Memoirs on Differential Equations and Mathematical Physics

Volume 86, 2022, 127–138

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EXISTENCE OF AN OPTIMAL ELEMENT FOR A CLASS OF NEUTRAL OPTIMAL PROBLEMS

Abstract. In the paper, for an optimal problem containing neutral differential equation with two types of control, whose right-hand side is linear with respect to the prehistory of the phase velocity, the existence theorems of optimal element are proved. Under the element, we imply the collection of delay parameters, initial vector and control functions.

2010 Mathematics Subject Classification. 49J25.

Key words and phrases. Neutral optimal problem, optimal element, existence theorem.

რეზიუმე. ნაშრომში ოპტიმალური ამოცანისთვის, რომელიც შეიცავს ნეიტრალურ დიფერენციალურ განტოლებას ორი ტიპის მართვით, რომლის მარჯვენა მხარე წრფივია ფაზური სიჩქარის წინაისტორიის მიმართ, დამტკიცებულია ოპტიმალური ელემენტის არსებობის თეორემები. ელემენტის ქვეშ იგულისხმება დაგვიანების პარამეტრების, საწყისი ვექტორისა და მართვის ფუნქციების ერთობლიობა.

1 Introduction

In the present paper, we consider an optimal problem for the controlled neutral differential equation

$$\dot{x}(t) = A(t, x(t), x(t-\theta), v(t))\dot{x}(t-\sigma) + f\big(t, x(t), x(t-\tau), u(t)\big), \quad x(t) \in \mathbb{R}^n, \quad t \in [t_0, t_1], \tag{1.1}$$

with the initial condition

$$x(t) = \varphi(t), \quad t < t_0, \quad x(t_0) = x_0, \tag{1.2}$$

where $v(t)$ and $u(t)$ are bounded piecewise continuous and measurable functions, respectively; $\varphi(t)$ is an absolutely continuous initial function with $|\dot{\varphi}(t)| \leq const$.

The neutral differential equation is a mathematical model of such a system whose behavior at a given moment depends on the velocity and state of the system in the past. Many real processes are described by neutral equations [1, 3, 4]. To illustrate this, here we consider a model of economic growth.

Let $p(t)$ be a quantity of a product produced at the moment t expressed in monetary units. The fundamental principle of economic growth has the form

$$p(t) = a(t) + i(t), \tag{1.3}$$

where $a(t)$ is a quantity of money for the salaries and social programs; $i(t)$ is a quantity of money for the induced investment (purchase of new technologies, etc). We consider the case where the functions $a(t)$ and $i(t)$ have the form

$$a(t) = \alpha(t, p(t), \widehat{u}(t)), \tag{1.4}$$

$$i(t) = \beta\big(t, p(t-\tau), \dot{p}(t), \dot{p}(t-\tau), \widehat{u}(t)\big) + \gamma \ddot{p}(t) + \varsigma\big(t, p(t-\theta), \dot{p}(t), \dot{p}(t-\theta), \widehat{v}(t)\big)\ddot{p}(t-\sigma), \tag{1.5}$$

where $\widehat{u}(t) \in [\widehat{u}_1, \widehat{u}_2]$ and $\widehat{v}(t) \in [\widehat{v}_1, \widehat{v}_2]$ are control functions (investment from the government or from the private firms), with $\widehat{u}_2 > \widehat{u}_1 > 0$ and $\widehat{v}_2 > \widehat{v}_1 > 0$; $\gamma > 0$ is a given number; $\theta > 0$, $\sigma > 0$ and $\tau > 0$ are the so-called delays. Formula (1.5) shows that the value of investment at the moment t depends: on the quantity of money at the moments $t-\tau$ and $t-\theta$ (in the past); on the velocity (production current) at the moments t, $t-\tau$ and $t-\theta$; on the acceleration at the moments t and $t-\sigma$. From formulas (1.3)–(1.5) we get the equation

$$\ddot{p}(t) = \frac{1}{\gamma}\Big[p(t) - \alpha(t, p(t), \widehat{u}(t)) - \beta\big(t, p(t-\tau), \dot{p}(t), \dot{p}(t-\tau), \widehat{u}(t)\big) - \varsigma\big(t, p(t-\theta), \dot{p}(t), \dot{p}(t-\theta), \widehat{v}(t)\big)\ddot{p}(t-\sigma)\Big]$$

which is equivalent to the following controlled neutral equation:

$$\begin{cases} \dot{x}^1(t) = x^2(t), \\ \dot{x}^2(t) = \dfrac{1}{\gamma}\Big[x^1(t) - \alpha(t, x^1(t), \widehat{u}(t)) - \beta\big(t, x^1(t-\tau), x^2(t), x^2(t-\tau), \widehat{u}(t)\big) \\ \qquad\qquad - \varsigma\big(t, x^1(t-\theta), x^2(t), x^2(t-\theta), \widehat{v}(t)\big)\dot{x}^2(t-\sigma)\Big]; \end{cases} \tag{1.6}$$

here, $x^1(t) = p(t)$.

