Memoirs on Differential Equations and Mathematical Physics

Volume 83, 2021, 43–54

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A BICUBIC SPLINES METHOD FOR SOLVING A TWO-DIMENSIONAL OBSTACLE PROBLEM

Abstract. The objective of this paper is to develop a numerical method for solving a bidimensional unilateral obstacle problem. This is based on the bicubic splines collocation method and the generalized Newton method. In this paper, we obtain an approximate expression for solving a bidimensional unilateral obstacle problem. We show that the approximate formula obtained by the bicubic splines collocation method is effective. Next, we prove the convergence of the proposed method. The method is applied to some test examples and the numerical results have been compared with the exact solutions. The obtained results show the computational efficiency of the method. It can be concluded that computational efficiency of the method is effective for the two-dimensional obstacle problem.

2010 Mathematics Subject Classification. 65L10, 34B15, 65M22.

Key words and phrases. Obstacle problem, bicubic splines collocation, nonsmooth equation, generalized Newton method.

რეზიუმე. ნაშრომის მიზანია რიცხვითი მეთოდის შემუშავება ორგანზომილებიანი ცალმხრივი დაბრკოლების ამოცანის ამოსახსნელად. იგი ეფუძნება ბიკუბური სპლაინების კოლოკაციის მეთოდს და ნიუტონის განზოგადებულ მეთოდს. ნაშრომში მიღებულია მიახლოებითი გამოსახულება ორგანზომილებიანი ცალმხრივი დაბრკოლების პრობლემის გადასაჭრელად. ვაჩვენებთ, რომ მიახლოებითი ფორმულა, რომელიც მიღებულია ბიკუბური სპლაინების კოლოკაციის მეთოდით, ეფექტურია. შემდეგ დავამტკიცებთ შემოთავაზებული მეთოდის კრებადობას. მეთოდი გამოყენებულია ზოგიერთ სატესტო მაგალითზე და რიცხვითი შედეგები შედარებულია ზუსტ ამოხსნებთან. მიღებული შედეგები აჩვენებს მეთოდის გამოთვლით ეფექტურობას. შეიძლება დავასკვნათ, რომ მეთოდის გამოთვლითი ეფექტურობა ძალაშია ორგანზომილებიანი ამოცანებისთვის დაბრკოლებით.

1 Introduction

In this paper, we consider the following unilateral obstacle problem:

$$\text{Find } u \in K \text{ such that } \int_\Omega \nabla u \cdot \nabla(v-u)\,dx + \int_\Omega f(v-u)\,dx \geq 0, \quad \forall\, v \in K, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open domain with $n \geq 2$, with a smooth boundary $\partial\Omega$, f is an element of $L^2(\Omega)$ and $K = \{v \in H_0^1(\Omega) \mid \ v \geq \psi \text{ a.e. in } \Omega\}$. The main point here is that we are considering an irregular obstacle function ψ which is an element of $H^1(\Omega)$ with $\psi \leq 0$ on $\partial\Omega$. It is well known that problem (1.1) admits a unique solution u, and if $\Delta\psi \in L^2(\Omega)$, then u is an element of $H^2(\Omega)$ (see [10,14]), and the solution u of problem (1.1) is an element of $H^2(\Omega)$ that can be characterized as (see [10], for instance)

$$\begin{cases} -\Delta u + f \geq 0 & \text{a.e. on } \Omega, \\ (-\Delta u + f)(u-\psi) = 0 & \text{a.e. on } \Omega, \\ u - \psi \geq 0 & \text{a.e. on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As a classical subject in the field of partial differential equations, the obstacle problem is aimed to find a solution which is constrained by a given obstacle to some extent. It has numerous applications in various fields including economics, engineering, biology, computer science, etc. There are several numerical solution methods of the obstacle problem (see, e.g., [1,6,9–11,13,17,26]). Numerical solution by penalty methods have been considered, e.g., in [9,24]. In this paper, we develop a numerical method for solving a two-dimensional obstacle problem by using the generalized tension splines collocation method and the generalized Newton method. First, problem (1.1) is approximated by a sequence of nonlinear equation problems by using the penalty method given in [14, 16]. Then we apply the GB-spline collocation method to approximate the solution of a boundary value problem of second order. The discret problem is formulated as to find the generalized tension splines coefficients of a nonsmooth system $\varphi(Y) = Y$, where $\varphi : \mathbb{R}^m \to \mathbb{R}^m$. In order to solve the nonsmooth equation, we apply the generalized Newton method (see, e.g., [4,5,25]). We prove that the generalized tension splines collocation method converges quadratically provided a property, coupling the penalty parameter ε and the discretization parameter h is satisfied.

