Memoirs on Differential Equations and Mathematical Physics

Volume 85, 2022, 21–33

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ON THE WELL-POSEDNESS OF THE WEIGHTED CAUCHY PROBLEM FOR SYSTEMS OF LINEAR IMPULSIVE DIFFERENTIAL EQUATIONS WITH SINGULARITIES

Abstract. The weighted Cauchy problem is considered for systems of linear impulsive differential equations with singularities. The singularity is considered in the sense that the matrix-and vector-functions corresponding to the impulsive system are not, in general, integrable at the initial point. Sufficient conditions are established for so-called H-well-posedness of the problem.

2010 Mathematics Subject Classification. 34A12, 34A30, 34A37, 34K26.

Key words and phrases. Impulsive differential equation, linear system, singularity, the weighted Cauchy problem, H-well-posedness.

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1 Statement of the problem and basic notation

Let $I = [a, b] \subset \mathbb{R}$ be a finite and closed interval non-degenerate in the point, $t_0 \in I$ and $I_{t_0} = I \setminus \{t_{t_0}\}$.

Consider the linear system of impulsive differential equations

$$\frac{dx}{dt} = P(t)x + q(t) \ \text{ for a.a. } \ t \in I_{t_0} \setminus T, \tag{1.1}$$

$$x(\tau_l+) - x(\tau_l-) = G(l)x(\tau_l) + g(l) \ \ (l = 1, 2, \dots), \tag{1.2}$$

where $T = \{\tau_1, \tau_2, \dots\}$, $\tau_l \in I_{t_0}$ $(l = 1, 2, \dots)$ are points of impulsive actions such that $\lim\limits_{l\to+\infty} \tau_l = t_0$,

$$P = (p_{ik})_{i,k=1}^n \in L_{loc}(I_{t_0}; \mathbb{R}^{n\times n}) \ \text{ and } \ q = (q_k)_{k=1}^n \in L_{loc}(I_{t_0}; \mathbb{R}^n)$$

are matrix-and vector-functions, respectively, and

$$G = (g_{ik})_{i,k=1}^n \in E(\mathbb{N}; \mathbb{R}^{n\times n}) \ \text{ and } \ g = (g_k)_{k=1}^n \in E(\mathbb{N}; \mathbb{R}^n)$$

are matrix- and vector-functions of the discrete argument.

Let $H = \operatorname{diag}(h_1, \dots, h_n) : I_{t_0} \to \mathbb{R}^{n\times n}$ be a diagonal matrix-function with continuous diagonal elements $h_k : I_{t_0} \to]0, +\infty[$ $(k = 1, \dots, n)$.

We consider the problem of finding a solution $x : I_{t_o} \to R^n$ of system (1.1), (1.2), satisfying the condition

$$\lim_{t\to t_0} (H^{-1}(t)x(t)) = 0. \tag{1.3}$$

Along with system (1.1), (1.2), we consider the perturbed singular system

$$\frac{dx}{dt} = \widetilde{P}(t)x + \widetilde{q}(t) \ \text{ for a.a. } \ t \in I_{t_0} \setminus T, \tag{1.4}$$

$$x(\tau_l+) - x(\tau_l-) = \widetilde{G}(l)x(\tau_l) + \widetilde{g}(l) \ \ (l = 1, 2, \dots), \tag{1.5}$$

under condition (1.3), where $\widetilde{P} = (\widetilde{p}_{ik})_{i,k=1}^n \in L_{loc}(I_{t_0}; \mathbb{R}^{n\times n})$ and $\widetilde{q} = (\widetilde{q}_k)_{k=1}^n \in L_{loc}(I_{t_0}; \mathbb{R}^n)$ are matrix- and vector-functions, respectively, and $\widetilde{G} = (\widetilde{g}_{ik})_{i,k=1}^n \in E(\mathbb{N}; \mathbb{R}^{n\times n})$ and $\widetilde{g} = (\widetilde{g}_k)_{k=1}^n \in E(\mathbb{N}; \mathbb{R}^n)$ are matrix- and vector-functions of the discrete argument.

As we know, the initial problem for systems of ordinary differential equations with singularities first have been fundamentally investigated by V. A. Chechik in [8], where the sufficient conditions for existence and uniqueness of solutions of the problem and some related questions are given.

The modified Cauchy and some other problems (among them the well-posedness question) for systems of ordinary differential equations with singularities, i.e., for problem (1.1), (1.3), are investigated, for example, in the papers [8–11] (see also the references therein).

The singularity of system (1.1) is considered in the sense that the matrix-function P or the vector-function q are not, in general, integrable at the point t_0, i.e., on some $]c, d[$ from I such that $t_0 \in]c, d[$. So, in general, the solution of problem (1.1), (1.3) is not continuous at the point t_0 and, therefore, it is not a solution in the classical sense. But its restriction to every interval from I_{t_0} is a solution of system (1.1) in the classical sense. To illustrate this we give the following example from [11].

Let $\alpha > 0$ and $\varepsilon \in]0, \alpha[$. Then the problem

$$\frac{dx}{dt} = -\frac{\alpha\, x}{t} + \varepsilon |t|^{\varepsilon-1-\alpha},$$

$$\lim_{t\to 0} (t^\alpha x(t)) = 0$$

has the unique solution $x(t) = |t|^{\varepsilon-\alpha} \operatorname{sgn} t$. This function is not a solution of the equation on the set $I = \mathbb{R}$, but its restrictions to $] - \infty, 0[$ and $]0, +\infty[$ are solutions of that equation.