In this paper, under an element we mean the collection of delay parameters θ, σ and τ, initial vector x_0, control functions $v(t)$ and $u(t)$.

The essential novelty here is the existence theorem of an optimal element $(\theta_0, \sigma_0, \tau_0, x_{00}, v_0(\cdot), u_0(\cdot))$ for the optimal problem containing equation (1.1), initial condition (1.2), the general boundary conditions

$$q^i(\theta, \sigma, \tau, x_0, x(t_1)) = 0, \quad i = 1, \dots, l,$$

and the functional

$$q^0(\theta, \sigma, \tau, x_0, x(t_1)) \to \min.$$

The existence theorems for various classes of neutral optimal problems with the fixed delay parameters are given in [5, 7]. For the neutral optimal problems, where $A(t, x(t), x(t-\theta), v(t)) \equiv A(t)$, the existence theorems are proved in [8–10].

The paper is organized as follows. In Section 2, the main theorem and its corollaries are formulated. In Section 3, some auxiliary assertions are given. The main Theorem is proved in Section 4.

2 Statement of problem and the existence theorems

Let $\mathbb{R}^n$ be the n-dimensional vector space of points $x = (x^1, \dots, x^n)^T$, where T means transposition; let $I = [t_0, t_1]$ be a fixed interval and let $\theta_2 > \theta_1 > 0$, $\sigma_2 > \sigma_1 > 0$, $\tau_2 > \tau_1 > 0$ be the given numbers with $t_1 - t_0 > \widehat{\tau} = \max\{\theta_2, \sigma_2, \tau_2\}$. Suppose that $O \subset \mathbb{R}^n$ is an open set and $V \subset \mathbb{R}^m$ and $U \subset \mathbb{R}^r$ are compact sets; the $n \times n$-dimensional matrix function $A(t, x, y, v)$ satisfies the standard conditions: it is continuous on the set $I \times O^2 \times V$ and continuously differentiable with respect to x and y; the function

$$f(t, x, y, u) = \big(f^1(t, x, y, u), \dots, f^n(t, x, y, u)\big)^T$$

satisfies the standard conditions on the set $I \times O^2 \times U$; further, denote by $\Delta = \Delta(I, V, k, L)$ the set of piecewise continuous functions $v : I \to V$ satisfying the conditions:

(a) for each function $v(\,\cdot\,) \in \Delta$, there exists a partition

$$t_0 = \xi_0 < \dots < \xi_{k+1} = t_1$$

of the interval I such that the restriction of the function $v(t)$ satisfies the Lipschitz condition on the open interval (ξ_i, ξ_{i+1}), $i = 0, \dots, k$, i.e.,

$$|v(t') - v(t'')| \leq L|t' - t''|, \ \ \forall\, t', t'' \in (\xi_i, \xi_{i+1}), \ \ i = 0, \dots, k;$$

(b) the numbers k and L do not depend on $v(\,\cdot\,)$.

By $\Omega = \Omega(I, U)$ we denote the set of measurable functions $u : I \to U$. Let

$$q^i : [\theta_1, \theta_2] \times [\sigma_1, \sigma_2] \times [\tau_1, \tau_2] \times X_0 \times O \to \mathbb{R}^1, \ \ i = 0, \dots, l,$$

be continuous functions, where $X_0 \subset O$ is a compact set.

To each element

$$w = (\theta, \sigma, \tau, x_0, v(\,\cdot\,), u(\,\cdot\,)) \in W = [\theta_1, \theta_2] \times [\sigma_1, \sigma_2] \times [\tau_1, \tau_2] \times X_0 \times \Delta \times \Omega$$

we assign the neutral differential equation

$$\dot{x}(t) = A\big(t, x(t), x(t - \theta), v(t)\big)\dot{x}(t - \sigma) + f\big(t, x(t), x(t - \tau), u(t)\big), \ \ t \in I, \tag{2.1}$$

with the initial condition

$$x(t) = \varphi(t), \ \ t \in [t_0 - \widehat{\tau}, t_0), \quad x(t_0) = x_0, \tag{2.2}$$

where $\varphi : [t_0 - \widehat{\tau}, t_0] \to O$ is a given absolutely continuous function with $|\dot{\varphi}(t)| \leq const$.

