Memoirs on Differential Equations and Mathematical Physics

Volume 83, 2021, 55–70

Rachid Guettaf, Arezki Touzaline

ANALYSIS OF A FRICTIONAL UNILATERAL CONTACT PROBLEM FOR PIEZOELECTRIC MATERIALS WITH LONG-TERM MEMORY AND ADHESION

Abstract. This paper deals with the study of a mathematical model that describes a frictional contact between a piezoelectric body and an obstacle. The material behavior is described with an electro-elastic constitutive law with long memory and the contact is modelled with Signorini conditions associated with the non-local friction law in which the adhesion between the contact surfaces is taken into account. We establish a variational formulation of the model in the form of a system involving the displacement, stress, electric displacement, electric potential and adhesion field. Under the assumption that the coefficient of friction is small enough, we prove the existence of a unique weak solution to the problem. The proof is based on arguments of variational inequalities, nonlinear evolutionary equations with monotone operators, differential equations and the Banach fixed-point theorem.

2010 Mathematics Subject Classification. 74M15, 74H10, 74F25, 49J40, 35D30.

Key words and phrases. Electro-elastic, adhesion, variational inequalities, fixed point, weak solution.

რეზიუმე. ნაშრომი ეძღვნება მათემატიკური მოდელის შესწავლას, რომელიც ხახუნის გათვალისწინებით აღწერს პიეზოელექტრულ სხეულსა და დაბრკოლებას შორის კონტაქტს. სხეულის მასალის ყოფაქცევა აღწერილია ელექტრო-დრეკადი ძირითადი თანაფარდობით, რომელიც ხანგრძლივ მეხსიერებას უზრუნველყოფს. კონტაქტის მოდელირება ხდება არალოკალურ ხახუნთან ასოცირებული სინიორინის პირობებით, რომლებშიც გათვალისწინებულია ადგეზია კონტაქტურ ზედაპირებს შორის. ჩამოყალიბებულია მოდელის ვარიაციული ფორმულირება სისტემის სახით, რომელიც შეიცავს გადაადგილებას, ძაბვას, ელექტრულ ძვრას, ელექტრულ პოტენციალს და ადგეზიურ ველს. იმ დაშვებით, რომ ხახუნის კოეფიციენტი საკმარისად მცირეა, ჩვენ ვამტკიცებთ ამოცანის ერთადერთი სუსტი ამონახსნის არსებობას. დამტკიცება ემყარება ვარიაციულ უტოლობებს, არაწრფივ ევოლუციურ განტოლებებს მონოტონური ოპერატორებით, დიფერენციალურ განტოლებებსა და ბანახის უძრავი წერტილის თეორემას.

1 Introduction

Contact problems involving deformable bodies are common in industry and in everyday life and play an important role in structural and mechanical systems, especially, the so-called piezoelectric materials, which consider the interaction of mechanical and electrical properties. Contact processes involve complicated surface phenomena and are modeled with highly nonlinear initial boundary value problems. Taking into account various conditions associated with more and more complex behavior laws lead to introducing new and nonstandard models, expressed by the aid of evolution variational inequalities. An early attempt to study contact problems within the framework of variational inequalities is due to Duvaut and Lions [5], to find the state of mathematical, mechanical, and numerical art (see [22, 26]). Several authors have studied unilateral frictional contact problems involving the Signorini state with or without adhesion (see, e.g., the references in [7, 9, 18, 26, 28]), as well as the models of viscoelastic adhesive materials and piezoelectric effect models (see [6, 12, 13, 15, 20]).

In this paper, we study a mathematical model that describes a problem of frictional and adhesive contact between a supposed long-memory electro-elastic body and a foundation. Recall that a frictionless contact problem with short memory has been studied in [25]. In the present work, we assume that the contact is modeled with a unilateral constraint and the law of non-local friction with adhesion. The bonding field evolution is described by a first-order differential equation. As in [10, 11], we use it as an internal surface variable with values between zero and one to describe the fractional density of active bonds. We refer the reader to the extensive bibliography on the subject in [4, 17, 22, 25].

The present paper aims to extend the results established in the study of a unilateral and frictional contact problem with adhesion. Novelty is the introduction of a non-local friction law in unilateral adhesive contact problem for an elastic body with long memory. We contribute to the solution of this problem by proposing a variational formulation for this model, then, we prove that under the assumption of the smallness of the coefficient of the friction and suitable regularity assumptions on the data, the problem admits a unique weak solution where we specify its regularity. The proof of this result requires proving several technical lemmas by arguments on variational inequalities, monotone operators, differential equations, and Banach's fixed-point theorem.

The paper is organized as follows. In Section 2, we state the mechanical model; we list the assumption on the problem data; we present some notations and give a variational formulation. Finally, in Section 3, under the assumption of the smallness of the coefficient of friction, we state and prove our main existence and uniqueness result.

2 Problem statement and variational formulation

First, we explain some notations used in this paper. We denote by $\mathbf{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d (d = 2, 3)$, while '$\cdot$' and $\|\cdot\|$ represent the inner product and the Euclidean norm on $\mathbf{S}_d$ and $\mathbb{R}^d$, respectively. Thus, for every $u, v \in \mathbb{R}^d$, $u \cdot v = u_i v_i$, $\|v\| = (v \cdot v)^{\frac{1}{2}}$ and for every σ, $\tau \in \mathbf{S}_d$, $\sigma \cdot \tau = \sigma_{ij}\tau_{ij}$, $\|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}$. Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $v_\nu = v \cdot \nu = v_i \nu_i$, $v_\tau = v - v_\nu \nu$, $\sigma_\nu = \sigma\nu \cdot \nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$.

We consider the following physical setting. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with the Lipschitz boundary $\partial\Omega = \Gamma$. The boundary Γ is partitioned into three disjoint measurable parts Γ_1, Γ_2, Γ_3 on the one hand, and on two disjoint measurable parts Γ_a and Γ_b on the other hand, such that $\text{meas}(\Gamma_1) > 0$, $\text{meas}(\Gamma_a) > 0$ and $\Gamma_3 \subset \Gamma_b$. Let $T > 0$ and let $[0, T]$ denote the time interval of interest. We assume the body is clamped on Γ_1 and therefore the displacement field vanishes there. A volume forces of density φ_0 act in Ω and surface tractions of density φ_2 act on Γ_2. The body is submitted to electrical constraints for which we assume the electric potential is zero on Γ_a, the body is subjected to an electric charge of density q_0 act on Ω and a surface electric charge of density q_0 act on Γ_b. On Γ_3, the body is in unilateral contact with adhesion following the nonlocal friction law with an insulator obstacle, the so-called foundation.

Thus, the formulation of the mechanical problem is written as follows.

Problem (P). *Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbf{S}_d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0,T] \to \mathbb{R}^d$ and a bonding field $\beta : \Gamma_3 \times [0,T] \to \mathbb{R}$ such that for all $t \in [0,T]$,*

$$\sigma(t) = \mathcal{B}\varepsilon(u(t)) + \int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\,ds - \mathcal{E}^* E(\varphi(t)), \tag{2.1}$$

$$D(t) = \mathcal{E}\varepsilon(u(t)) + \mathcal{C}E(\varphi(t)), \tag{2.2}$$

$$\operatorname{Div}\sigma(t) + \varphi_0(t) = 0 \ \text{ in } \Omega, \tag{2.3}$$

$$\operatorname{div} D(t) + q_0(t) = 0 \ \text{ in } \Omega, \tag{2.4}$$

$$u(t) = 0 \ \text{ on } \Gamma_1, \tag{2.5}$$

$$\sigma\nu(t) = \varphi_2(t) \ \text{ on } \Gamma_2, \tag{2.6}$$

$$u_\nu(t) \le 0, \ \ \sigma_\nu(t) - \gamma_\nu\beta^2 R_\nu(u_\nu(t)) \le 0, \ \ u_\nu(t)(\sigma_\nu(t) - \gamma_\nu\beta^2(t)R_\nu(u_\nu(t))) = 0 \ \text{ on } \Gamma_3, \tag{2.7}$$