Numerical methods to approximate the solution of boundary value problems have been considered by several authors. We only mention the papers [3, 15] and the references therein, which use the bicubic spline collocation method for solving the boundary value problems.

The present paper is organized as follows. In Section 2, we present the penalty method to approximate the obstacle problem by a sequence of second order boundary value problems, we also construct a bicubic spline to approximate the solution of the boundary problem, and we present the generalized Newton method. In Section 3, we show the convergence of the generalized tension spline to the solution of the boundary problem and provide an error estimate. Some numerical results are given in Section 4 to validate our methodology. The study ends with conclusions and remarks in Section 5.

2 Bicubic spline collocation method

In this section, we construct a bicubic spline which approximates the solution u_ε of problem (2.1), with Ω being the interval $I \times J = (a,b)^2 \subset \mathbb{R}^2$. We denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^{(n+1)(n+1)}$, by $\|\cdot\|_\infty$ the uniform norm, by $\otimes$ Kronecker product (tensor product) and by $\odot$ the biproduct of matrices.

By using the penalty method (see [14, p. 110], [16]), an approximate solution u_ε of problem (1.1) can be characterized as the following boundary value problem (see [14, p. 107], [16]):

$$\begin{cases} -\Delta u_\varepsilon = \max(-\Delta\psi + f, 0)\theta_\varepsilon(u_\varepsilon - \psi) - f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where θ_ε is a sequence of Lipschitz functions which tend to the function θ defined by

$$\theta_\varepsilon(t) = \begin{cases} 1, & t \leq 0, \\ 1 - \dfrac{t}{\varepsilon}, & 0 \leq t \leq \varepsilon, \\ 0, & t \geq \varepsilon. \end{cases} \tag{2.2}$$

If we put

$$J_\varepsilon(x, y, u_\varepsilon(x, y)) = \max(-\Delta\psi(x, y) + f(x, y), 0)\Theta_\varepsilon$$

with

$$\Theta_\varepsilon = \theta_\varepsilon(u_\varepsilon(x, y) - \psi(x, y)) - f(x, y),$$

then problem (2.1) becomes

$$\begin{cases} -\Delta u_\varepsilon = J_\varepsilon(\,\cdot\,, u_\varepsilon) & \text{on } \Omega, \\ u_\varepsilon(a, y) = u_\varepsilon(x, b) = 0, & x, y \in (a, b). \end{cases} \tag{2.3}$$

It is easy to see that J_ε is a nonlinear continuous function on u_ε; and for any two functions u_ε and v_ε, J_ε satisfies the following Lipschitz condition:

$$\big|J_\varepsilon(x, y, u_\varepsilon(x, y)) - J_\varepsilon(x, y, v_\varepsilon(x, y))\big| \leq L_\varepsilon |u_\varepsilon(x, y) - v_\varepsilon(x, y)| \ \text{ a.e. on } (x, y) \in \Omega, \tag{2.4}$$

where

$$L_\varepsilon = \frac{1}{\varepsilon}\, \| - \Delta\psi + f\|_\infty = \frac{1}{\varepsilon} \max_{(x,y)\in\Omega} | - \Delta\psi(x, y) + f(x, y)|.$$