Definition 2.1. Let $w = (\theta, \sigma, \tau, x_0, v(\,\cdot\,), u(\,\cdot\,)) \in W$. A function $x(t) = x(t; w) \in O$, $t \in I_1 = [t_0 - \widehat{\tau}, t_1]$, is called a solution corresponding to the element w if it satisfies condition (2.2), is absolutely continuous on the interval I and satisfies equation (2.1) almost everywhere (a.e.) on I.

Definition 2.2. An element $w \in W$ is said to be admissible if there exists a corresponding solution $x(t) = x(t; w)$ satisfying the condition

$$q(\theta, \sigma, \tau, x_0, x(t_1)) = 0, \tag{2.3}$$

where $q = (q^1, \dots, q^l)$.

We denote the set of admissible elements by W_0. Now we consider the functional

$$J(w) = q^0(\theta, \sigma, \tau, x_0, x(t_1; w)).$$

Definition 2.3. An element $w_0 = (\theta_0, \sigma_0, \tau_0, x_{00}, v_0(\,\cdot\,), u_0(\,\cdot\,)) \in W_0$ is said to be optimal if

$$J(w_0) = \inf_{w \in W_0} J(w). \tag{2.4}$$

(2.1)–(2.4) is called the neutral optimal problem.

Theorem 2.1. *There exists an optimal element $w_0 \in W_0$ if the following conditions hold:*

2.1. $W_0 \neq \varnothing$;

2.2. *there exists a compact set $K \subset O$ such that for an arbitrary $w \in W_0$,*

$$x(t; w) \in K, \quad t \in I;$$

2.3. *for each fixed $(t, x, y) \in I \times K^2$, the set*

$$G(t, x, y) = \big\{ f(t, x, y, u) : \ u \in U \big\}$$

is convex.

Remark 2.1. Let U be a convex set and

$$f(t, x, y, u) = B(t, x, y) + C(t, x, y)u.$$

Then condition 2.3 of Theorem 2.1 holds.

Now we consider the optimal problem with the integral functional and with fixed ends

$$\begin{gathered}
\dot{x}(t) = A\big(t, x(t), x(t-\theta), v(t)\big)\dot{x}(t-\sigma) + f\big(t, x(t), x(t-\tau), u(t)\big), \quad t \in I, \\
x(t) = \varphi(t), \quad t \in [t_0 - \widehat{\tau}, t_0), \quad x(t_0) = x_0, \quad x(t_1) = x_1, \\
\int\limits_{t_0}^{t_1} \Big[a^0\big(t, x(t), x(t-\theta), v(t)\big)\dot{x}(t-\sigma) + f^0\big(t, x(t), x(t-\tau), u(t)\big) \Big]\, dt \to \min .
\end{gathered}$$

Here, $a^0(t, x, y, v) : I \times O^2 \times V \to \mathbb{R}^n$ and $f^0(t, x, y, u) : I \times O^2 \times U \to \mathbb{R}^1$ are continuous functions, $x_0, x_1 \in O$ are fixed points.

Evidently, this problem is equivalent to the following problem:

$$\begin{gathered}
\dot{x}^0(t) = a^0\big(t, x(t), x(t-\theta), v(t)\big)\dot{x}(t-\sigma) + f^0\big(t, x(t), x(t-\tau), u(t)\big), \\
\dot{x}(t) = A\big(t, x(t), x(t-\theta), v(t)\big)\dot{x}(t-\sigma) + f\big(t, x(t), x(t-\tau), u(t)\big), \quad t \in I, \\
x^0(t_0) = 0, \quad x(t) = \varphi(t), \quad t \in [t_0 - \widehat{\tau}, t_0), \quad x(t_0) = x_0, \quad x(t_1) = x_1, \\
x^0(t_1) \to \min,
\end{gathered}$$

which is a particular case to the similar problem (2.1)–(2.4) in the space $\mathbb{R}^{1+n}$. For the last posed neutral optimal problem, by Z_0 we denote the set of admissible elements $z = (\theta, \sigma, \tau, v(\,\cdot\,), u(\,\cdot\,)) \in Z = [\theta_1, \theta_2] \times [\sigma_1, \sigma_2] \times [\tau_1, \tau_2] \times \Delta \times \Omega$ and by $z_0 = (\theta_0, \sigma_0, \theta_0, v_0(\,\cdot\,), u_0(\,\cdot\,))$ we denote an optimal element (see Definitions 2.2 and 2.3). Let us introduce the function $F = (f^0, f)^T$.