$$\dot{\beta}(t) = -[\beta(t)((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau\|R_\tau(u_\tau(t))\|^2) - \epsilon_a]_+ \ \text{ on } \Gamma_3, \tag{2.8}$$

$$\varphi(t) = 0 \ \text{ on } \Gamma_a, \tag{2.9}$$

$$D\nu(t) = q_2(t) \ \text{ on } \Gamma_b, \tag{2.10}$$

$$\beta(0) = \beta_0 \ \text{ on } \Gamma_3, \tag{2.11}$$

$$\begin{cases} \left\|\sigma_\tau(t) + \gamma_\tau\beta^2(t)R_\tau(u_\tau(t))\right\| \leqslant \mu|R\sigma_\nu(u(t))|, \\ \left\|\sigma_\tau(t) + \gamma_\tau\beta^2(t)R_\tau(u_\tau(t))\right\| < \mu|R\sigma_\nu(u(t))| \Longrightarrow u_\tau = 0 \\ \left\|\sigma_\tau(t) + \gamma_\tau\beta^2(t)R_\tau(u_\tau(t))\right\| = \mu|R\sigma_\nu(u(t))| \Longrightarrow \exists\,\lambda \geqslant 0 \ \text{ such that} \\ \qquad \sigma_\tau(t) + \gamma_\tau^2\beta^2(t)R_\tau(u_\tau(t)) = -\lambda u_\tau(t) \end{cases} \text{ on } \Gamma_3. \tag{2.12}$$

We now describe the equations and conditions involved in our model above.

First, equations (2.1) and (2.2) present an elastic constitutive law with long memory in which u is the displacement field, $D = (D_1, \dots, D_d)$ is the electric displacement field, $\sigma = (\sigma_{ij})$ is the stress tensor, $\varepsilon(u)$ denote the linearised deformation tensor defined by $\varepsilon(u) = (\varepsilon_{ij}(u))$, $\varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$; $\mathcal{B}$ is an operator of elasticity, $\mathcal{F}$ is the tensor of relaxation, $\mathcal{E} = (e_{ijk})$ is the third order piezoelectric operator, $\mathcal{E}^* = (e^*_{ijk})$ is its transpose. $E(\varphi) = -\nabla\varphi$ is the electric field, where $\nabla\psi = (\partial_i\psi)$ and $\mathcal{C} = (\mathcal{C}_{ij})$ is a positive definite symmetric tensor, called the electric permittivity. More details on the constitutive equations of forms (2.1) and (2.2) can be found in [1] and [2]. Next, (2.3) is the equation of motion describing the evolution of the displacement u where $\operatorname{Div}\sigma = (\partial_j\sigma_{ij})$ and (2.4) is the equation describing the evolution of the electric displacement D. Conditions (2.5) and (2.6) are the displacement and traction boundary conditions, whereas (2.7) are the Signorini contact conditions with adhesion, with zero gap, in which γ_ν denotes an adhesion coefficient which may be dependent on $x \in \Gamma_3$. R_ν and R_τ are the truncation operators defined by

$$R_\nu(s) = \begin{cases} L & \text{if } \ s < L, \\ -s & \text{if } \ -L \le s \le 0, \\ 0 & \text{if } \ s > L, \end{cases} \qquad R_\tau(s) = \begin{cases} s & \text{if } \ |s| \le L, \\ L\,\dfrac{s}{|s|} & \text{if } \ |s| > L, \end{cases}$$

where $L > 0$ is the characteristic length of the bond.

The differential equation (2.8) describes the evolution of the bonding field β. Here, γ_ν, γ_τ and ϵ_a are positive coefficients of adhesion, where $[r]_+ = \max\{0, r\}$. In (2.9), we assume that the potential vanishes on Γ_a, and we express the fact that the electric charge density q_2 is imposed on Γ_b by (2.10). Finally, (2.11) is the initial condition and (2.12) represent Coulomb's law of dry friction with adhesion, where μ denotes the coefficient of friction.

Now, to obtain a variational formulation of Problem (P), we will use the spaces

$$H = L^2(\Omega)^d, \quad Q = \{\tau = (\tau_{ij});\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\},$$
$$H_1 = \{u = (u_i) :\ u_i \in H^1(\Omega),\ \ i = \overline{1,d}\}, \quad Q_1 = \{\sigma \in Q :\ \operatorname{Div}\sigma \in H\}$$

H, Q, H_1, H_d are the real Hilbert spaces endowed with the respective inner products

$$(u,v)_H = \int\limits_\Omega u_i v_i \, dx, \quad \langle \sigma, \tau \rangle_Q = \int\limits_\Omega \sigma_{ij}\tau_{ij} \, dx,$$

$$(u,v)_{H_1} = \langle u,v\rangle_H + (\varepsilon(u), \varepsilon(v))_Q, \quad (\sigma, \tau_{H_d}) = \langle \sigma, \tau \rangle_Q + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_H.$$

We denote respectively the norms associated with $\|\cdot\|_H$, $\|\cdot\|_Q$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{H_d}$.

Recall that the following *Green's* formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_Q + (\operatorname{Div}\sigma, v)_H = \int\limits_\Gamma \sigma\nu \cdot v \, da, \ \ \forall\, v \in H_1 \tag{2.13}$$

where da is the measure surface element.

The displacement fields will be sought in the space $V = \{v \in H_1 : \ \gamma v = 0 \text{ a.e. on } \Gamma_1\}$.

Since $\operatorname{meas}(\Gamma_1) > 0$, the Korn inequality holds, i.e., there exists a constant $C_0 > 0$ such that

$$\|\varepsilon(v)\|_Q \geqslant C_0 \|v\|_{H_1}, \ \ \forall\, v \in V,$$

and V is a Hilbert space with the inner product $(u,v)_V = (\varepsilon(u), \varepsilon(v))_Q$ and the associated norm $\|\cdot\|_V$.

For $v \in H_1$, we use the same symbol v for its trace on Γ. Given the Sobolev trace theorem, there is a constant $C_\Omega > 0$ such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leqslant C_\Omega \|v\|_V, \ \ \forall\, v \in V. \tag{2.14}$$

We use the set of admissible displacements fields given by $U_{ad} = \{v \in V : \ v_\nu \leq 0 \text{ a.e. on } \Gamma_3\}$.

For the electric displacement field, we need the following two Hilbert spaces:

$$W = \big\{\psi \in H^1 : \ \gamma\psi = 0 \text{ a.e on } \Gamma_a\big\}, \quad W_a = \big\{D = (D_i) : \ D_i \in L^2(\Omega), \ \operatorname{div} D \in L^2(\Omega)\big\}$$

endowed, respectively, with the inner products

$$(\psi, \phi)_W = (\nabla\psi, \nabla\phi)_H, \quad (D, E)_{Wa} = (D,E)_H + (\operatorname{div} D, \operatorname{div} E)_{L^2(\Omega)},$$

and we denote the norms associated with $\|\cdot\|_W$ and $\|\cdot\|_{W_a}$.