The existence and uniqueness question of the considered in the paper problem is investigated in [3,5,6]. As for the well-posedness question, as we know, such a problem for the impulsive differential problem (1.1), (1.2); (1.3) has not been investigated. So, the present research is quite relevant.

The theory of the regular impulsive differential equations has been investigated in earlier papers (see, for example, [1, 2, 4, 7, 12–14] and the references therein). As for the singular case, the corresponding theory, as we know, is far enough from deep research. Some boundary value problems for linear impulsive systems with singularities are investigated in [3].

In the present paper, we give sufficient conditions for so-called H-well-posedness of problem (1.1), (1.2); (1.3). The analogous results for the Cauchy problem for ordinary differential systems with singularities (1.1) belong to I. Kiguradze [9–11].

In the paper, the use is made of the following notation and definitions.

- $\mathbb{N} = \{1, 2, \dots\}$.
- $\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}_+ = [0, +\infty[\,$, $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.
- $\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{i,j})_{i,j=1}^{n,m}$ with the norm
$$\|X\| = \max_{j=1,\dots,m} \sum_{i=1}^{n} |x_{ij}|.$$
- $O_{n\times m}$ (or O) is the zero $n \times m$-matrix. 0_n (or 0) is the zero n-vector.
- If $X = (x_{ik})_{i,k=1}^{n,m} \in \mathbb{R}^{n\times m}$, then
$$|X| = \big(|x_{ik}|\big)_{i,k=1}^{n,m}, \quad [X]_+ = \frac{|X| + X}{2}\,, \quad [X]_- = \frac{|X| - X}{2}\,.$$
- $\mathbb{R}_+^{n\times m} = \big\{(x_{ij})_{i,j=1}^{n,m} :\ x_{ij} \geq 0 \ (i = 1, \dots, n;\ j = 1, \dots, m)\big\}$.
- $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.
- If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X; I_n is the identity $n \times n$-matrix.
- The inequalities between the matrices are understood componentwisely.
- A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.
- $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a, b] \to \mathbb{R}^{n\times m}$ at the point t.
- $AC([a, b]; \mathbb{R}^{n\times m})$ is the set of all absolutely continuous matrix-functions $X : [a, b] \to \mathbb{R}^{n\times m}$.
- $AC_{loc}(I_{t_0,T}; \mathbb{R}^{n\times m})$, where $I_{t_0,T} = I_{t_0} \setminus T$, is the set of all matrix-functions whose restrictions to every $[a, b] \subset I_{t_0,T}$ belong to $AC([a, b]; \mathbb{R}^{n\times m})$.
- If $\alpha < \beta$, then $N_{\alpha,\beta} = \{l \in \mathbb{N} :\ \alpha \leq \tau_l < \beta\}$.
- $L([a, b]; \mathbb{R}^{n\times m})$ is the set of all integrable matrix-functions $X : [a, b] \to \mathbb{R}^{n\times m}$.
- $L_{loc}(I_{t_0}; \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I_{t_0} \to D$ whose restrictions to every closed interval $[a, b]$ from I_{t_0} belong to $L([a, b]; \mathbb{R}^{n\times m})$.
- $E(M; \mathbb{R}^{n\times m})$, where $M \subset \mathbb{N}$, is the set of all discrete matrix-functions from M into $\mathbb{R}^{n\times m}$.
- A vector-function $x \in AC_{loc}(I_{t_0,T}; \mathbb{R}^n)$ is said to be a solution of system (1.1), (1.2) if
$$x'(t) = P(t)x(t) + q(t) \ \text{ for a.a. } \ t \in I_{t_0,T}$$
and there exist one-sided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1, 2, \dots)$ satisfying equalities (1.2).

We consider problem (1.1), (1.2); (1.3) only in the case $t_0 = b$. Similarly, we can consider the case $t_0 = a$. The general case $t_0 \in]a, b[$ will be reduced to the given two cases.

In the considered case, without loss of generality, we can assume that the solution x of the impulsive differential system (1.1), (1.2) is continuous from the left in the points of the impulsive actions τ_l $(l = 1, 2, \dots)$, i.e., $x(\tau_l) = x(\tau_l-)$ $(l = 1, 2, \dots)$.

We assume that
$$\det(I_n + G(l)) \neq 0 \ \ (l = 1, 2, \dots). \tag{1.6}$$
The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e., for the case when $P \in L(I; \mathbb{R}^{n\times n})$ and $q \in L(I; \mathbb{R}^n)$ (see [4, 12–14]).

Remark 1.1. In the case $t_0 = a$, we assumed that the solutions are continuous from the right in the impulsive actions, then we would assume that the condition

$$\det(I_n - G(l)) \neq 0 \ \ (l = 1, 2, \dots)$$

holds instead of condition (1.6). In the general case, i.e., where $t_0 \in]\min I, \sup I[$, we assumed that the solutions are continuous from the left in the impulsive actions τ_l $(l = 1, 2, \dots)$ for which $\tau_l < t_0$, and continuous from the right if $\tau_l > t_0$. In this case, we would assume that

$$\det(I_n + (-1)^j G(l)) \neq 0 \ \text{ for } \ (-1)^j(\tau_l - t_0) < 0 \ \ (j = 1, 2; \ l = 1, 2, \dots).$$