Theorem 2.2. *There exists an optimal element $z_0 \in Z_0$ if the following conditions hold:*

2.4. $Z_0 \neq \varnothing$;

2.5. *there exists a compact set $K_0 \subset \mathbb{R}^1 \times O$ such that for an arbitrary $z \in Z_0$,*

$$(x^0(t; z), x(t; z))^T \in K_0, \quad t \in [t_0, t_1];$$

2.6. *for each fixed $(t, x, y) \in I \times K_0^2$, the set*

$$\big\{ F(t, x, y, u) : \ u \in U \big\}$$

is convex.

Theorem 2.2 follows from a theorem similar to Theorem 2.1 formulated for the space R^{1+n}.

Now we consider the optimal problem for the economic growth model (1.6) with the initial condition

$$x^1(t)=\varphi^1(t),\quad x^2(t)=\varphi^2(t),\ \ t\in[t_0-\widehat{\tau},t_0),\quad x^1(t_0)=x_0^1,\quad x^2(t_0)=x_0^2$$

and with the functional

$$-x^1(t_1)\to\min.$$

Here, $\varphi^1(t)$ and $\varphi^2(t)$ are absolutely continuous functions with $|\dot{\varphi}^1(t)|\le const$ and $|\dot{\varphi}^2(t)|\le const$; x_0^1 and x_0^2 are the fixed numbers. It is assumed that the functions involving in equation (1.6) satisfy the standard conditions on the corresponding sets. In this case, by E_0 we denote the set of admissible elements

$$e=(\theta,\sigma,\tau,\widehat{v}(\,\cdot\,),\widehat{u}(\,\cdot\,))\in E=[\theta_1,\theta_2]\times[\sigma_1,\sigma_2]\times[\tau_1,\tau_2]\times\Delta(I,[\widehat{v}_1,\widehat{v}_2],k_0,L_0)\times\Omega(I,[\widehat{u}_1,\widehat{u}_2])$$

and by $e_0=(\theta_0,\sigma_0,\theta_0,\widehat{v}_0(\,\cdot\,),\widehat{u}_0(\,\cdot\,))$ we denote the optimal element.

Theorem 2.3. *There exists an optimal element $e_0\in E_0$ if the following conditions hold:*

2.7. $E_0\neq\varnothing$;

2.8. *there exists a compact set $K_1\subset\mathbb{R}^1$ such that for an arbitrary $e\in E_0$,*

$$x^2(t;e)\in K_1,\ \ t\in I;$$

2.9. *for each fixed $(t,x^1,x^2,y^1,y^2)\in I\times K_1^4$, the set*

$$\big\{\alpha(t,x^1,\widehat{u})+\beta(t,y^1,x^2,y^2,\widehat{u}):\ \widehat{u}\in[\widehat{u}_1,\widehat{u}_2]\big\}$$

is convex.

It is clear that Theorem 2.3 is a simple corollary of Theorem 2.1.

3 Auxiliary assertions

Theorem 3.1. *Let $v_j(\,\cdot\,)\in\Delta, j=1,2,\dots$. Then there exists a subsequence of the sequence $\{v_j(\,\cdot\,)\}_{j=1}^\infty$ such that it converges to a function $v_0(\,\cdot\,)\in\Delta$ for each $t\in I$, except for not more than k points.*

Proof. By assumption, the function $v_j(t)$, $t\in(\xi_i^j,\xi_{i+1}^j)$, satisfies the Lipschitz condition with $\xi_0^j=t_0$, $\xi_{k+1}^j=t_1$ (see conditions (a) and (b) in the previous section). By virtue of the Cauchy criterion, from this follows the existence of one-sided limits

$$\lim_{t\to\xi_i^j-}v_j(t)=v_{j,i}^-,\ \ i=1,\dots,(k+1),\qquad \lim_{t\to\xi_i^j+}v_j(t)=v_{j,i}^+,\ \ i=0,\dots,k.$$

On the interval I, we set the continuous function

$$\vartheta_{j,i}(t)=\begin{cases}v_{j,i}^+, & t_0\le t\le\xi_i^j,\\ v_j(t), & t\in(\xi_i^j,\xi_{i+1}^j),\\ v_{j,i}^-, & \xi_{i+1}^j\le t\le t_1,\end{cases}$$

and the piecewise continuous function

$$\vartheta_j(t)=\sum_{i=0}^{k-1}\chi\big(t;[\xi_i^j,\xi_{i+1}^j)\big)\vartheta_{j,i}(t)+\chi\big(t;[\xi_k^j,\xi_{k+1}^j]\big)\vartheta_{j,k}(t).$$

Here, $\chi(t;I_0)$ denotes the characteristic function of an interval I_0.

Obviously,

$$\vartheta_j(t) = \vartheta_{j,i}(t) = v_j(t), \ \ t \in (\xi_i^j, \xi_{i+1}^j), \tag{3.1}$$

therefore, $\vartheta_j(\,\cdot\,) \in \Delta$.