Since $\operatorname{meas}(\Gamma_a) > 0$, the *Friedrichs–Poincaré* inequality holds and we have a constant $C_F > 0$ such that

$$\|\nabla\psi\|_W \geq C_F \|\psi\|_{H^1(\Omega)}, \ \ \forall\, \psi \in W.$$

Moreover, if $D \in W_a$ is sufficiently regular, the following *Green's* formula holds:

$$(D, \nabla\psi)_H + (\operatorname{div} D, \psi)_{L^2(\Omega)} = \int\limits_{\Gamma_b} D\nu \cdot \psi \, da, \ \ \forall\, \psi \in W. \tag{2.15}$$

We will also need the space Q_∞ of fourth order tensors defined by

$$Q_\infty = \Big\{\mathcal{A} = (\mathcal{A}_{ijkh}); \ \mathcal{A}_{ijkh} = \mathcal{A}_{jikh} = \mathcal{A}_{khij} \in L^\infty(\Omega)\Big\}.$$

Q_∞ is a Banach space with the norm defined by

$$\|\mathcal{A}\|_{Q_\infty} = \max_{0 \leq i,j,k,h \leq d} \|\mathcal{A}_{ijkh}\|_{L^\infty(\Omega)}.$$

Let $T > 0$. For every real Hilbert space X, we use the usual notation for the spaces $L^p(0,T;X)$, $k \in [0,\infty]$ and $W^{1,\infty}(0,T;X)$. Recall that the norm of the space $W^{1,\infty}(0,T;X)$ is defined by $\|u\|_{W^{1,\infty}(0,T;X)} = \|u\|_{L^\infty(0,T;X)} + \|\dot{u}\|_{L^\infty(0,T;X)}$, where $\dot{u}$ denotes the first derivative of u with respect to time. We also use the space of continuous functions $C([0,T];X)$ with the norm $\|x\|_{C([0,T];X)} = \max\limits_{t\in[0,T]} \|x(t)\|_X$.

Finally, we introduce the space of bonding field denoted as $\mathbf{B}$ by

$$\mathbf{B} = \Big\{\beta:\ [0,T] \to L^2(\Gamma_3);\ 0 \le \beta(t) \le 1,\ \forall\, t \in [0,T],\ a.e.\ \ on\ \Gamma_3\Big\}.$$

For the study of Problem (P) we adopt the following assumptions on the data.

The operator $\mathcal{B}$ and the tensors $\mathcal{F}$, $\mathcal{C}$, $\mathcal{E}$ and $\mathcal{E}^*$ satisfy the following hypotheses:

$$\begin{cases} \text{(a)} & \mathcal{B} : \Omega \times S_d \to S_d, \\ \text{(b)} & \mathcal{B} \in Q_\infty \ \text{ and there exists a constant } \ M_{\mathcal{B}} > 0 \ \text{ such that} \\ & \qquad \|\mathcal{B}(x,\xi_1) - \mathcal{B}(x,\xi_2)\| \le M_{\mathcal{B}}\|\xi_1 - \xi_2\|,\ \ \forall\, \xi_1,\xi_2 \in S_d,\ \ \text{a.e. in } \Omega, \\ \text{(c)} & \text{There exists a constant } \ m_{\mathcal{B}} > 0 \ \text{ such that } \ \mathcal{B}\xi\cdot\xi \geqslant m_{\mathcal{B}}\|\xi\|^2,\ \ \forall\, \xi \in S_d \ \text{ a.e. in } \Omega, \\ \text{(d)} & \text{The function } \ x \to \mathcal{B}(x,\xi) \ \text{ is measurable on } \Omega \text{ a.e } \xi \in S_d; \end{cases} \tag{2.16}$$

$$\mathcal{F} \in C([0,T];Q_\infty); \tag{2.17}$$

$$\begin{cases} \text{(a)} & \mathcal{C} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ \text{(b)} & \mathcal{C}(x,E) = (c_{ij}(x)E_j),\ \ \forall\, E = (E_{ij}) \in \mathbb{R}^d \ \text{ a.e. in } \Omega, \quad c_{ij} = c_{ji} \in L^\infty(\Omega), \\ \text{(c)} & \text{There exists a constant } \ m_{\mathcal{C}} > 0 \ \text{ such that} \\ & \qquad c_{ij}(x)E_iE_j \geqslant m_{\mathcal{C}}\|E\|^2 \ \ \forall\, \xi \in S_d \ \text{ a.e. in } \Omega; \end{cases} \tag{2.18}$$

$$\begin{cases} \text{(a)} & y\mathcal{E} : \Omega \times S_d \to \mathbb{R}^d, \\ \text{(b)} & \mathcal{E}(x,\xi) = (e_{ijk}(x)\xi_{ij}),\ \ \forall\, \xi = (\xi_{ij}) \in \mathbf{S}_d \ \text{ a.e. in } \Omega, \\ \text{(c)} & e_{ijk} = e_{ikj} \in L^\infty(\Omega); \end{cases} \tag{2.19}$$

$$\mathcal{E}\sigma\cdot v = \sigma\cdot\mathcal{E}^* v,\ \ \forall\, \sigma \in \mathbf{S}_d,\ \ \forall\, v \in \mathbb{R}^d. \tag{2.20}$$

where the components of the tensor $\mathcal{E}^*$ are given by $e^*_{ijk} = e_{kij}$.

In addition, we assume that adhesion coefficients satisfy

$$\gamma_\tau, \gamma_\nu, \epsilon_a \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\tau, \gamma_\nu, \epsilon_a \geqslant 0 \ \text{ a.e. } x \in \Gamma_3, \tag{2.21}$$

and the following regularity on φ_0 and q_0:

$$\varphi_0 \in C([0,T];H),\ \ \varphi_2 \in C([0,T];L^2(\Gamma_2)^d), \tag{2.22}$$

$$q_0 \in C([0,T];H),\ \ q_2 \in C([0,T];L^2(\Gamma_b)^d). \tag{2.23}$$

To reflect that the foundation is isolated, we assume

$$q_0(t) = 0 \ \text{ on } \Gamma_3,\ \ \forall\, t \in [0,T]. \tag{2.24}$$

The initial data β_0 satisfy

$$\beta_0 \in L^2(\Gamma_3),\ \ 0 \le \beta_0 \le 1 \ \text{ a. e. on } \Gamma_3. \tag{2.25}$$

The friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3),\ \ \mu(x) \ge 0 \ \text{ a. e. on } \Gamma_3. \tag{2.26}$$

Finally, R is linear and continuous mapping, where

$$R : H^{-\frac{1}{2}}(\Gamma) \to L^2(\Gamma_3). \tag{2.27}$$

By the representation theorem of *Riesz–Fréchet*, for all $t \in [0,T]$, we define $f(t) \in V$ and $q(t) \in W$ as follows:

$$(f(t),v)_V = \int_\Omega \varphi_0(t)\cdot v\,dx + \int_{\Gamma_2} \varphi_2(t)\cdot v\,da,\ \ \forall\, v \in V,$$

$$(q(t),\psi)_V = \int_\Omega q_0(t)\cdot\psi\,dx + \int_{\Gamma_2} q_2(t)\cdot\psi\,da,\ \ \forall\, \psi \in W,$$

which imply that $f \in C([0,T];H)$ and $q \in C([0,T];W)$. Next, we consider V_0, the subset of regularity defined by $V_0 = \{v \in H_1 : \ \operatorname{Div}\sigma(v) \in H\}$. Let us denote by $j_{ad} : L^\infty(\Gamma_3) \times V_0 \times V \to \mathbb{R}$ and $j_{fr} : V_0 \times V \to \mathbb{R}$, respectively, the functionals given by

$$j_{ad}(\beta,u,v) = \int\limits_{\Gamma_3} \big(-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + \gamma_\tau \beta^2 R_\tau(u_\tau)\cdot v_\tau \big)\, da,$$

$$j_{fr}(u,v) = \int\limits_{\Gamma_3} \mu |R\sigma_\nu(u)| \|v_\tau\|\, da, \ \ \forall\, (u,v) \in V_0 \times V.$$

If (v,φ) is a solution of Problem (P_V) stated below, then $\sigma(t) = \sigma(u(t),\varphi(t)) \in Q$ a.e. $t \in [0,T]$ and therefore

$$j_{fr}(u(t),v) = \int\limits_{\Gamma_3} \mu |R\sigma_\nu(u(t))| \|v_\tau\|\, da, \ \ \forall\, v \in V.$$

Using the *Green's* formula (2.13) and (2.15), we prove that if u, σ, φ and D are regular and satisfy equations and conditions (2.1)–(2.12), then

$$(\sigma(t), \varepsilon(u(t)))_Q + j_{ad}(\beta(t),u(t),v) + j_{fr}(u(t),v) - j_{fr}(u(t),u(t)) \ge (f(t), v-u(t))_V, \quad \forall\, v \in V, t \in [0,T], \tag{2.28}$$

$$(D(t), \nabla\psi)_H + (q(t),\psi)_W = 0, \ \ \forall\, \psi \in W. \tag{2.29}$$

Taking $\sigma(t)$ in (2.28) by the expression given by (2.1), and $D(t)$ by the expression given by (2.2) , we derive the following variational formulation of Problem (P).