Let $P_0 \in L_{loc}(I_{t_0}; \mathbb{R}^{n\times n})$ and $G_0 \in E(\mathbb{N}; \mathbb{R}^{n\times n})$. Then a matrix-function $C_0 : I_{t_0} \times I_{t_0} \to \mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous impulsive differential system

$$\frac{dx}{dt} = P_0(t)x, \tag{1.7}$$

$$x(\tau_l+) - x(\tau_l-) = G_0(l)x(\tau_l) \ \ (l = 1, 2, \dots), \tag{1.8}$$

if for every interval $J \subset I_{t_0}$ and $\tau \in J$, the restriction of the matrix-function $C_0(., \tau) : I_{t_0} \to \mathbb{R}^{n\times n}$ to J is the fundamental matrix of system (1.7), (1.8) satisfying the condition $C_0(\tau, \tau) = I_n$. Therefore, C_0 is the Cauchy matrix of system (1.7), (1.8) if and only if the restriction of C_0 to $J \times J$, for every interval $J \subset I_{t_0}$, is the Cauchy matrix of the system in the sense of definition given in [11].

2 Formulation of the main results

Definition 2.1. Problem (1.1), (1.2); (1.3) is said to be H-well-posed if it has a unique solution x and for every $\varepsilon > 0$ there exists $\eta > 0$ such that problem (1.4), (1.5); (1.3) has a unique solution y and the estimate

$$\|H(t)\,(x(t) - y(t))\| < \varepsilon \ \text{ for } \ t \in I \tag{2.1}$$

holds for every matrix-functions $\widetilde{P} \in L_{loc}(I_{t_0,T}; R^{n\times n})$, $\widetilde{G} \in E(\mathbb{N}; \mathbb{R}^{n\times n})$ and vector-functions $\widetilde{q} \in L_{loc}(I_{t_0,T}; R^n)$, $\widetilde{g} \in E(\mathbb{N}; \mathbb{R}^n)$ such that

$$\det(I_n + \widetilde{G}(l)) \neq 0 \ \ (l = 1, 2, \dots), \tag{2.2}$$

$$\int\limits_t^{t_0} \big\|H^{-1}(\tau)(\widetilde{P}(s) - P(s))H(s)\big\|\,ds + \sum_{l\in N_{t,t_0}} \big\|H^{-1}(\tau_l)(\widetilde{G}(l) - G(l))H(\tau_l)\big\| < \eta \ \text{ for } \ t \in I, \tag{2.3}$$

and

$$\int\limits_t^{t_0} \big\|H^{-1}(\tau)(\widetilde{q}(s) - q(s))\big\|\,ds + \sum_{l\in N_{t,t_0}} \big\|H^{-1}(\tau_l)(\widetilde{g}(l) - g(l))\big\| < \eta \ \text{ for } \ t \in I. \tag{2.4}$$

Theorem 2.1. *Let there exist a matrix-function $P_0 \in L_{loc}(I_{t_0,T}; R^{n\times n})$, a discrete matrix-function $G_0 \in E(\mathbb{N}; \mathbb{R}^{n\times n})$ and constant matrices $B_0, B \in \mathbb{R}_+^{n\times n}$ such that the conditions*

$$\det(I_n + G_0(l)) \neq 0 \ \ (l = 1, 2, \dots), \tag{2.5}$$

$$r(B) < 1 \tag{2.6}$$

hold, and the estimates

$$|C_0(t, \tau)| \le H(t)B_0H^{-1}(\tau) \ \ \textit{for} \ \ t \le \tau, \ \ \tau \notin T, \tag{2.7}$$

$$\big|C_0(t, \tau_l)\,(I_n + G_0(l))^{-1}\big| \le H(t)B_0H^{-1}(\tau_l) \ \ (l = 1, 2, \dots), \tag{2.8}$$

and

$$\int_t^{t_0} |C_0(t,\tau)(P(s)-P_0(s))H(s)|\,ds \\ + \sum_{l\in N_{t,t_0}} \big|C_0(t,\tau_l)(I_n+G_0(l))^{-1}(G(l)-G_0(l))H(\tau_l)\big| \le H(t)B \tag{2.9}$$

are satisfied on the interval $[t_0-\delta,t_0[$ *for some* $\delta>0$, *where* C_0 *is the Cauchy matrix of system* (1.7), (1.8). *Let, moreover,*

$$\lim_{t\to t_0}\bigg(\int_t^{t_0} \big\|H^{-1}(s)C_0(t,s)q(s)\big\|\,ds + \sum_{l\in N_{t,t_0}} \big\|H^{-1}(\tau_l)C_0(t,\tau_l)(I_n+G_0(l))^{-1}g(l)\big\|\bigg) = 0. \tag{2.10}$$

Then problem (1.1), (1.2); (1.3) *is* H*-well-posed.*

Theorem 2.2. *Let there exist a constant matrix* $B=(b_{ik})_{i,k=1}^n\in\mathbb{R}_+^{n\times n}$ *such that condition* (2.6) *holds, and the estimates*

$$c_i(t,\tau)\le b_0\,\frac{h_i(t)}{h_i(\tau)} \ \ \text{for} \ \ t\le\tau \ \ (i=1,\dots,n), \tag{2.11}$$

$$\bigg|\int_t^{t_0} c_i(t,\tau)h_i(\tau)[p_{ii}(\tau)]_-\,d\tau + \sum_{l\in N_{t,t_0}} c_i(t_l,\tau_l)h_i(\tau_l)[g_{ii}(l)]_-\bigg| \le b_{ii}h_i(t) \ \ (i=1,\dots,n) \tag{2.12}$$