For each fixed $i = 0, \dots, k$, the sequence $\{\vartheta_{j,i}(t)\}_{j=1}^{\infty}$ satisfies the conditions

$$\vartheta_{j,i}(t) \in K, \ \ t \in I; \quad \big|\vartheta_{j,i}(t') - \vartheta_{j,i}(t'')\big| \le L|t' - t''|, \ \ \forall\, t', t'' \in I.$$

Therefore, for each fixed $i = 0, \dots, k$, the sequence $\{\vartheta_{j,i}(t)\}_{j=1}^{\infty}$ is uniformly bounded and equicontinuous on I. Thus, by virtue of the Arzelà–Ascoli lemma, from $\{\vartheta_{j,i}(t)\}_{j=1}^{\infty}$ we can pick out a uniformly convergent subsequence and again denote it by $\{\vartheta_{j,i}(t)\}_{j=1}^{\infty}$.

Thus

$$\lim_{j\to\infty} \vartheta_{j,i}(t) = \psi_i(t) \ \text{ uniformly for } \ t \in I$$

and $\psi_i(t)$ satisfies the Lipshitz condition with the constant L.

Without loss of generality, we assume that

$$\lim_{j\to\infty} \xi_i^j = \xi_i, \ \ i = 1, \dots, k.$$

Consequently, we have

$$\lim_{j\to\infty} \chi\big(t; [\xi_i^j, \xi_{i+1}^j)\big) = \lim_{j\to\infty} \chi\big(t; [t_0, \xi_1^j)\big) = \chi\big(t; [\xi_0, \xi_1)\big),$$
$$\lim_{j\to\infty} \chi\big(t; [\xi_i^j, \xi_{i+1}^j)\big) = \chi\big(t; [\xi_i, \xi_{i+1})\big), \ \ t \ne \xi_i, \ \ i = 1, \dots, k-1,$$

and

$$\lim_{j\to\infty} \chi\big(t; [\xi_k^j, \xi_{k+1}^j]\big) = \lim_{j\to\infty} \chi\big(t; [\xi_k^j, t_1]\big) = \chi\big(t; [\xi_k, \xi_{k+1}]\big), \ \ t \ne \xi_k.$$

It is not difficult to see that

$$\lim_{j\to\infty} \vartheta_j(t) = \vartheta_0(t) = \sum_{i=0}^{k-1} \chi\big(t; [\xi_i, \xi_{i+1})\big)\psi_i(t) + \chi\big(t; [\xi_k, \xi_{k+1}]\big)\psi_k(t)$$

for each $t \in I$ except for not more than k points ξ_i, $i = 1, \dots, k$, besides $\vartheta_0(\,\cdot\,) \in \Delta$. Taking into account (3.1), we can conclude that

$$\lim_{j\to\infty} v_j(t) = v_0(t) := \vartheta_0(t), \ \ t \in I, \ \ t \ne \xi_i, \ \ i = 1, \dots, k. \qquad \square$$

Theorem 3.2. *Let $x_i(t) \in K$, $t \in I_1$, $i = 1, 2, \dots$, be a solution corresponding to the element $w_i = (\theta_i, \sigma_i, \tau_i, x_{0i}, v_i(\,\cdot\,), u_i(\,\cdot\,)) \in W$, $i = 1, 2, \dots$, and*

$$\lim_{i\to\infty} \sigma_i = \sigma_0. \tag{3.2}$$

Then there exists a number $M > 0$ such that for a sufficiently large i_0,

$$|\dot{x}_i(t)| \le M, \ \ t \in I_1, \ \ i \ge i_0. \tag{3.3}$$

Proof. Let $t \in [t_0 - \widehat{\tau}, t_0)$, then $|\dot{x}_i(t)| = |\dot{\varphi}(t)| \le M_0 = const$. It is not difficult to see that for a sufficiently large i_0, we have

$$\Big[\frac{t_1 - t_0}{\sigma_i}\Big] = \Big[\frac{t_1 - t_0}{\sigma_0}\Big] = d, \ \ i \ge i_0$$

(see (3.2)), i.e.,

$$t_0 + d\sigma_i \le t_1 < t_0 + (d+1)\sigma_i,$$

where $[\alpha]$ means the integer part of a number α.