Problem (P_V)**.** *Find a displacement field $u \in C([0,T];V)$, an electric potential $\varphi \in C([0,T];W)$ and a bonding field $\beta \in W^{1,\infty}([0,T];L^2(\Gamma_3)) \cap \mathbf{B}$ such that $u(t) \in U_{ad} \cap V_0$ for all $t \in [0,T]$ and*

$$\begin{aligned}\big(\mathcal{B}\varepsilon(u(t)), \varepsilon(v-u(t))\big)_Q &+ \Big(\int\limits_0^t \mathcal{F}(t-s)\varepsilon(u(s))\, ds, \varepsilon(v-u(t)) \Big)_Q \\ &+ \big(\mathcal{E}^*\nabla\varphi(t), \varepsilon(v-u(t))\big)_Q + j_{ad}(\beta(t),u(t),v-u(t)) \\ &+ j_{fr}(u(t),v) - j_{fr}(u(t),u(t)) \ge (f(t),v-u(t))_V, \ \ \forall\, v \in U_{ad}, \ \ t \in [0,T],\end{aligned} \tag{2.30}$$

$$(\mathcal{C}\nabla\varphi(t), \nabla\psi)_H - (\mathcal{E}\varepsilon(u(t), \nabla\psi)_H = (q(t),\psi)_W, \ \ \forall\, \psi \in W, \ \ t \in [0,T], \tag{2.31}$$

$$\dot\beta(t) = -\Big[\beta(t)((\gamma_\nu R_\nu u_\nu(t))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2) - \epsilon_a\Big]_+, \ \ t \in [0,T], \tag{2.32}$$

$$\beta(0) = \beta_0. \tag{2.33}$$

3 Existence and uniqueness

Our main existence and uniqueness result that we state and prove is the following

Theorem 3.1. *Assume that assumptions* (2.16)–(2.27) *hold. Then there exists a constant $\mu_0 > 0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then Problem (P_V) has a unique solution (u,φ,β).*

We carry out the proof of Theorem 3.1 in several steps. We define intermediate problems and prove their unique solvability, and then we construct a contraction mapping whose unique fixed point is the solution of Problem (P_V). First, we consider the closed subset $Z = \{\theta \in C([0,T];\ L^2(\Gamma_3)) \cap B;\ \theta(0) = \beta_0\}$, where the Banach space $C([0,T];L^2(\Gamma_3))$ is endowed with the norm

$$\|\theta\|_k = \max_{t\in[0,T]} \big[e^{-kt}\|\theta\|_{L^2(\Gamma_3)}\big], \ \ k > 0.$$

For a given $\beta \in Z$, we consider the following auxiliary problem.

Problem (P_V^β)**.** Find a displacement field $u_\beta \in C([0,T];V)$ and an electric potential $\varphi_\beta \in C([0,T];W)$ such that $u_\beta(t) \in U_{ad} \cap V_0$ for all $t \in [0,T]$ and

$$\big(\mathcal{B}\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t))\big)_Q + \bigg(\int_0^t \mathcal{F}(t-s)\varepsilon(u_\beta(s))\, ds, \varepsilon(v - u_\beta(t))\bigg)_Q$$
$$+\big(\mathcal{E}^*\nabla\varphi_\beta(t), \varepsilon(v - u_\beta(t))\big)_Q + j_{ad}(\beta(t), u_\beta(t), v - u_\beta(t))$$
$$+j_{fr}(u_\beta(t), v) - j_{fr}(u_\beta(t), u_\beta(t)) \geq (f(t), v - u_\beta(t))_V, \ \ \forall\, v \in U_{ad}, \ \ t \in [0,T], \tag{3.1}$$
$$\big(\mathcal{C}\nabla\varphi_\beta(t), \nabla\psi\big)_H - \big(\mathcal{E}\varepsilon(u_\beta(t), \nabla\psi)\big)_H = (q(t), \psi)_W, \ \ \forall \psi \in W, \ \ t \in [0,T]. \tag{3.2}$$

We have the following result.

Theorem 3.2. *Problem* (P_V^β) *has a unique solution* $(u_\beta, \varphi_\beta) \in C([0,T]; V \times W)$.

We consider the product Hilbert space $X = V \times W$ with the inner product defined by

$$\langle x, y\rangle = \big\langle (u,\varphi), (v,\psi)\big\rangle = \langle u, v\rangle + \langle \varphi, \psi\rangle, \ \ x, y \in X,$$

and the associated norm $\|\cdot\|_X$. In the sequel, let $X_1 = U_{ad} \times W$.

To prove Theorem 3.2 for all $\eta \in C([0,T];Q)$ and $t \in [0,T]$, we consider the following problem.

Problem (P_η^1)**.** Find $x_{\beta\eta} \in C([0,T];X)$ such that $x_{\beta\eta}(t) \in X_1$ for all $t \in [0,T]$ and

$$\big(\mathcal{B}\varepsilon(u_{\beta\eta}(t)), \varepsilon(v - u_{\beta\eta}(t))\big)_Q + \big(\mathcal{E}^*\nabla\varphi_{\beta\eta}(t), \varepsilon(v - u_{\beta\eta}(t))\big)_Q + \big(\mathcal{C}\nabla\varphi_{\beta\eta}(t), \nabla\psi\big)_H - \big(\mathcal{E}\varepsilon(u_{\beta\eta}(t), \nabla\psi)\big)_H$$
$$+ (\eta(t), \varepsilon(v - u_{\beta\eta}(t)))_Q + j_{ad}\big(\beta(t), u_{\beta\eta}(t), v - u_{\beta\eta}(t)\big) + j_{fr}(u_{\beta\eta}(t), v) - j_{fr}(u_{\beta\eta}(t), u_{\beta\eta}(t))$$
$$\geq (f(t), v - u_{\beta\eta}(t))_V + (q(t), \psi)_W, \ \ \forall v \in U_{ad}, \ \ \forall\, \psi \in W, \ \ t \in [0,T]. \tag{3.3}$$

Since *Riesz's* representation theorem implies that there exists an element $f_\eta(t) \in X$ defined for all $x = (u, \varphi)$ by

$$\langle f_\eta(t), x\rangle = (f(t), u)_V + (q(t), \varphi)_W - (\eta(t), \varepsilon(v))_Q,$$

we introduce the operator $\Lambda_\beta : [0,T] \times X \to X$ defined as

$$\langle \Lambda_\beta(t)x, X\rangle = (\mathcal{B}\varepsilon(u), \varepsilon(v))_Q + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_Q$$
$$+ (\mathcal{C}\nabla\varphi, \nabla\psi)_H - (\mathcal{E}\varepsilon(u), \nabla\psi)_H + j_{ad}(\beta(t), u, v), \ \text{ for all } \ x = (u, \varphi), \ \ y = (v, \psi) \in X$$

denoted by $\widetilde{X} = X \times X$, we introduce $\widetilde{j}_{fr} : \widetilde{X} \to \mathbb{R}$ defined by

$$\widetilde{j}_{fr}(y, x) = j_{fr}(u, v) \ \text{ for all } \ x = (u, \varphi), \ \ y = (v, \psi) \in X.$$

Then Problem (P_η^1) is equivalent to

Problem (P_η^2)**.** Find $x_{\beta\eta} : [0,T] \to X_1$ such that

$$\big\langle \Lambda_\beta(t) x_{\beta\eta}(t), y - x_{\beta\eta}(t)\big\rangle + \widetilde{j}_{fr}(y, x_{\beta\eta}(t)) - \widetilde{j}_{fr}(x_{\beta\eta}(t), x_{\beta\eta}(t))$$
$$\geq \langle f_\eta(t), y - x_{\beta\eta}(t)\rangle, \ \ \forall\, y \in X, \ \ t \in [0,T]. \tag{3.4}$$

Remark. The two precedent Problems (P_η^1) and (P_η^2) are equivalent in the way that if $x_{\beta\eta} = (u_\beta, \varphi_{\beta\eta}) \in C([0,T];X)$ is a solution of one of the problems, it is also a solution of the other problem.