and

$$\bigg|\int_t^{t_0} c_i(t,\tau)h_k(\tau)|p_{ik}(\tau)|\,d\tau + \sum_{l\in N_{t,t_0}} c_i(t,\tau_l)\,[g_{ii}(l)]_+\cdot(1+[g_{ii}(l)]_+)^{-1}h_k(\tau_l)g_{ik}(l)\bigg| \\ \le b_{ik}h_i(t) \ \ (i\ne k;\ \ i,k=1,\dots,n) \tag{2.13}$$

hold on $[t_0-\delta,t_0[$ *for some* $b_0>0$ *and* $\delta>0$. *Let, moreover,*

$$\lim_{t\to t_0}\bigg(\int_t^{t_0}\frac{c_i(t,\tau)}{h(t)}q(\tau)\,d\tau + \sum_{l\in N_{t,t_0}}\frac{c_i(t,\tau_l)}{h_i(t)}\,[g_{ii}(l)]_+\cdot\big(1+[g_{ii}(l)]_+\big)^{-1}g(l)\bigg) = 0 \ \ (i=1,\dots,n), \tag{2.14}$$

where c_i *is the Cauchy function of the impulsive differential equation*

$$\frac{dx}{dt} = p_{0i}(t)x,$$
$$x(\tau_l+)-x(\tau_l-) = g_{0i}(l)x(\tau_l) \ \ (l=1,2,\dots);$$

here

$$p_{0i}(t)\equiv[p_{ii}(t)]_+,\quad g_{0i}(l)\equiv[g_{ii}(l)]_+ \ \ (i=1,\dots,n).$$

Then problem (1.1), (1.2); (1.3) *is* H*-wellposed.*

Remark 2.1. The Cauchy functions $c_i(t,\tau)$ $(i=1,\dots,n)$ have the form

$$c_i(t,\tau) = \begin{cases} \exp\bigg(\displaystyle\int_\tau^t [p_{ii}(s)]_+\,ds\bigg)\displaystyle\prod_{l\in T_{\tau,t}}\big(1+[g_{ii}(l)]_+\big) & \text{if } t>\tau, \\ \exp\bigg(\displaystyle\int_\tau^t [p_{ii}(s)]_+\,ds\bigg)\displaystyle\prod_{l\in T_{t,\tau}}\big(1+[g_{ii}(l)]_+\big)^{-1} & \text{if } t<\tau, \\ 1 & \text{if } t=\tau \end{cases} \tag{2.15}$$

for $t,\tau\in I$.

Remark 2.2. In Theorems 2.1 and 2.2, the strict inequality (2.6) cannot be replaced by the non-strict one. We give the corresponding example from [11] for ordinary differential equations, i.e., when $G(l) \equiv O_{n\times n}$ and $g(l) \equiv 0_n$.

On the interval $]-1,0[$ consider the problem

$$\frac{dx}{dt} = \frac{x}{t} + \frac{1}{|\ln|t||}\,, \tag{2.16}$$

$$\lim_{t\to 0} \frac{x(t)}{t} = 0. \tag{2.17}$$

Every solution of equation (2.16) has the form

$$x(t) = ct + t\ln|\ln|t|| \ \ (c \in \mathbb{R}).$$

So, problem (2.16), (2.17) is not solvable. On the other hand, the Cauchy function $c(t,\tau)$ has the form $c(t,\tau) = t\tau^{-1}$ for $t \le \tau < 0$ and the conditions of Theorem 2.1, except of condition (2.6), are fulfilled (on $]-1,0[$) for $n=1$, $P(t) \equiv t^{-1}$, $q(t) \equiv |\ln|t||^{-1}$ and $h(t) \equiv t$ only for the case where $B \ge 1$, i.e., when $r(B) \ge 1$.

Remark 2.3. By Definition 2.1, problem (1.1), (1.2); (1.3) is H-well-posed if and only if it has a unique solution x and for every sequences of matrix-and vector-functions,

$$P_m = (p_{mik})_{i,k=1}^n \in L_{loc}(I_{t_0};\mathbb{R}^{n\times n}), \quad q_m = (q_{mk})_{k=1}^n \in L_{loc}(I_{t_0};\mathbb{R}^n) \ \ (m = 1,2,\dots)$$

and

$$G_m = (g_{mik})_{i,k=1}^n \in E(\mathbb{N};\mathbb{R}^{n\times n}), \quad g_m = (g_{mk})_{k=1}^n \in E(\mathbb{N};\mathbb{R}^n) \ \ (m = 1,2,\dots),$$

such that

$$\det(I_n + G_m(l)) \ne 0 \ \ (l = 1,2,\dots) \ \text{ for every sufficiently large m},$$

and the conditions

$$\lim_{m\to+\infty} \bigg(\int_t^{t_0} \big\|H^{-1}(\tau)(P_m(s) - P(s))H(s)\big\|\,ds + \sum_{l\in N_{t,t_0}} \big\|H^{-1}(\tau_l)(G_m(l) - G(l))H(\tau_l)\big\| \bigg) = 0,$$

$$\lim_{m\to+\infty} \bigg(\int_t^{t_0} \big\|H^{-1}(\tau)(q_m(s) - q(s))\big\|\,ds + \sum_{l\in N_{t,t_0}} \big\|H^{-1}(\tau_l)(g_m(l) - g(l))\big\| \bigg) = 0$$

hold uniformly on I, the impulsive differential system

$$\frac{dx}{dt} = P_m(t)x + q_m(t) \ \text{ for a.a. } \ t \in I_{t_0} \setminus T, \tag{1.1$_m$}$$

$$x(\tau_l+) - x(\tau_l-) = G_m(l)x(\tau_l) + g_m(l) \ \ (l = 1,2,\dots) \tag{1.2$_m$}$$

has a unique solution x_m satisfying condition (1.3) for every sufficiently large m and

$$\lim_{m\to+\infty} \big\|H(t)\,(x_m(t) - x(t))\big\| = 0 \tag{2.18}$$

uniformly on I.