If $t \in [t_0, t_0 + \sigma_i)$, then

$$|\dot{x}_i(t)| = \Big|A\big(t, x_i(t), x_i(t-\theta_i), v_i(t)\big)\dot{\varphi}(t-\sigma_i) + f\big(t, x_i(t), x_i(t-\tau_i), u_i(t)\big)\Big| \le \|A\|M_0 + N = M_1,$$

where

$$\|A\| = \sup\Big\{|A(t,x,y,v)| : \ (t,x,y,v) \in I \times K \times \big(K \cup \varphi([t_0-\widehat{\tau}, t_0]), V\big)\Big\},$$
$$N = \sup\Big\{|f(t,x,y,u)| : \ (t,x,y,u) \in I \times K \times \big(K \cup \varphi([t_0-\widehat{\tau}, t_0]), U\big)\Big\}.$$

Let $t \in [t_0 + \sigma_i, t_0 + 2\sigma_i)$, then

$$|\dot{x}_i(t)| \le \|A\| \, |\dot{x}_i(t-\sigma_i)| + N \le \|A\|M_1 + N = M_2.$$

Continuing this process, we obtain

$$|\dot{x}_i(t)| \le \|A\|M_{j-1} + N = M_j, \ \ t \in [t_0 + (j-1)\sigma_i, t_0 + j\sigma_i), \ \ j = 3, \dots, d.$$

Moreover, if $t_0 + d\sigma_i < t_1$, then we have

$$|\dot{x}_i(t)| \le M_{d+1}, \ \ t \in [t_0 + d\sigma_i, t_1].$$

It is clear that for $M = \max\{M_0, \dots, M_{d+1}\}$ condition (3.3) is fulfilled. □

Theorem 3.3 ([2,6])**.** *Let $g(t,u) \in \mathbb{R}^n$ be a continuous function on the set $I \times U$ and let the set*

$$G(t) = \big\{g(t,u) : \ u \in U\big\}$$

be convex and

$$g_i(\,\cdot\,) \in L_1(I), \ \ g_i(t) \in G(t) \ \ a.e. \ on \ I, \ \ i = 1, 2, \dots .$$

Moreover,

$$\lim_{i\to\infty} g_i(t) = g(t) \ \ weakly \ on \ I.$$

Then

$$g(t) \in G(t) \ \ a.e. \ on \ I$$

and there exists a measurable function $u(t) \in U$, $t \in I$ such that

$$g(t, u(t)) = g(t) \ \ a.e. \ on \ I.$$

4 Proof of Theorem 2.1

Let

$$w_i = \big(\theta_i, \sigma_i, \tau_i, x_{0i}, v_i(\,\cdot\,), u_i(\,\cdot\,)\big) \in W_0, \ \ i = 1, 2, \dots ,$$

be a minimizing sequence, i.e.,

$$\lim_{i\to\infty} J(w_i) = \widehat{J} = \inf_{w \in W_0} J(w).$$

Without loss of generality, we assume that

$$\lim_{i\to\infty} \theta_i = \theta_0, \quad \lim_{i\to\infty} \sigma_i = \sigma_0, \quad \lim_{i\to\infty} \tau_i = \tau_0, \quad \lim_{i\to\infty} x_{0i} = x_{00}$$

and

$$\lim_{i\to\infty} v_i(t) = v_0(t)$$

for each $t \in I$, except for not more than k points.

Let $x_i(t)$, $t \in I_1$, be a solution corresponding to the element $w_i \in W_0$. By assumption, $x_i(t) = \varphi(t)$, $t \in [t_0 - \widehat{\tau}, t_0)$, and $x_i(t) \in K$, $t \in I$. For $\forall\, t', t'' \in I$, we have

$$|x_i(t') - x_i(t'')| \le \bigg|\int\limits_{t''}^{t'} |\dot{x}_i(t)|\, dt\bigg| \le M|t' - t''|, \ \ i \ge i_0$$

(see Theorem (3.2)).

The sequence $x_i(t)$, $t \in I$, $i \ge i_0$, is uniformly bounded and equicontinuous. By the Arzelà–Ascoli lemma, from this sequence we can extract a subsequence, which will again be denoted by $x_i(t)$, $i \ge i_0$, such that

$$\lim_{i\to\infty} x_i(t) = y_0(t) \ \text{ uniformly in } I.$$

Thus

$$\lim_{i\to\infty} x_i(t) = x_0(t) \ \text{ uniformly in } I_1,$$

where

$$x_0(t) = \begin{cases} \varphi(t), & t \in [t_0 - \widehat{\tau}, t_0), \\ y_0(t), & t \in I. \end{cases}$$