We now have the following

Lemma 3.1. *There exists a constant* $\mu_0 > 0$ *such that if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$*, Problem* (P_η^2) *has a unique solution* $x_{\beta\eta} \in C([0,T];X)$.

We prove Lemma 3.1 by steps. The functional j_{ad} is linear over the third term and therefore

$$j_{ad}(\beta, u, -v) = -j_{ad}(\beta, u, v). \tag{3.5}$$

Using the properties of truncation operators, we deduce that there exists $c > 0$ such that

$$j_{ad}(\beta_1, u_1, u_2 - u_1) + j_{ad}(\beta_2, u_2, u_1 - u_2) \leq c \int_{\Gamma_3} |\beta_1 - \beta_2| \, \|u_1 - u_2\|_V \, ds. \tag{3.6}$$

Taking $\beta = \beta_1 = \beta_2$ in the last inequality, we obtain

$$j_{ad}(\beta, u_1, u_2 - u_1) + j_{ad}(\beta, u_2, u_1 - u_2) \leq 0. \tag{3.7}$$

Choosing $u_1 = v$ and $u_2 = 0$ in (3.7) and using (3.5) and the equality $R_\nu(0) = R_\tau(0) = 0$, we obtain

$$j_{ad}(\beta, v, v) \geq 0. \tag{3.8}$$

Similar computations based on the properties of R_ν and R_τ show that there exists a constant $c > 0$ such that

$$|j_{ad}(\beta, u_1, v) - j_{ad}(\beta, u_2, v)| \leq c\|u_1 - u_2\|_V \|v\|_V. \tag{3.9}$$

For $t \in [0, T]$ and for all $x_1 = (u_1, \varphi_1)$ and $x_2 = (u_2, \varphi_2)$, using (3.4), we have

$$\begin{aligned}\big\langle \Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2, x_1 - x_2 \big\rangle &= \big(\mathcal{B}\varepsilon(u_1) - \mathcal{B}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2)\big)_Q \\ &+ \big(\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2, \varepsilon(u_1) - \varepsilon(u_2)\big)_Q + \big(\mathcal{C}\nabla\varphi_1 - \mathcal{C}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2\big)_H \\ &- \big(\mathcal{E}\varepsilon(u_1) - \mathcal{E}\varepsilon(u_2), \nabla\varphi_1 - \nabla\varphi_2\big)_H + j_{ad}(\beta, u_1, u_2) - j_{ad}(\beta, u_2, u_1),\end{aligned}$$

and, by (2.20), we have

$$\big(\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_2, \varepsilon(u_1) - \varepsilon(u_2)\big)_Q = \big(\mathcal{E}\varepsilon(u_1) - \mathcal{E}\varepsilon(u_2), \nabla\varphi_1 - \nabla\varphi_2\big)_H.$$

Then, by (3.8), (2.16)(c) and (2.18)(c) we deduce

$$\begin{aligned}\big\langle \Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2, x_1 - x_2 \big\rangle &\geq \big(\mathcal{B}\varepsilon(u_1) - \mathcal{B}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2)\big)_Q \\ &+ \big(\mathcal{C}\nabla\varphi_1 - \mathcal{C}\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2\big)_H \geq m_{\mathcal{B}}\|u_1 - u_2\|_V^2 + m_{\mathcal{C}}\|\varphi_1 - \varphi_2\|_W^2.\end{aligned}$$

Then the operator $\Lambda_\beta(t)$ is strongly monotone, and for $C_m = \min(m_{\mathcal{B}}, m_{\mathcal{C}})$ it satisfies

$$\big\langle \Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2, x_1 - x_2 \big\rangle \geq C_m\|x_1 - x_2\|_X^2, \ \ \forall\, x, y \in X. \tag{3.10}$$

For $y = (v, \psi)$, using (2.14), (2.16)(b), (2.18) and (3.9), we get

$$\big\langle \Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2, y \big\rangle \leq c\Big(\|u_1 - u_2\|_V\big(\|v\|_V + \|\psi\|_W\big) + \|\varphi_1 - \varphi_2\|_W\big(\|v\|_V + \|\psi\|_W\big)\Big),$$

thus, $\Lambda_\beta(t)$ is a Lipschitz continuous operator and there exists a constant $L_0 > 0$ such that

$$\|\Lambda_\beta(t)x_1 - \Lambda_\beta(t)x_2\| \leq L_0\|x_1 - x_2\|_X, \ \ \forall\, x, y \in X.$$

Next, let the non-empty subset $L^2_+(\Gamma_3)$ be defined by

$$L^2_+(\Gamma_3) = \big\{g \in L^2(\Gamma_3);\ g \geqslant 0 \text{ a.e. on } \Gamma_3\big\}.$$

For each $g \in L^2_+(\Gamma_3)$, we define the functional $h(g, \cdot\,) : X \to \mathbb{R}$ by

$$h(g, y) = \int_{\Gamma_3} \mu g \|w_\tau\| \, da, \ \ \forall\, y = (w, \varphi) \in X,$$

and introduce an intermediate problem as follows.

Problem (P_1^g)**.** Find $x_{\beta\eta} : [0, T] \to X_1$ such that

$$\big\langle \Lambda_\beta(t)x_{\beta\eta g}(t), y - x_{\beta\eta g}(t) \big\rangle + h(g, y) - h(g, x_{\beta\eta g}(t)) \geqslant (f, y - x_{\beta\eta g}(t))_V, \ \ \forall\, y \in X. \tag{3.11}$$

Lemma 3.2. *Problem* (P_1^g) *has a unique solution.*

Proof. The functional $h(g,\cdot)$ is convex and lower semi-continuous, Λ_β is Lipschitz continuous and strongly monotone, we deduce that Problem (P_1^g) has a unique solution (see [13]). $\square$

Now, to prove Lemma 3.1, for each $t \in [0,T]$ we define on $L^2_+(\Gamma_3)$ the map $\Psi_t : g \longmapsto \Psi_t(g) = |R\sigma_\nu(u_{\beta\eta g}(t))|$. Then we show the following

Lemma 3.3. *There exists a constant* $\mu_1 > 0$ *such that if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$*, the mapping* Ψ *has a unique fixed point* g^**, and* $x_{\beta\eta g^*}$ *is a unique solution to Problem* (P_η^2)*.*

Proof. For $i = 1, 2$, define the following

Problem $(P^2_{\eta g i})$. *Find* $x_{\beta\eta g i} = (u_{\beta\eta g_i}, \varphi_{\beta\eta g_i}) \in X_1$ *such that*

$$\langle \Lambda_\beta(t) x_{\beta\eta g i}, y\rangle + h(g_i, y) - h(g_i, x_{\beta\eta g i}) \geqslant (f, y - x_{\beta\eta g i})_V, \quad \forall\, y \in V.$$

Take $y = x_{\beta\eta g_2}$ in inequality (3.11) written for $g = g_1$, then take $y = x_{\beta\eta g_1}$ in (3.11) written for $g = g_2$, by adding the resulting inequalities, we get

$$\big\langle \Lambda_\beta(t)(x_{\beta\eta g_1} - x_{\beta\eta g_2}), x_{\beta\eta g_1} - x_{\beta\eta g_2}\big\rangle \leq h(g_1, x_{\beta\eta g_1}) - h(g_1, x_{\beta\eta g_2}) + h(g_2, x_{\beta\eta g_2}) - h(g_2, x_{\beta\eta g_1}).$$

