set of all Hilbert–Schmidt operators is denoted by $\mathcal{C}_2(H)$.

Remark 4.2. The Hilbert–Schmidt norm, also known as the Frobenius norm of the operator A, is denoted by $\|\cdot\|_2$ and is the square root of the left-hand side of (4.1).

Proposition 4.2. *Let A be an operator in $\mathcal{C}_2(H)$.*

(i) $\|A\|_2 = \Big(\sum_n \|Ae_n\|^2\Big)^{\frac{1}{2}}$ *for any basis* $\{e_n\}$.

(ii) $\|A\| \leq \|A\|_2$.

(iii) $\|A^*\|_2 = \|A\|_2$.

(iv) *If* $T \in \mathcal{L}(H)$*, then* $AT, TA \in \mathcal{C}_2(H)$ *and*

$$\max\big\{\|AT\|_2, \|TA\|_2\big\} \leq \|T\| \cdots \|A\|_2$$

Proof. The proof can be found in [20]. □

Now let $\mathbb{R}^n$ be a space with positive measure and $H_1 = H_2 = L^2(\mathbb{R}^n)$. In this situation, the operators $A \in \mathcal{C}_2(H_1, H_2)$ are described as follows.

Theorem 4.1. *The operators* $A \in \mathcal{C}_2(L^2(\mathbb{R}^n))$ *are exactly those which can be represented as*

$$Au(x) = \int\limits_{\mathbb{R}^n} k(x,y)u(y)\,dy, \tag{4.2}$$

with a kernel $k \in L^2(\mathbb{R}^{2n})$*. We then also have*

$$\|A\|_{HS} = \|k\|_{L^2(\mathbb{R}^n)}. \tag{4.3}$$

We have the following results about the Hilbert–Schmidtness of the Fourier integral operator.

Proposition 4.3. *Let* $F_{a,s}$ *be the Fourier integral operators defined by*

$$F_{a,s}u(x) = \iint e^{i[s(x,\xi)-\langle y,\xi\rangle]}a(x,\xi)u(y)\,dy\,\widehat{d\xi},$$

where $a \in \Gamma^m(\mathbb{R}^{2n})$ *and* s *satisfies* (G_1)*,* (G_2) *and* (G_3)*. Then for any* $m \in \omega(\mathbb{R}^{2n})$ *such that* $m \in L^2(\mathbb{R}^{2n})$*,* $F_{a,s}$ *can be extended as a Hilbert–Schmidt operator on* $L^2(\mathbb{R}^n)$*.*

Proof. First, let us observe that the Fourier integral operator $F_{a,s}$ can be written as

$$F_{a,s}u(x) = \int e^{is(x,\xi)}a(x,\xi)\mathcal{F}(u(\xi))\,d\xi,$$

where $\mathcal{F}$ is the Fourier transform.

We put

$$F_{a,s}u = A_{a,s}\mathcal{F}(u). \tag{4.4}$$

Clearly, we have

$$A_{a,s}u(x) = \int\limits_{\mathbb{R}^n} e^{is(x,\xi)}a(x,\xi)u(\xi)\,d\xi.$$

This follows from (4.4) and, using *(iv)* in Proposition 4.2, we have

$$\begin{aligned}\|F_{a,s}\|_{HS} &= \|A_{a,s}\mathcal{F}\|_{HS}\\ &\leq \|A_{a,s}\|_{HS}\|\mathcal{F}\|_{\mathcal{L}(L^2(\mathbb{R}^n))}.\end{aligned}$$

Now, it is enough to prove that $A_{a,s} \in \mathcal{C}_2(L^2(\mathbb{R}^n))$. Let us first note that $A_{a,s}$ has the same integral representation as (4.2) with the kernel $k_{a,s}$.

In fact, straightforward computation shows us that

$$A_{a,s}u(x) := \int\limits_{\mathbb{R}^n} k_{a,s}(x,\xi)u(\xi)\,d\xi,$$

where

$$k_{a,s}(x,\xi) := e^{is(x,\xi)}a(x,\xi).$$

Let us proof that $k_{a,s} \in L^2(\mathbb{R}^{2n})$,

$$\begin{aligned}|k_{a,s}(x,\xi)| &= |e^{is(x,\xi)}a(x,\xi)| \\ &= |a(x,\xi)| \\ &\leq C_{0,0}m(x,\xi),\end{aligned}$$

then

$$|k_{a,s}(x,\xi)|^2 \leq C_{0,0}^2 m^2(x,\xi).$$

So, for all $m \in L^2(\mathbb{R}^{2n})$,

$$k_{a,s} \in L^2(\mathbb{R}^{2n}),$$

and from (4.3) we have

$$\|A_{a,s}\|_{HS} = \|k_{a,s}\|_{L^2(\mathbb{R}^{2n})} < +\infty,$$

which proves that $F_{a,s} \in \mathcal{C}_2(L^2(\mathbb{R}^n))$. □

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