3 Auxiliary propositions

We give the lemma on a priori estimate of solutions of system (1.1), (1.2) (see Lemma 3.2 below). To prove the lemma, we use the Cauchy formula for the representation of solutions of impulsive differential systems.

Lemma 3.1 (Variation-of-constants formula)**.** *Let $G_* \in E(\mathbb{N};\mathbb{R}^{n\times n})$ be such that*

$$\det(I_n + G_*(l)) \neq 0 \ \ (l = 1,2,\dots).$$

Then every solution of the system

$$\frac{dx}{dt} = P_*(t)x + q_*(t) \ \ \text{for a.a. } t \in I,$$
$$x(\tau_l+) - x(\tau_l-) = G_*(l)x(\tau_l) + g_*(l) \ \ (l = 1,2,\dots),$$

where $P_* \in L_{loc}(I;\mathbb{R}^{ns\times n})$, $q_* \in L_{loc}(I;\mathbb{R}^n)$ and $G_* \in E(\mathbb{N};\mathbb{R}^{n\times n})$, $g_* \in E(\mathbb{N};\mathbb{R}^n)$, admits the representation

$$x(t) = C_*(t,\tau)x(\tau) + \int\limits_{\tau}^{t} C_*(t,s)q_*(s)\,ds + \sum_{l\in N_{\tau,t}} C_*(t,\tau_l)(I_n + G_*(l))^{-1}g_*(l) \ \ \text{for } \tau < t, \ \ \tau, t \in I, \tag{3.1}$$

where C_* is the Cauchy matrix of the homogeneous system

$$\frac{dx}{dt} = P_*(t)x \ \ \text{for a.a. } t \in I,$$
$$x(\tau_l+) - x(\tau_l-) = G_*(l)x(\tau_l) \ \ (l = 1,2,\dots).$$

Representation (3.1) is proved, for example, in [4,13,14].

Lemma 3.2. *Let the matrix-functions $P_0 \in L_{loc}(I_{t_0};\mathbb{R}^{n\times n})$, $G_0 \in E(\mathbb{N};\mathbb{R}^{n\times n})$ and the constant matrices B_0 and B from $\mathbb{R}_+^{n\times n}$ be such that conditions* (2.5)–(2.9) *hold for some $\delta > 0$, where C_0 is the Cauchy matrix of system* (1.7), (1.8)*. Let, moreover,*

$$\gamma(t) = \sup\bigg\{ \int\limits_{\tau}^{t_0} \big\|H^{-1}(\tau)C_0(\tau,s)q(s)\big\|\,ds$$
$$+ \sum_{l\in N_{\tau,t_0}} \big\|H^{-1}(\tau)C_0(\tau,\tau_l)(I_n + G_0(l))^{-1}g(l)\big\| : \ t \le \tau \le t_0 \bigg\} < +\infty \ \ \text{for } t \in [t_0 - \delta, t_0[\,. \tag{3.2}$$

Then every solution x of system (1.1), (1.2) *admits the estimate*

$$\|H^{-1}(t)x(t)\| \le \rho\big(\|B_0\|\,\|H^{-1}(\tau_0)x(\tau_0)\| + \gamma(t)\big) \ \ \text{for } t \in J, \ \ t < \tau_0, \tag{3.3}$$

where $\rho = \|(I_n - B)^{-1}\|$, while $J \subset [t_0 - \delta, t_0[$ and $\tau_0 \in J$ are an arbitrary interval and an arbitrary point, respectively.

This lemma is proved in [6].

4 Proof of the main results

Proof of Theorem 2.1. By conditions (2.5)–(2.9), problem (1.1), (1.2) has a unique solution x (see [6]). On the other hand, because I is the finite and closed interval, there exists $\overline{\rho} \in \mathbb{R}_+^{n\times n}$ such that

$$|x(t)| \le H(t)\overline{\rho} \ \ \text{for } t \in I. \tag{4.1}$$

Let us denote by B_1 the $n\times n$-matrix whose every element equals to 1. Due to (2.6), there exists $\eta_0 \in]0,1[$ such that

$$r(\widetilde{B}) < 1, \tag{4.2}$$

where $\widetilde{B} = B + \eta_0 B_0\, B_1$.

We introduce the functions

$$v(t) = \|B_0\| \, \big\|H^{-1}(t)(\widetilde{P}(t) - P(t))H(t)\big\| \ \text{ for } \ t \in [a, t_0 - \delta],$$
$$\alpha_l = \|B_0\| \, \big\|H^{-1}(\tau_l)|G(l) - G_0(l)|H(\tau_l)\big\| \ \ (l = 1, 2, \dots).$$

Let $\varepsilon > 0$ be an arbitrary fixed number. Then, taking into account (2.3), we get that there exists $\eta \in]0, \eta_0[$ such that

$$\rho_0 < \frac{\varepsilon}{2} \tag{4.3}$$

and

$$\rho_1 \rho_2 \exp(\rho_3) < \varepsilon, \tag{4.4}$$

where

$$\rho_0 = \eta \, (1 + \|\overline{\rho}\|)\big(1 + \|(I_n - \widetilde{B})^{-1}\|\big) \, \|B_0\|, \quad \rho_1 = \eta \, (1 + \|\overline{\rho}\|)\|(I_n - \widetilde{B})^{-1}\| \, \|B_0\|,$$
$$\rho_2 = \int\limits_a^{t_0-\delta} v(s) \, ds, \quad \rho_3 = \prod_{l \in N_{a, t_0-\delta}} (1 + \alpha_l).$$

In addition, it is evident that $\rho_3 + \rho_4 < +\infty$.