Then using (2.14) and (3.10), we have

$$C_m \|x_{\beta\eta g_1}(t) - x_{\beta\eta g_2}(t)\|_X^2 \leqslant C_\Omega \|g_1 - g_2\|_{L^2(\Gamma_3)} \int\limits_{\Gamma_3} \mu\big(|u_{\beta\eta g 1\tau}(t)| - |u_{\beta\eta g 2\tau}(t)|\big)\, da. \tag{3.12}$$

Using (2.27), it follows that there exists a constant c_0 such that

$$\|\Psi(g_1) - \Psi(g_2)\|_{L^2(\Gamma_3)} \leqslant c_0 \big\|\sigma_\nu(u_{\beta\eta g_1}(t)) - \sigma_\nu(u_{\beta\eta g_2}(t))\big\|_{H^{-\frac{1}{2}}(\Gamma)}. \tag{3.13}$$

Moreover, using (2.16), we prove that there is a constant $c_1 > 0$ such that

$$\big\|\sigma_\nu(u_{\beta\eta g_1}(t)) - \sigma_\nu(u_{\beta\eta g_2}(t))\big\|_{H^{-\frac{1}{2}}(\Gamma)} \leq c_1 \|x_{\beta\eta g_1}(t) - x_{\beta\eta g_2}(t)\|_X. \tag{3.14}$$

Hence, taking into account (2.14) and combining (3.12), (3.13) and (3.14), after some calculus we find

$$\|\Psi(g_1) - \Psi(g_2)\|_{L^2(\Gamma_3)} \leqslant \frac{c_0 c_1 C_\Omega}{C_m} \|\mu\|_{L^\infty(\Gamma_3)} \|g_1 - g_2\|_{L^2(\Gamma_3)}.$$

Let $\mu_1 = \frac{C_m}{c_0 c_1 C_\Omega}$, then we deduce that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_1$, Ψ is a contraction and, so, it admits a unique fixed point denoted by g^*.

Keeping in mind that there is a unique element $x_{\beta\eta g^*}$ satisfying the inequality

$$\big\langle \Lambda_\beta(t) x_{\beta\eta g^*}, y - x_{\beta\eta g^*}\big\rangle + h(\Psi(g^*), y) - h(\Psi(g^*), x_{\beta\eta g^*}) \geqslant (f, y - x_{\beta\eta g^*})_V, \quad \forall\, y \in X,$$

and $h \circ \Psi = j$, we prove that $x_{\beta\eta}(t) = x_{\beta\eta g^*}$ is a unique solution of Problem (P_η^2). We shall now see that $x_{\beta\eta} \in C([0,T]; X)$. Indeed, let t_1, $t_2 \in [0,T]$, take $y = x_{\beta\eta}(t_2)$ in (3.3) written for $t = t_1$ and take $y = x_{\beta\eta}(t_1)$ in the same inequality written for $t = t_2$. Using (2.16), (2.27) and the properties of R_ν and R_τ, we prove that there exists a constant $c > 0$ such that

$$\|x_{\beta\eta}(t_1) - x_{\beta\eta}(t_2)\|_X \leq c\big(\|\beta(t_1) - \beta(t_2)\|_{L^2(\Gamma_3)} + \|f(t_1) - f(t_2)\|_H + \|\eta(t_1) - \eta(t_2)\|_Q\big).$$

Then, as $f \in C([0,T]; H)$, $\eta \in C([0,T]; Q)$ and $\beta \in C([0,T]; L^2(\Gamma_3))$, we immediately conclude that $x_{\beta\eta} \in C([0,T]; X)$. We also have that $u_{\beta\eta}(t) \in U_{ad} \cap V_0$, $\forall\, t \in [0,T]$. Indeed, for each $t \in [0,T]$, denote $\sigma(u_{\beta_\eta}(t)) = \mathcal{B}\varepsilon(u_{\beta_\eta}(t)) - \mathcal{E}^* E(\varphi_{\beta_\eta}(t)) + \eta(t)$ and using *Green's* formula with the regularity $\varphi_0(t) \in H$, we get $\operatorname{div} \sigma(u_{\beta_\eta}(t))) \in H$ and then $u_{\beta\eta}(t) \in V_0$. $\square$

Now, we define the operator $\digamma_\beta : C([0,T];Q) \to C([0,T];Q)$ by

$$\digamma_\beta \eta(t) = \int_0^t \mathcal{F}(t-s)\varepsilon(u_{\beta\eta}(s))\,ds, \quad \forall\, \eta \in C(0,T;Q), \quad t \in [0,T].$$

We have the following

Lemma 3.4. *The operator $\digamma_\beta$ has a unique fixed point η_β.*

Proof. Let $\eta_1, \eta_1 \in C([0,T];Q)$. By a standard computation based on (2.17) and (3.3), we prove that there exists a constant $c_2 > 0$ such that

$$\|\digamma_\beta \eta_1(t) - \digamma_\beta \eta_2(t)\|_Q \le c_2 \int_0^t \|\eta_1(t) - \eta_2(t)\|_Q\,ds, \quad \forall\, t \in [0,T].$$

By iteration, for any positive integer n we deduce the estimate

$$\|\digamma_\beta^n \eta_1 - \digamma_\beta^n \eta_2\|_{C([0,T];Q)} \le \frac{c_2^n T^n}{n!}\,\|\eta_1 - \eta_2\|_{C([0,T];Q)}.$$

As $\lim\limits_{n\to+\infty} \frac{c_2^n T^n}{n!} = 0$, it follows that for a positive integer n sufficiently large, $\digamma_\beta^n$ is a contraction on the space $C([0,T];Q)$. Then, by using the Banach fixed point theorem, $\digamma_\beta^n$ has a unique fixed point $\eta_\beta \in C([0,T];Q)$ which is also a unique fixed point of $\digamma_\beta$, i.e.,

$$\digamma_\beta \eta_\beta(t) = \eta_\beta(t), \quad \forall\, t \in [0,T]. \qquad \square$$

Next, we denote $u_\beta = u_{\beta\eta}$ and $\varphi_\beta = \varphi_{\beta\eta}$ and deduce that the couple (u_β, φ_β) is a solution of Problem (P_V^β). The uniqueness follows from the fixed point of the operator $\digamma$, which completes the proof of Theorem 3.2.

In the following step, we use u_β, the solution obtained by Theorem 3.2, to state the following Cauchy problem.

Problem (P_{ad})**.** *Find a bonding field $\theta_\beta : [0,T] \to L^\infty(\Gamma_3)$ such that*

$$\dot\theta_\beta(t) = -\Big[\theta_\beta(t)\big((\gamma_\nu R_\nu u_{\beta\nu}(t))^2 + \gamma_\tau \|R_\tau(u_{\beta^*\tau}(t))\|^2\big) - \epsilon_a\Big]_+ \quad a.e.\ t \in [0,T], \tag{3.15}$$

$$\theta_\beta(0) = \beta_0. \tag{3.16}$$

Lemma 3.5. *Problem (P_{ad}) has a unique solution θ_β which satisfies $\theta_\beta \in W^{1,\infty}([0,T];L^\infty(\Gamma_2)) \cap Z$.*

Proof. Consider the mapping $\mathcal{F} : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$\mathcal{F}_\beta(t,\theta) = -\Big[\theta\big((\gamma_\nu R_\nu u_{\beta\nu}(t))^2 + \gamma_\tau \|R_\tau(u_{\beta\tau}(t))\|^2\big) - \epsilon_a\Big].$$

For all $t \in [0,T]$ and $\theta \in L^2(\Gamma_3)$, it follows from the properties of the truncation operators R_ν and R_τ that $\mathcal{F}_\beta$ is Lipschitz continuous uniformly in time with respect to β. Moreover, for any $\theta \in L^2(\Gamma_3)$, the mapping $t \to \mathcal{F}_\beta(t,\theta)$ belongs to $L^\infty(0,T;L^2(\Gamma_3))$. Using now a version of the Cauchy–Lipschitz theorem (see [15]), we obtain a unique function $\theta_\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ satisfying (3.15) and (3.16). We note that the restriction $0 \le \theta_\beta \le 1$ is implicitly included in the variational Problem P_V and, therefore, from the definition of the sets B and Z, we find that $\theta_\beta \in Z$, which concludes the proof of lemma. $\square$

Consider the mapping $\Phi : Z \to Z$ defined by $\Phi\beta = \theta_\beta$.