Now, let

$$\Pi_x = \big\{a = x_{-3} = \cdots = x_0 < x_1 < \cdots < x_{n+1} = \cdots = x_{n+3} = b\big\},$$
$$\Pi_y = \{a = y_{-3} = \cdots = y_0 < y_1 < \cdots < y_{n+1} = \cdots = y_{n+3} = b\}$$

be the subdivisions of the intervals I and J, respectively, with $x_i = a + ih$ and $y_j = a + jh$, where $0 \leq i, j \leq n$ and $h = (b - a)/n$. The partition $\Pi_{xy} = \Pi_x \otimes \Pi_y$ subdivides Ω into smaller rectangles in the plane:

$$T = \Big\{(x, y) : \ x_i \leq x \leq x_{i+1}, \ y_j \leq y \leq y_{j+1}, \ i, j = -3, \ldots, n - 1\Big\}.$$

Denote by

$$S_4^{bicu}(\Omega, \Pi_{xy}) = S_4^{cub}(I, \Pi_x) \otimes S_4^{cub}(J, \Pi_y)$$

a bicubic spline with respect to the partition Π_{xy} with $S_4^{cub}(I, \Pi_x)$ (resp. $S_4^{cub}(J, \Pi_y)$), the space of piecewise polynomials of degree 3 over the subdivision Π_x (resp. Π_y) and of class $\mathcal{C}^2$ everywhere on I (resp. J).

Moreover, let $\{B^x_{-3}, B^x_{-2}, \ldots, B^x_{n-1}\}$ (resp. $\{B^y_{-3}, \ldots, B^y_{n-1}\}$) be a B-spline basis of $S_4^{cub}(I, \Pi_x)$ (resp. $S_4^{cub}(J, \Pi_y)$). By applying the tensor product method (see [19]), we obtain the following bicubic spline interpolation.

Proposition 2.1 (see [19])**.** *Let u_ϵ be a solution of problem* (2.3). *Then there exists a unique bicubic spline interpolant $S_\epsilon \in S_4^{bicu}(\Omega, \Pi_{xy})$ of u_ϵ which satisfies*

$$S_\epsilon(\tau_i^x, \tau_j^y) = u_\epsilon(\tau_i^x, \tau_j^y), \ \ i, j = 0, \ldots, n + 2,$$

where

$$\tau_0^x = x_0, \ \ \tau_i^x = \frac{x_i + x_{i-1}}{2}, \ \ 1 \leq i \leq n, \ \ \tau_{n+1}^x = x_{n-1}, \ \ \tau_{n+2}^x = x_n,$$
$$\tau_0^y = y_0, \ \ \tau_j^y = \frac{y_j + y_{j-1}}{2}, \ \ 1 \leq j \leq n, \ \ \tau_{n+1}^y = y_{n-1}, \ \ \tau_{n+2}^y = y_n.$$

If we put

$$S_\varepsilon(x,y)=\sum_{p,q=-3}^{n-1} c_{p,q,\varepsilon}B_p^x(x)B_q^y(y),$$

then by using the boundary conditions of problem (2.3) we obtain

$$c_{-3,q,\varepsilon}=S_\varepsilon(a,y)=u_\varepsilon(a,y)=0,\;\; q=-3,\dots,n-1,$$

and

$$c_{p,n-1,\varepsilon}=S_\varepsilon(x,b)=u_\varepsilon(x,b)=0,\;\; p=-3,\dots,n-1.$$

Hence

$$S_\varepsilon(x,y)=\sum_{p,q=-2}^{n-2} c_{p,q,\varepsilon}B_p^x(x)B_q^y(y).$$

Furthermore, for any $u_\varepsilon\in H^4(\Omega)$, where $H^4(\Omega)=\{u\in L^2(\Omega);\partial^\alpha u\in L^2(\Omega),|\alpha|\le 4\}$ is the Sobolev space (see [8]), we have

$$-\Delta S_\varepsilon(\tau_i^x,\tau_j^y)=J_\varepsilon(\tau_i^x,\tau_j^y,u_\varepsilon)+O(1),\;\; i,j=1,\dots,n+1. \tag{2.5}$$