Let $\widetilde{P} \in L_{loc}(I_{t_0}; \mathbb{R}^{n\times n})$ and $\widetilde{G} \in E(\mathbb{N}; \mathbb{R}^{n\times n})$ be matrix-functions satisfying conditions (2.2) and (2.3).

Then by (2.7), due to (2.3) and (2.9), we find

$$\begin{aligned}
&\int\limits_t^{t_0} \big|C_0(t,\tau)(\widetilde{P}(s) - P_0(s))\big|H(s) \, ds + \sum_{l \in N_{t,t_0}} \Big|C_0(t,\tau_l)(I_n + G_0(l))^{-1}(\widetilde{G}(l) - G_0(l))\Big|H(\tau_l)] \\
&\qquad \le \int\limits_t^{t_0} \big|C_0(t,\tau)(P(s) - P_0(s))\big|H(s) \, ds + \int\limits_t^{t_0} \big|C_0(t,\tau)(\widetilde{P}(s) - P(s))\big|H(s) \, ds \\
&\qquad\qquad + \sum_{l \in N_{t,t_0}} \Big|C_0(t,\tau_l)(I_n + G_0(l))^{-1}(G(l) - G_0(l))\Big|H(\tau_l) \\
&\qquad\qquad\qquad + \sum_{l \in N_{t,t_0}} \Big|C_0(t,\tau_l)(I_n + G_0(l))^{-1}(\widetilde{G}(l) - G(l))\Big|H(\tau_l) \\
&\qquad\qquad \le H(t)B + H(t)B_0 \int\limits_t^{t_0} H^{-1}(s)|\widetilde{P}(s) - P(s)|H(s) \, ds \\
&\qquad + H(t)B_0 \sum_{l \in N_{t,t_0}} H^{-1}(\tau_l)|(\widetilde{G}(l) - G(l))|H(\tau_l) \le H(t)(B + \eta B_0 B_1) \ \text{ for } \ t \in [t_0 - \delta, t_0[
\end{aligned}$$

and, therefore,

$$\begin{aligned}
&\int\limits_t^{t_0} \big|C_0(t,\tau)(\widetilde{P}(s) - P_0(s))\big|H(s) \, ds \\
&\qquad + \sum_{l N_{t,t_0}} \Big|C_0(t,\tau_l)(I_n + G_0(l))^{-1}(\widetilde{G}(l) - G_0(l))\Big|H(\tau_l) \le H(t)\widetilde{B} \ \text{ for } \ t \in [t_0 - \delta, t_0[\,.
\end{aligned} \tag{4.5}$$

So, the matrix-functions $\widetilde{P}$, $\widetilde{G}$ and the constant matrix $\widetilde{B}$ satisfy conditions (2.6) and (2.9) as well.

Analogously, for the vector-functions $\widetilde{q} \in L_{loc}(I_{t_0}; \mathbb{R}^{n\times n})$ and $\widetilde{u} \in E(\mathbb{N}; \mathbb{R}^{n\times n})$, satisfying conditions (2.4), we show that

$$\int\limits_t^{t_0} \big\|H^{-1}(t)C_0(t,s)\widetilde{q}(s)\big\|\,ds + \sum_{l\in N_{t,t_0}} \Big\|H^{-1}(t)C_0(t,\tau_l)(I_n+G_0(l))^{-1}\widetilde{g}(l)\big\|$$

$$\leq \int\limits_t^{t_0} \big\|H^{-1}(t)C_0(t,s)q(s)\big\|\,ds + \sum_{l\in N_{t,t_0}} \big\|H^{-1}(t)C_0(t,\tau_l)(I_n+G_0(l))^{-1}g(l)\big\|$$

$$+ \int\limits_t^{t_0} \big\|H^{-1}(t)C_0(t,s)(\widetilde{q}(s)-q(s))\big\|\,ds + \sum_{l\in N_{t,t_0}} \Big\|H^{-1}(t)C_0(t,\tau_l)(I_n+G_0(l))^{-1}(\widetilde{g}(l)-g(l))\Big\|$$

$$\leq \int\limits_t^{t_0} \big\|H^{-1}(t)C_0(t,s)q(s)\big\|\,ds + \sum_{l\in N_{t,t_0}} \Big\|H^{-1}(t)C_0(t,\tau_l)(I_n+G_0(l))^{-1}g(l)\Big\|$$

$$+ \|B_0\| \int\limits_t^{t_0} \big\|H^{-1}(s)(\widetilde{q}(s)-q(s))\big\|\,ds + \|B_0\| \sum_{l\in N_{t,t_0}} \big\|H^{-1}(\tau_l)(\widetilde{g}(l)-g(l))\big\| \ \text{ for } \ t\in[t_0-\delta,t_0[.$$

Hence, in view of conditions (2.8) and (2.10), it follows that the vector-functions $\widetilde{q}$ and $\widetilde{g}$ satisfy condition (2.10) as well.