The third step consists in the following result.

Lemma 3.6. *There exists a unique element $\beta^* \in Z$ such that $\Phi\beta^* = \beta^*$.*

Proof. Indeed, let β_i, $i = 1, 2$, be two elements of Z. Denote by u_{β_i}, φ_{β_i}, θ_{β_i} the functions obtained in Theorem 3.2 and Lemma 3.5 and denote $\theta_{\beta_i} = \theta_i$. It follows from (3.15) that

$$\theta_i(t) = \beta_0 - \int_0^t \Big[\beta_i(s)\big((\gamma_\nu R_\nu u_{\beta_i\nu}(s))^2 + \gamma_\tau \|R_\tau(u_{\beta_i\tau}(s))\|^2\big) - \epsilon_a\Big]_+ ds$$

and there exists a constant $c > 0$ such that

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \Big\|\beta_1(s)R_\nu(u_{\beta_{1\nu}}(s))^2 - \beta_2(s)R_\nu(u_{\beta_{2\nu}}(s))^2\Big\|_{L^2(\Gamma_3)} ds$$
$$+ \int_0^t \Big\|\beta_1(s)\|R_\tau(u_{\beta_{1\tau}}(s))\|^2 - \beta_2(s)\|R_\tau(u_{\beta_{2\tau}}(s))\|^2\Big\|_{L^2(\Gamma_3)} ds.$$

Using the properties of the operators R_ν and R_τ, we get

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Gamma_3)} \leq c_3 \bigg(\int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} ds + \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_{L^2(\Gamma_3)^d} ds\bigg) \tag{3.17}$$

for some constant $c_3 > 0$. □

Now, to continue the proof, we need to prove the following

Lemma 3.7. *There exists a constant $\mu_2 > 0$ such that if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_2$, we have*

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_{L^2(\Gamma_3)^d} \leq c\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}, \quad \forall t \in [0, T].$$

Proof. Let $t \in [0; T]$. We take $\psi = \psi - \varphi_\beta(t)$ in (3.2) and by adding with (3.1) we get

$$\begin{aligned}
&\big(\mathcal{B}\varepsilon(u_\beta(t)), \varepsilon(v - u_\beta(t))\big)_Q + \bigg(\int_0^t \mathcal{F}(t-s)\varepsilon(u(s))\, ds, \varepsilon(v - u(t))\bigg)_Q \\
&\quad + \big(\mathcal{E}^*\nabla\varphi_\beta(t), \varepsilon(v - u_\beta(t))\big)_Q + j_{ad}(\beta(t), u_\beta(t), v - u_\beta(t)) + \big(\mathcal{C}\nabla\varphi_\beta(t), \nabla\psi - \nabla\varphi_\beta(t)\big)_H \\
&\qquad - \big(\mathcal{E}\varepsilon(u_\beta(t), \nabla\psi - \nabla\varphi_\beta(t))\big)_H + j_{fr}(u_\beta(t), v) - j_{fr}(u_\beta(t), u_\beta(t)) \\
&\qquad \geq (f(t), v - u_\beta(t))_V + (q(t), \psi - \varphi_\beta(t))_W, \quad \forall\, v \in U_{ad}, \ \ \forall\, \psi \in W, \ \ t \in [0, T].
\end{aligned} \tag{3.18}$$

Taking $v = u_{\beta_2}(t)$ and $\psi = \varphi_{\beta_2}$ in (3.18) satisfied by $(u_{\beta_1}(t), \varphi_{\beta_1})$, and then taking $v = u_{\beta_1}(t)$ and $\psi = \varphi_{\beta_1}$ in the same inequality satisfied by $(u_{\beta_2}(t), \varphi_{\beta_2})$, by adding the resulting inequalities and using (2.20), we obtain

$$\begin{aligned}
&\Big(\mathcal{B}\varepsilon(u_{\beta_1}(t)) - \mathcal{B}\varepsilon(u_{\beta_2}(t)), \varepsilon(u_{\beta_1}(t)) - \varepsilon(u_{\beta_2}(t))\Big)_Q + \Big(\mathcal{C}\nabla\varphi_{\beta_1}(t) - \mathcal{C}\nabla\varphi_{\beta_2}(t), \nabla\varphi_{\beta_1}(t) - \nabla\varphi_{\beta_2}(t)\Big)_H \\
&\qquad \leq \bigg(\int_0^t \mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(t)) - \varepsilon(u_{\beta_2}(t))\big)\, ds, \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t))\bigg)_Q \\
&\quad + j_{ad}\big(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)\big) + j_{fr}(u_{\beta_1}(t), u_{\beta_2}(t)) + j_{ad}\big(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)\big) \\
&\qquad + j_{fr}(u_{\beta_2}(t), u_{\beta_1}(t)) - j_{fr}(u_{\beta_1}(t), u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t), u_{\beta_2}(t)).
\end{aligned}$$

Using (2.16)(c) and (2.18)(c), we deduce

$$\begin{aligned}
&m_{\mathcal{B}}\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 + m_{\mathcal{C}}\|\varphi_{\beta_1}(t) - \varphi_2(t)\|_W \\
&\qquad \leq \bigg(\int_0^t \mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(s)) - \varepsilon(u_{\beta_2}(s))\big)\, ds, \varepsilon(u_{\beta_2}(t)) - \varepsilon(u_{\beta_1}(t))\bigg)_Q \\
&\quad + j_{ad}\big(\beta_1(t), u_{\beta_1}(t), u_{\beta_2}(t) - u_{\beta_1}(t)\big) + j_{fr}(u_{\beta_1}(t), u_{\beta_2}(t)) + j_{ad}\big(\beta_2(t), u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t)\big) \\
&\qquad + j_{fr}(u_{\beta_2}(t), u_{\beta_1}(t)) - j_{fr}(u_{\beta_1}(t), u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t), u_{\beta_2}(t)),
\end{aligned}$$

thus

$$
\begin{aligned}
m_{\mathcal{B}}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2 \le \Big(\int_0^t \mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\big)\,ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t))\Big)_Q \\
+ j_{ad}\big(\beta_1(t),u_{\beta_1}(t),u_{\beta_2}(t)-u_{\beta_1}(t)\big) + j_{ad}\big(\beta_2(t),u_{\beta_2}(t),u_{\beta_1}(t)-u_{\beta_2}(t)\big) \\
+ j_{fr}(u_{\beta_1}(t),u_{\beta_2}(t)) - j_{fr}(u_{\beta_1}(t),u_{\beta_1}(t)) + j_{fr}(u_{\beta_2}(t),u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t),u_{\beta_2}(t)). \quad (3.19)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\Big(\int_0^t \mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\big)\,ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t))\Big)_Q \\
&\qquad \le \Big(\int_0^t \|\mathcal{F}(t-s)\|_{Q_\infty}\|u_{\beta_1}(s)-u_{\beta_2}(s)\|_V\,ds\Big)\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V \\
&\qquad\qquad \le c_4\Big(\int_0^t \|u_{\beta_1}(s)-u_{\beta_2}(s)\|_V\,ds\Big)\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V
\end{aligned}
$$

for some positive constant c_4. Using Young's inequality, we find that

$$
\begin{aligned}
&\Big(\int_0^t \mathcal{F}(t-s)\big(\varepsilon(u_{\beta_1}(s))-\varepsilon(u_{\beta_2}(s))\big)\,ds, \varepsilon(u_{\beta_2}(t))-\varepsilon(u_{\beta_1}(t))\Big)_Q \\
&\qquad \le \frac{c_4^2}{m_{\mathcal{B}}}\Big(\int_0^t \|u_{\beta_1}(s)-u_{\beta_2}(s)\|_V^2\,ds\Big) + \frac{m_{\mathcal{B}}}{4}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2. \quad (3.20)
\end{aligned}
$$