The bicubic spline collocation method, presented in this paper, constructs numerically a bicubic spline $\widetilde{S}_\varepsilon=\sum\limits_{p,q=-3}^{n-1}\widetilde{c}_{p,q,\varepsilon}B_p^xB_q^y$ which satisfies equation (2.3) at the points (τ_i^x,τ_j^y), $i,j=0,\dots,n+2$. It is easy to see that

$$\widetilde{c}_{-3,q,\varepsilon}=\widetilde{c}_{p,n-1,\varepsilon}=0\;\text{ for }\; p,q=-3,\dots,n-1$$

and the coefficients $\widetilde{c}_{p,q,\varepsilon}$, $p,q=-2,\dots,n-2$, satisfy the following nonlinear system with $(n+1)^2$ equations:

$$\sum_{p,q=-2}^{n-2}\widetilde{c}_{p,q,\varepsilon}\Delta B_p^x(\tau_i^x)B_q^y(\tau_j^y)=-J_\varepsilon\Big(\tau_i^x,\tau_j^y,\sum_{p,q=-2}^{n-2}\widetilde{c}_{p,q,\varepsilon}B_p^x(\tau_i^x)B_q^y(\tau_j^y)\Big)\;\text{ for }\; i,j=1,\dots,n+1. \tag{2.6}$$

Since

$$\Delta B_p^x(\tau_i^x)B_q^y(\tau_j^y)=B_p^x(\tau_i^x)\Delta B_q^y(\tau_j^y)+B_q^y(\tau_j^y)\Delta B_p^x(\tau_i^x),$$

relations (2.5) and (2.6) can be written in the matrix form, respectively, as follows:

$$\begin{aligned} 2(A_h\odot B_h)C_\varepsilon&=-F_\varepsilon-\widehat{E}_\varepsilon,\\ 2(A_h\odot B_h)\widetilde{C}_\varepsilon&=-F_{\widetilde{C}_\varepsilon}, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} A_h\odot B_h&=\frac{1}{2}\,(A_h\otimes B_h+B_h\otimes A_h),\\ C_\varepsilon&=\big[(c_{-2,q,\varepsilon})_{-2\le q\le n-2},\dots,(c_{n-2,q,\varepsilon})_{-2\le q\le n-2}\big]^T,\\ \widetilde{C}_\varepsilon&=\big[(\widetilde{c}_{-2,q,\varepsilon})_{-2\le q\le n-2},\dots,(\widetilde{c}_{n-2,q,\varepsilon})_{-2\le q\le n-2}\big]^T, \end{aligned}$$

for any integer i such that $1\le i\le n+1$,

$$\begin{aligned} F_\varepsilon&=\Big[J_\varepsilon\big(\tau_i^x,\tau_1^y,u_\varepsilon(\tau_i^x,\tau_1^y)\big),\dots,J_\varepsilon\big(\tau_i^x,\tau_{n+1}^y,u_\varepsilon(\tau_i^x,\tau_{n+1}^y)\big)\Big]^T,\\ F_{\widetilde{C}_\varepsilon}&=\Big[J_\varepsilon\big(\tau_i^x,\tau_1^y,\widetilde{S}_\varepsilon(\tau_i^x,\tau_1^y)\big),\dots,J_\varepsilon\big(\tau_i^x,\tau_{n+1}^y,\widetilde{S}_\varepsilon(\tau_i^x,\tau_{n+1}^y)\big)\Big]^T, \end{aligned}$$

and $\widehat{E}_\varepsilon$ is a vector, where each component is of order $O(1)$. It is well known that $A_h = \frac{1}{h^2} A$ and $B_h = B$, where A and B are the matrices independent of h given as follows:

$$A = \begin{bmatrix} \frac{-15}{4} & \frac{1}{4} & \frac{1}{2} & 0 & \dots & & & 0 \\ \frac{3}{4} & \frac{-3}{4} & \frac{-1}{2} & \frac{1}{2} & 0 & \dots & & 0 \\ 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & \dots & & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{-3}{4} & \frac{3}{4} \\ 0 & \dots & & & 0 & \frac{1}{2} & \frac{1}{4} & \frac{-15}{4} \\ 0 & \dots & & & 0 & 1 & \frac{-5}{2} & \frac{3}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{57}{96} & \frac{25}{96} & \frac{1}{48} & 0 & \dots & & & 0 \\ \frac{3}{96} & \frac{45}{96} & \frac{23}{48} & \frac{1}{48} & 0 & \dots & & 0 \\ 0 & \frac{1}{48} & \frac{23}{48} & \frac{45}{96} & \frac{3}{96} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{1}{48} & \frac{23}{48} & \frac{45}{96} & \frac{3}{96} & 0 \\ 0 & \dots & & 0 & \frac{1}{48} & \frac{25}{96} & \frac{57}{96} \\ 0 & \dots & & 0 & \frac{1}{6} & \frac{7}{12} & \frac{1}{4} \end{bmatrix}.$$

Then relation (2.7) becomes

$$\begin{aligned} (A \odot B)C_\varepsilon &= -\frac{1}{2}\, h^4 F_\varepsilon - E_\varepsilon, \\ (A \odot B)\widetilde{C}_\varepsilon &= -\frac{1}{2}\, h^2 F_{\widetilde{C}_\varepsilon} \end{aligned} \tag{2.8}$$

with E_ε being a vector, where each of its components is of order $O(h^2)$.

As the matrices A and B are invertible (see [18]), then $A \odot B$ is invertible (see [12]) and

$$(A \odot B)^{-1} = A^{-1} \odot B^{-1}. \tag{2.9}$$

Proposition 2.2. *Assume that the penalty parameter ε and the discretization parameter h satisfy the following relation:*

$$h^2 \| -\Delta\psi + f\|_\infty \|A^{-1} \odot B^{-1}\|_\infty < 2\varepsilon. \tag{2.10}$$

Then there exists a unique bicubic spline which approximates the exact solution u_ε of problem (2.3).

Proof. From relation (2.8), we have

$$\widetilde{C}_\varepsilon = -\frac{1}{2}\,h^2 A^{-1} \odot B^{-1} F_{\widetilde{C}_\varepsilon}.$$

Let $\varphi : \mathbb{R}^{(n+1)(n+1)} \to \mathbb{R}^{(n+1)(n+1)}$ be a function defined by

$$\varphi(Y) = -\frac{1}{2}\,h^2 A^{-1} \odot B^{-1} F_Y.$$

To prove the existence of bicubic spline collocation, it suffices to prove that φ admits a unique fixed point. Indeed, let Y_1 and Y_2 be two vectors of $\mathbb{R}^{(n+1)(n+1)}$. Then we have

$$\|\varphi(Y_1) - \varphi(Y_2)\| \leq \frac{1}{2}\,h^2 \|A^{-1} \odot B^{-1}\|_\infty \|F_{Y_1} - F_{Y_2}\|_\infty. \tag{2.11}$$

Using relation (2.4) and the fact that $\sum\limits_{p,q=-2}^{n-2} B_p^x B_q^y \leq 1$, we get

$$\begin{aligned}\Big| J_\varepsilon\big(\tau_i^x, \tau_j^y, S_{Y_1}(\tau_i^x, \tau_j^y)\big) - J_\varepsilon\big(\tau_i^x, \tau_j^y, S_{Y_2}(\tau_i^x, \tau_j^y)\big)\Big| \\ \leq L_\varepsilon \big| S_{Y_1}(\tau_i^x, \tau_j^y) - S_{Y_2}(\tau_i^x, \tau_j^y)\big| \leq L_\varepsilon \|Y_1 - Y_2\|_\infty,\end{aligned}$$

where $L_\varepsilon = \frac{1}{\varepsilon}\,\| -\Delta\psi + f\|_\infty$. Then we obtain

$$\|F_{Y_1} - F_{Y_2}\|_\infty \leq L_\varepsilon \|Y_1 - Y_2\|_\infty.$$

From relation (2.11), we conclude that

$$\|\varphi(Y_1) - \varphi(Y_2)\| \leq L_\varepsilon\,\frac{1}{2}\,h^2 \|A^{-1} \odot B^{-1}\|_\infty \|Y_1 - Y_2\|_\infty.$$