Thus, according to Theorem 2.1, the last two conditions together with inequality (4.2) guarantee the unique solvability of problem $(1.4),(1.5);(1.3)$. Let y be a solution of the problem.

Let us assume

$$z(t)\equiv x(t)-y(t) \ \text{ and } \ u(t)\equiv H^{-1}(t)z(t).$$

Then z will be a solution of the impulsive system

$$\frac{dz}{dt} = \widetilde{P}(t)z + q_*(t) \ \text{ for a.a. } \ t\in I_{t_0}\setminus T, \tag{4.6}$$

$$z(\tau_l+) - z(\tau_l-) = \widetilde{G}(l)z(\tau_l) + g_*(l) \ \ (l=1,2,\dots), \tag{4.7}$$

under the condition

$$\lim_{s_0\to t_0-} (H^{-1}(s_0)\, z(s_0)) = 0, \tag{4.8}$$

where

$$q_*(t)\equiv (P(t)-\widetilde{P}(t))x(t) + q(t) - \widetilde{q}(t), \quad g_*(l)\equiv (G(l)-\widetilde{G}(l))x(\tau_l) + g(l) - \widetilde{g}(l).$$

According to Lemma 3.2, conditions (4.2), (4.5) and (4.6)–(4.8) guarantee the estimate

$$u(t) \leq \|(I_n-\widetilde{B})^{-1}\|\,\gamma(t) \ \text{ for } \ t\in[t_0-\delta,t_0[, \tag{4.9}$$

where

$$\gamma(t) = \sup\big\{\xi_*(\tau):\ t\leq\tau<t_0\big\} \ \text{ for } \ t\in[t_0-\delta,t_0[\,, \tag{4.10}$$

and

$$\xi_*(\tau)\equiv \int\limits_\tau^{t_0} \big\|H^{-1}(\tau)C_0(\tau,s)q_*(s)\big\|\,ds + \sum_{l\in N_{\tau,t_0}} \Big\|H^{-1}(\tau)C_0(\tau,\tau_l)(I_n+G_0(l))^{-1}g_*(l)\Big\|.$$

Hence, due to $(2.7),(2.8)$ and (4.1), we conclude

$$\xi_*(t) \leq \|B_0\|\bigg(\int\limits_t^{t_0} \big\|H^{-1}(s)|q_*(s)|\,\big\|\,ds + \sum_{l\in N_{t,t_0}} \big\|H^{-1}(\tau_l)|g_*(l)|\,\big\|\bigg)$$

$$\leq \|B_0\| \int\limits_t^{t_0} \Big\|H^{-1}(s)\big(|(P(s)-\widetilde{P}(s))x(s)| + |q(s)-\widetilde{q}(s)|\big)\,\Big\|\,ds$$

$$
\begin{aligned}
&+ \|B_0\| \sum_{l\in N_{t,t_0}} \Big\| H^{-1}(\tau_l)\big(|(G(l)-\widetilde{G}(l))x(\tau_l)| + |(g(l)-\widetilde{g}(l))|\big) \Big\| \\
&\le \overline{\rho}\|B_0\| \bigg(\int_t^{t_0} \big\| H^{-1}(s)|P(s)-\widetilde{P}(s)|H(s)| \big\|\, ds + \sum_{l\in N_{t,t_0}} \big\| H^{-1}(\tau_l)|G(l)-\widetilde{G}(l)|H(\tau_l)| \big\| \bigg) \\
&\quad + \|B_0\| \bigg(\int_t^{t_0} \big\| H^{-1}(s)|q(s)-\widetilde{q}(s)| \big\|\, ds + \sum_{l\in N_{t,t_0}} \big\| H^{-1}(\tau_l)|g(l)-\widetilde{g}(l)| \big\| \bigg)
\end{aligned}
$$

and, therefore, due to $(2.3), (2.4)$ and (4.10), we find

$$\gamma(t) \le \eta\,(1+\|\overline{\rho}\|)\|B_0\| \ \text{ for } \ t\in[t_0-\delta, t_0[$$

and, in view of (4.3),

$$u(t) \le \eta\,(1+\|\overline{\rho}\|)\,\|(I_n-\widetilde{B})^{-1}\|\,\|B_0\| < \rho_0 < \frac{\varepsilon}{2} \ \text{ for } \ t\in[t_0-\delta, t_0[\,. \tag{4.11}$$

Since z is a solution of the impulsive system $(4.6), (4.7)$, it is evident that it is a solution of the system

$$
\begin{gathered}
\frac{dz}{dt} = P_0(t)z + q_0(t) \ \text{ for a. a. } \ t\in I_{t_0}\setminus T, \\
z(\tau_l+) - z(\tau_l-) = G_0(l)z(\tau_l) + g_0(l) \ \ (l=1,2,\dots)
\end{gathered}
$$

as well, where

$$q_0(t) \equiv (\widetilde{P}(t)-P_0(t))z(t) + q_*(t), \quad g_0(l) \equiv (\widetilde{G}(l)-G_0(l))z(\tau_l) + g_*(l).$$