Using (3.6) and Young's inequality, we deduce that there exists a positive constant c_5 such that

$$
j_{ad}(\beta_1,u_1,u_2-u_1)+j_{ad}(\beta_2,u_2,u_1-u_2) \le c_5\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}^2 + \frac{m_{\mathcal{B}}}{4}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2. \quad (3.21)
$$

Moreover, we have

$$
\begin{aligned}
&j_{fr}(u_{\beta_1}(t),u_{\beta_2}(t)) - j_{fr}(u_{\beta_1}(t),u_{\beta_1}(t)) + j_{fr}(u_{\beta_2}(t),u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t),u_{\beta_2}(t)) \\
&\qquad \le \int_{\Gamma_3} \mu R|\sigma_\nu(u_{\beta_{1\nu}}(t))-\sigma_\nu(u_{\beta_{2\nu}}(t))|\,\|u_{\beta_1}(t)-u_{\beta_2}(t)\|\,da.
\end{aligned}
$$

Keeping in mind (3.14) and using (2.14), we get

$$
\begin{aligned}
&j_{fr}(u_{\beta_1}(t),u_{\beta_2}(t)) - j_{fr}(u_{\beta_1}(t),u_{\beta_1}(t)) + j_{fr}(u_{\beta_2}(t),u_{\beta_1}(t)) - j_{fr}(u_{\beta_2}(t),u_{\beta_2}(t)) \\
&\qquad \le c_1 C_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2. \quad (3.22)
\end{aligned}
$$

We now combine inequalities (3.19), (3.20), (3.21) and (3.22) to deduce

$$
\begin{aligned}
&m_{\mathcal{B}}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2 \\
&\qquad \le c_5\|\beta_1-\beta_2\|_{L^2(\Gamma_3)}^2 + \frac{m_{\mathcal{B}}}{4}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2 + c_1C_\Omega^2\|\mu\|_{L^\infty(\Gamma_3)}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2 \\
&\qquad\qquad + \frac{c_4^2}{m_{\mathcal{B}}}\Big(\int_0^t \|u_{\beta_1}(s)-u_{\beta_2}(s)\|_V\,ds\Big)^2 + \frac{m_{\mathcal{B}}}{4}\|u_{\beta_1}(t)-u_{\beta_2}(t)\|_V^2.
\end{aligned}
$$

Hence, we have

$$\Big(\frac{m_{\mathcal{B}}}{2} - c_1 C_\Omega^2 \|\mu\|_{L^\infty(\Gamma_3)}\Big) \|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2$$
$$\leq c_5 \|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)} + \frac{c_4^2}{m_{\mathcal{B}}} \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V^2 \, ds.$$

Further, if

$$\|\mu\|_{L^\infty(\Gamma_3)} < \mu_2 = \frac{m_{\mathcal{B}}}{2c_1 C_\Omega^2},$$

we deduce that there exists a constant $c_8 > 0$ such that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \leq c_8 \bigg(\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)} + \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|_V^2 \, ds \bigg).$$

Hence, using Cornwall's argument, it follows that there exists a constant $c_9 > 0$ such that

$$\|u_{\beta_1}(t) - u_{\beta_2}(t)\|_V^2 \leq c_9 \|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Gamma_3)}, \quad \forall\, t \in [0,T]. \tag{3.23}$$

Now, to end the proof of Lemma 3.6 we use (3.17) and (3.23) to obtain

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_9 \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} \, ds, \quad \forall\, t \in [0,T],$$

where $c_7 > 0$. We have

$$e^{-kt} \|\Phi\beta_1(t) - \Phi\beta_2(t)\|_{L^2(\Gamma_3)} \leq c_9 e^{-kt} \int_0^t e^{ks} e^{-ks} \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)} \, ds,$$

then

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_k \leq c_9 e^{-kt} \|\beta_1(t) - \beta_2(t)\|_k \int_0^t e^{ks} \, ds, \quad \forall\, t \in [0,T].$$

So, we deduce that

$$\|\Phi\beta_1(t) - \Phi\beta_2(t)\|_k \leq \frac{c_{10}}{k} \|\beta_1(t) - \beta_2(t)\|_k, \quad \forall\, t \in [0,T], \tag{3.24}$$

where $c_{10} > 0$. Inequality (3.24) shows that for $k > c_{10}$, Φ is a contraction on Z. Then Φ has a unique fixed point which satisfies (3.15) and (3.16). □

Thus, we have all the ingredients to prove Theorem 3.1.

Existence. Consider β^*, the fixed point of the operator Φ, and $x^* = (u^*, \varphi^*)$, the solution of Problem $(P_V^{\beta^*})$, i.e., $u^* = u_{\beta^*}$ and $\varphi^* = \varphi_{\beta^*}$.

By (3.1), (3.2), (3.15) and (3.16), we conclude that the triple $(u^*, \varphi^*, \beta^*)$ is a solution to Problem (P_V).

Uniqueness. The uniqueness arises from the uniqueness of the fixed point of the operator Φ, which completes the proof of Theorem 3.1.

Indeed, let (u, φ, β) be a solution of Problem (P_V), it follows from (3.1) and (3.2) that u is a solution of Problem (P_V^β) and, by Theorem 3.2, this problem has a unique solution (u_β, φ_β), where $u_\beta = u$ and $\varphi_\beta = \varphi$.

Taking $u = u_\beta$ and $\varphi = \varphi_\beta$ in Problem (P_V), we deduce that β is a solution of Problem (P_{ad}). From the result of Lemma 3.5, Problem (P_{ad}) has a unique solution β^*, so we find $\beta^* = \beta$, and then we conclude that $(u^*, \varphi^*, \beta^*)$ is a unique solution to Problem (P_V).

Let now σ^* and D^* be the functions defined by (2.1) and (2.2), respectively, which correspond to (u^*, φ^*). Then it results from (2.16)–(2.20) that $\sigma^* \in C([0,T];Q)$ and $D^* \in C([0,T];H)$. Using also a standard argument, it follows from (2.30) and (2.31) that

$$\begin{aligned} \operatorname{Div}\sigma^*(t) + \varphi_0(t) = 0 \ \text{ in } \Omega, \\ \operatorname{div} D^*(t) + q_0(t) = 0 \ \text{ in } \Omega. \end{aligned}$$

Therefore, using (2.22) and (2.23), we deduce that $\operatorname{Div}\sigma^*(u^*(t), \varphi^*(t)) \in H$ for each $t \in [0,T]$ and $\operatorname{div} D^* \in C([0,T]; L^2(\Omega))$, which implies that $\sigma^* \in C([0,T];Q_1)$ and $D^* \in C([0,T];W_a)$. The triple $(u^*, \varphi^*, \beta^*)$ which satisfies (2.30)–(2.33) is called a weak solution of Problem (P). We conclude that under stated assumptions, Problem (P) has a unique weak solution $(u^*, \varphi^*, \beta^*, \sigma^*, D^*)$ with the regularity $u^* \in C([0,T];V)$, $\varphi^* \in C([0,T];W)$, $\beta^* \in W^{1,\infty}((0,T;L^2(\Gamma_3))) \cap B$, $\sigma^* \in C([0,T];Q_1)$ and $D^* \in C([0,T];W_a)$.

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(Received 27.08.2020)

Authors' addresses:

Rachid Guettaf

Faculty of Mathematics, University of Science and Technology Houari Boumedienne, B.P. 32. El Alia Bab Ezzouar, Algiers, ALGERIA.

E-mail: `ra_guettaf@univ-boumerdes.dz`

Arezki Touzaline

Faculty of Mathematics, University of Science and Technology Houari Boumedienne, B.P. 32. El Alia Bab Ezzouar, Algiers, ALGERIA.

E-mail: `ttouzaline@yahoo.fr`