Thus we have

$$\|\varphi(Y_1) - \varphi(Y_2)\| \leq k \|Y_1 - Y_2\|_\infty,$$

with $k = \frac{1}{2}\,h^2 \|A^{-1} \odot B^{-1}\|_\infty$, by relation (2.10). Hence the function φ admits a unique fixed point. □

In order to calculate the coefficients of the generalized tension spline collocation given by the nonsmooth system

$$\widetilde{C_\varepsilon} = \varphi(\widetilde{C}_\varepsilon),$$

we propose the generalized Newton method defined by

$$\widetilde{C}_\varepsilon^{(k+1)} = \widetilde{C}_\varepsilon^{(k)} - (I_{n+1} - V_k)^{-1}\big(\widetilde{C}_\varepsilon^{(k)} - \varphi(\widetilde{C}_\varepsilon^{(k)})\big),$$

where $I_{(n+1)(n+1)}$ is the unit matrix of order $(n+1)(n+1)$ and V_k is the generalized Jacobian of the function $\widetilde{C}_\varepsilon \mapsto \varphi(\widetilde{C}_\varepsilon)$ (see, e.g., [4, 5, 25]).

3 Convergence of the method

Theorem 3.1. *If we assume that the penalty parameter ε and the discretization parameter h satisfy the relation*

$$h^2 \| -\Delta\psi + f\|_\infty \|A^{-1} \odot B^{-1}\|_\infty < \varepsilon. \tag{3.1}$$

then the bicubic spline $\widetilde{S}_\varepsilon$ converges to the solution u_ε. Moreover, the error estimate $\|u_\varepsilon - \widetilde{S}_\varepsilon\|_\infty$ is of order $O(h^2)$.

Proof. From (2.8) and (2.9), we have

$$C_\varepsilon - \widetilde{C}_\varepsilon = -\frac{1}{2}\, h^4 A^{-1} \odot B^{-1}(F_\varepsilon - F_{\widetilde{C}_\varepsilon}) - A^{-1} \odot B^{-1} E_\varepsilon.$$

Since E_ε is of order $O(h^2)$, there exists a constant K_1 such that $\|E_\varepsilon\|_\infty \leq k_1 h^2$. Hence, we get

$$\|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty \leq \frac{1}{2}\, h^2 \|A^{-1} \odot B^{-1}\|_\infty \|F_\varepsilon - F_{\widetilde{C}_\varepsilon}\|_\infty + K_1 \|A^{-1} \odot B^{-1}\|_\infty h^2. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}
&\Big| J_\varepsilon\big(\tau_i^x, \tau_j^y, u_\varepsilon(\tau_i^x, \tau_j^y)\big) - J_\varepsilon\big(\tau_i^x, \tau_j^y, \widetilde{S}(\tau_i^x, \tau_j^y)\big) \Big| \\
&\qquad \leq L_\varepsilon \big| u_\varepsilon(\tau_i^x, \tau_j^y) - \widetilde{S}_\varepsilon(\tau_i^x, \tau_j^y) \big| \leq L_\varepsilon \big| u_\varepsilon(\tau_i^x, \tau_j^y) - S_\varepsilon(\tau_i^x, \tau_j^y) \big| + L_\varepsilon \big| S_\varepsilon(\tau_i^x, \tau_j^y) - \widetilde{S}_\varepsilon(\tau_i^x, \tau_j^y) \big|.
\end{aligned}$$