According to Lemma 3.1, applying to the last system, we find

$$
\begin{aligned}
H^{-1}(t)z(t) &= H^{-1}(t)C_0(t,t_0-\delta)z(t_0-\delta) \\
&\quad - \int_t^{t_0-\delta} H^{-1}(t)C_0(t,s)q_0(s)\,ds - \sum_{l\in N_{t,t_0-\delta}} H^{-1}(t)C_0(t,\tau_l)(I_n+G_0(l))^{-1}g_0(l) \\
&= H^{-1}(t)C_0(t,t_0-\delta)z(t_0-\delta) - \int_t^{t_0-\delta} H^{-1}(t)C_0(t,s)(\widetilde{P}(s)-P_0(s))z(s)\,ds \\
&\quad - \int_t^{t_0-\delta} H^{-1}(t)C_0(t,s)q_*(s)\,ds + \sum_{l\in N_{t,t_0-\delta}} H^{-1}(t)C_0(t,\tau_l)(I_n+G_0(l))^{-1}(\widetilde{G}(l)-G_0(l))z(\tau_l) \\
&\quad - \sum_{l\in N_{t,t_0-\delta}} H^{-1}(t)C_0(t,\tau_l)(I_n+G_0(l))^{-1}g_*(l) \ \text{ for } \ t\in[t_0-\delta,t_0[\,.
\end{aligned}
\tag{4.12}
$$

By $(2.7), (2.8)$ and (4.1), for every $t\in[t_0-\delta, t_0[$, we get

$$
\begin{gathered}
\big\|H^{-1}(t)C_0(t,t_0-\delta)z(t_0-\delta)\big\| \le \|B_0\|\,\|H^{-1}(t_0-\delta)z(t_0-\delta)\|, \\
\bigg\| \int_t^{t_0-\delta} H^{-1}(t)C_0(t,s)(\widetilde{P}(s)-P_0(s))z(s)\,ds \bigg\| \le \|B_0\| \int_t^{t_0-\delta} \big\|H^{-1}(s)|\widetilde{P}(s)-P_0(s)|H(s)\big\| u(s)\,ds,
\end{gathered}
$$

$$\Big\| \int\limits_t^{t_0-\delta} H^{-1}(t)C_0(t,s)q_*(s)\,ds \Big\|$$
$$\leq \|\overline{\rho}\|\,\|B_0\| \int\limits_{t_0-\delta}^{t} \big\| H^{-1}(s)|\widetilde{P}(s)-P(s)|H(s)\big\|\,ds + \|B_0\| \int\limits_t^{t_0-\delta} \big\| H^{-1}(s)|\widetilde{q}(s)-q(s)|\,\big\|\,ds,$$
$$\Big\| \sum_{l\in N_{t,t_0-\delta}} H^{-1}(t)C_0(t,\tau_l)(I_n+G_0(l))^{-1}(\widetilde{G}(l)-G_0(l))z(\tau_l) \Big\|$$
$$\leq \|B_0\| \sum_{l\in N_{t,t_0-\delta}} \big\| H^{-1}(\tau_l)|(\widetilde{G}(l)-G_0(l))|\,\big\| u(\tau_l)$$

and

$$\Big\| \sum_{l\in N_{t,t_0-\delta}} H^{-1}(t)C_0(t,\tau_l)(I_n+G_0(l))^{-1}g_*(\tau_l) \Big\|$$
$$\leq \|\overline{\rho}\|\,\|B_0\| \sum_{l\in N_{t,t_0-\delta}} \big\| H^{-1}(\tau_l)|\widetilde{G}(l)-G(l)|H(\tau_l)\big\| + \|B_0\| \sum_{l\in N_{t,t_0-\delta}} \big\| H^{-1}(\tau_l)|\widetilde{g}(l)-g(l)|\,\big\|.$$

In view of these estimates, due to (2.3) and (2.4), from (4.12) we conclude

$$u(t) \leq (1+\|\overline{\rho}\|)\|B_0\|\eta + \|B_0\|u(t_0-\delta) + \|B_0\| \int\limits_t^{t_0-\delta} \big\| H^{-1}(s)|\widetilde{P}(s)-P_0(s)|H(s)\big\| u(s)\,ds$$
$$+ \|B_0\| \sum_{l\in N_{t,t_0-\delta}} \big\| H^{-1}(\tau_l)|\widetilde{G}(l)-G_0(l)|\,\big\| u(\tau_l) \ \text{ for } \ t\in[a,t_0-\delta].$$

Therefore, if we take into account (4.11), we find

$$u(t) \leq \rho_1 + \int\limits_t^{t_0-\delta} v(s)u(s)\,ds + \sum_{l\in N_{t,t_0-\delta}} \alpha_l u(\tau_l) \ \text{ for } \ t\in[a,t_0-\delta]. \tag{4.13}$$

Further, according to the Gronwall–Bellman inequality (see [14, Lemma 2.1]), we have

$$u(t) \leq \rho_1 \prod_{l\in N_{t,t_0-\delta}} (1+\alpha_l)\exp\bigg(\int\limits_t^{t_0-\delta} v(s)\,ds \bigg) \ \text{ for } \ t\in[a,t_0-\delta]. \tag{4.14}$$

From (4.14) we obtain $u(t) \leq \rho_1\rho_2\exp(\rho_3)$ for $t\in[a,t_0-\delta]$. According to (4.4), we find $u(t)<\varepsilon$ for $t\in[a,t_0-\delta]$. Thus estimate (2.1) holds. □

Proof of Theorem 2.2. This theorem is a particular case of Theorem 2.1. Thus it is easy to check that under conditions of Theorem 2.2, problem (1.1), (1.2); (1.3) is uniquely solvable (see [6]). □

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(Received 09.12.2020; accepted 02.11.2021)

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