Further, by the Dunford–Pettis theorem, from the sequence $\dot{x}_i(\,\cdot\,) \in L_1(I_1)$, $i \ge i_0$, we can extract a subsequence, which will again be denoted by $\dot{x}_i(t)$, $i \ge i_0$, such that

$$\lim_{i\to\infty} \dot{x}_i(t) = \gamma(t) \ \text{ weakly in } I_1.$$

Obviously, on the interval I, we get

$$x_0(t) = \lim_{i\to\infty} x_i(t) = \lim_{i\to\infty} \bigg[x_{0i} + \int\limits_{t_0}^{t} \dot{x}_i(s)\, ds\bigg] = x_{00} + \int\limits_{t_0}^{t} \gamma(s)\, ds.$$

Thus $\dot{x}_0(t) = \gamma(t)$, i.e.,

$$\lim_{i\to\infty} \dot{x}_i(t) = \dot{x}_0(t) \ \text{ weakly in } I_1.$$

We have

$$x_i(t) = x_{0i} + z_{1i}(t) + z_{2i}(t), \ \ t \in I, \ \ i \ge i_0, \tag{4.1}$$

where

$$z_{1i}(t) = \int\limits_{t_0}^{t} A\big(s, x_i(s), x_i(s - \theta_i), v_i(s)\big)\dot{x}_i(s - \sigma_i), \quad z_{2i}(t) = \int\limits_{t_0}^{t} f\big(s, x_i(s), x_i(s - \tau_0), u_i(s)\big)\, ds.$$

First of all, we transform the expression $z_{1i}(t)$ for $t \in I$ and obtain

$$z_{1i}(t) = z_{1i}^1(t) + z_{1i}^2,$$

where

$$z_{1i}^1(t) = \int\limits_{t_0}^{t} \Big[A\big(s, x_i(s), x_i(s - \theta_i), v_i(s)\big) - A\big(s, x_0(s), x_0(s - \theta_0), v_0(s)\big)\Big]\dot{x}_i(s - \sigma_i)\, ds$$

and

$$z_{1i}^2(t) = \int\limits_{t_0}^{t} A\big(s, x_0(s), x_0(s - \theta_0), v_0(s)\big)\dot{x}_i(s - \sigma_i)\, ds.$$

It is not difficult to see that

$$\lim_{i\to\infty}\Big[A\big(s,x_i(s),x_i(s-\theta_i),v_i(s)\big)-A\big(s,x_0(s),x_0(s-\theta_0),v_0(s)\big)\Big]=0$$

a.e. on I and

$$\lim_{i\to\infty}\dot x_i(s-\sigma_i)=\dot x_0(t-\sigma_0),\ \text{ weakly in } I.$$

Therefore,

$$\lim_{i\to\infty}z^1_{1i}(t)=0,$$

$$\lim_{i\to\infty}z^2_{1i}(t)=\int\limits_{t_0}^{t}A\big(s,x_0(s),x_0(s-\theta_0),v_0(s)\big)\dot x_0(s)\,ds.$$

Thus

$$\lim_{i\to\infty}z_{1i}(t)=\int\limits_{t_0}^{t}A\big(s,x_0(s),x_0(s-\theta_0),v_0(s)\big)\dot x_0(s-\sigma_0)\,ds. \tag{4.2}$$

Now we transform the expression $z_{2i}(t)$ for $t\in I$ and get

$$z_{2i}(t)=z^1_{2i}(t)+z^2_{2i}(t),$$

where

$$z^1_{2i}(t)=\int\limits_{t_0}^{t}f\big(s,x_0(s),x_0(s-\tau_0),u_i(s)\big)\,ds$$

and

$$z^2_{2i}(t)=\int\limits_{t_0}^{t}\Big[f\big(s,x_i(s),x_i(s-\tau_i),u_i(s)\big)-f\big(s,x_0(s),x_0(s-\tau_0),u_i(s)\big)\Big]\,ds.$$

From the sequence

$$f_i[s]=f\big(s,x_0(s),x_0(s-\tau_0),u_i(s)\big)\in G(s,x_0(s),x_0(s-\tau_0)),\ \ i\ge i_0,\ \ s\in I,$$

we extract a subsequence, which will again be denoted by $f_i[s]$, $i\ge i_0$, such that

$$\lim_{i\to\infty}f_i[s]=f_0[s]\ \text{ weakly in the space } L_1(I).$$

By Theorem 3.3,

$$f_0[s]\in G(s,x_0(s),x_0(s-\tau_0))$$

and there exists a function $u_0(\cdot)\in\Omega$ such that

$$f_0[s]=f\big(s,x_0(s),x_0(s-\tau_0),u_0(s)\big).$$

Consequently,

$$\lim_{i\to\infty}z^1_{2i}(t)=\int\limits_{t_0}^{t}f_0[s]\,ds=\int\limits_{t_0}^{t}f\big(s,x_0(s),x_0(s-\tau),u_0(s)\big)\,ds. \tag{4.3}$$

Next,

$$\lim_{i\to\infty}\Big[f\big(s,x_i(s),x_i(s-\tau_i),u\big)-f\big(s,x_0(s),x_0(s-\tau_0),u\big)\Big]=0$$

a.e. for $s\in I$ and uniformly for $u\in U$, i.e.,

$$\lim_{i\to\infty}z^2_{2i}(t)=0,\ \ t\in I.$$

Thus

$$\lim_{i\to\infty} z_{2i}(t) = \int_{t_0}^{t} f\big(s, x_0(s), x_0(s-\tau), u_0(s)\big)\, ds. \tag{4.4}$$

From (4.1), taking into account (4.2)–(4.4), we obtain

$$\begin{aligned}\lim_{i\to\infty} x_i(t) &= x_0(t)\\ &= x_{00} + \int_{t_0}^{t} \Big[A\big(s, x_0(s), x_(s-\theta_0), v_0(s)\big)\dot{x}(s-\sigma_0) + f\big(s, x_0(s), x_0(s-\tau), u_0(s)\big)\Big]\, ds, \ \ t \in I.\end{aligned}$$

The function $x_0(t)$, $t \in I_1$, on the interval $[t_0 - \widehat{\tau}, t_0]$ satisfies the initial condition

$$x_0(t) = \varphi(t), \ \ t \in [t_0 - \widehat{\tau}, t_0), \quad x_0(t_0) = x_{00},$$

and on the interval $[t_0, t_1]$ satisfies the differential equation

$$\dot{x}_0(t) = A\big(t, x_0(t), x_0(t-\theta_0), v_0(t)\big)\dot{x}_0(t-\sigma_0) + f\big(t, x_0(t), x_0(t-\tau_0), u_0(t)\big).$$

Clearly, the function $x_0(t)$ is the solution corresponding to the element

$$w_0 = (\theta_0, \sigma_0, \tau_0, x_{00}, v_0(\,\cdot\,), u_0(\,\cdot\,)) \in W$$

and satisfying the condition

$$q(\theta_0, \sigma_0, \tau_0, x_{00}, x_0(t_1)) = 0,$$

i.e., $w_0 \in W_0$ and $x_0(t) = x(t; w_0)$. Moreover,

$$\widehat{J} = \lim_{i\to\infty} q^0(\theta_i, \sigma_i, \tau_i, x_{0i}, x_i(t_1)) = q^0(\theta_0, \sigma_0, \tau_0, x_{00}, x_0(t_1)) = J(w_0).$$

Thus the optimality of the element w_0 is proved.

References

[1] R. D. Driver, A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics. 1963 *Internat. Sympos. Nonlinear Differential Equations and Nonlinear Mechanics*, pp. 474–484, *Academic Press, New York*.

[2] A. F. Filippov, On certain questions in the theory of optimal control. *J. SIAM Control Ser. A* **1** (1962), 76–84.

[3] K. P. Hadeler, Neutral delay equations from and for population dynamics. *The 8th Colloquium on the Qualitative Theory of Differential Equations*, No. 11, 18 pp., Proc. Colloq. Qual. Theory Differ. Equ., 8, *Electron. J. Qual. Theory Differ. Equ., Szeged*, 2008.

[4] Dzh. Kheǐl, *Theory of Functional-Differential Equations.* (Russian) "Mir", Moscow, 1984.

[5] B. Sh. Mordukhovich, *Approximation Methods in Problems of Optimization and Control.* (Russian) "Nauka", Moscow, 1988.

[6] C. Olech, Existence theorems for optimal problems with vector-valued cost function. *Trans. Amer. Math. Soc.* **136** (1969), 159–180.

[7] T. A. Tadumadze, Existence of a solution in optimal problems of neutral type. (Russian) *Soobshch. Akad. Nauk Gruzin. SSR* **97** (1980), no. 1, 33–36.

[8] T. Tadumadze, Existence of optimal initial data and continuity of the integral functional minimum with respect to perturbations for a class of neutral differential equation. *Semin. I. Vekua Inst. Appl. Math. Rep.* **36(37)** (2010/11), 37–40, 109.

[9] T. Tadumadze and N. Gorgodze, Variation formulas of a solution and initial data optimization problems for quasi-linear neutral functional differential equations with discontinuous initial condition. *Mem. Differ. Equ. Math. Phys.* **63** (2014), 1–77.

[10] T. Tadumadze and A. Nachaoui, On the existence of an optimal element in quasi-linear neutral optimal problems. *Semin. I. Vekua Inst. Appl. Math. Rep.* **40** (2014), 50–67.

(Received 21.07.2021; accepted 25.10.2021)

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