Since S_ε is the bicubic spline interpolation of u_ε, there exists a constant K_2 such that

$$\|u_\varepsilon - S_\varepsilon\|_\infty \leq K_2 h^2. \tag{3.3}$$

Using the fact that

$$|S_\varepsilon - \widetilde{S}_\varepsilon| \leq \|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty \sum_{p,q=-2}^{n-2} B_p^x B_q^y \leq \|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty, \tag{3.4}$$

we obtain

$$|F_\varepsilon - F_{\widetilde{C}_\varepsilon}| \leq L_\varepsilon \|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty + L_\varepsilon K_2 h^4.$$

By using relation (3.2) and assumption (3.1), it is easy to see that

$$\begin{aligned}
\|C_\varepsilon - \widetilde{C}_\varepsilon\|_\infty &\leq \frac{\frac{1}{2}\, h^2 \|A^{-1} \odot B^{-1}\|_\infty}{1 - L_\varepsilon \frac{1}{2}\, h^2 \|A^{-1} \odot B^{-1}\|_\infty} \,(K_2 L_\varepsilon h^2 + 2K_1) \\
&\leq h^2 \|A^{-1} \odot B^{-1}\|_\infty (K_2 L_\varepsilon h^2 + 2K_1).
\end{aligned} \tag{3.5}$$

Thus

$$\|u_\varepsilon - \widetilde{S}_\varepsilon\|_\infty \leq \|u_\varepsilon - S_\varepsilon\|_\infty + \|S_\varepsilon - \widetilde{S}_\varepsilon\|_\infty.$$

Therefore, from relations (3.3), (3.4) and (3.5), we deduce that $\|u_\varepsilon - \widetilde{S}_\varepsilon\|_\infty$ is of order $O(h^2)$. Hence, the proof is complete. □

Remark 3.1. Theorem 3.1 provides a relation coupling the penalty parameter ε and the discretization parameter h, which guarantees the quadratic convergence of the bicubic spline collocation $\widetilde{S}_\varepsilon$ to the solution u_ε of the penalty problem.

We have the interesting properties.

Theorem 3.2 ([14, p. 110], [16])**.** *Let u denote the solution of the variational inequality problem* (1.1) *and u_ε, $\varepsilon > 0$, denote the solution of the penalty problem* (2.1) *with θ_ε defined by relation* (2.2)*. Then $\{u_\varepsilon\}$ is a nondecreasing sequence and*

$$u(x,y) \leq u_\varepsilon(x,y) \leq u(x,y) + \varepsilon, \quad (x,y) \in \Omega, \quad \textit{for } \varepsilon > 0.$$

Theorem 3.3. *Suppose that $u(x,y)$ is the solution of* (1.1) *and $u_{bc}(x,y)$ is the approximate solution by our presented method. Then we have*

$$\|u(x,y) - u_{bc}(x,y)\|_\infty \leq \epsilon + kh^2, \quad (x,y) \in \Omega, \quad \textit{for } \varepsilon > 0,$$

where k is a finite constant. Therefore, for sufficiently small ϵ and h, the solution of presented scheme (2.8) *converges to the solution of the variational inequality problem* (1.1) *in the discrete L_∞-norm and the rates of convergence are $O(\epsilon + h^2)$.*

4 Numerical examples

In this section, we give the numerical experiments in order to validate the theoretical results presented in this paper. We report numerical results for solving a two-dimensional obstacle problem by using the bicubic spline method to approximate the solution of the penalty problem (2.3), and the generalized Newton method [23] to determine the coefficients of the bicubic spline collocation.

As a numerical experiment, the example by Bartels and Carstensen [2] with $\Omega = (-1.5, 1.5)^2$ is considered, however, with an additional mass term. For the obstacle $\psi = 0$ and volume force $f = 2$, the exact solution is

$$u(x,y) = \begin{cases} -\dfrac{r^2}{2} - \ln(r) - \dfrac{1}{2} & \text{if } \ r = |x|_2 \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

As a stopping criteria for the generalized Newton iterations, we have considered that the absolute value of the difference between the input coefficients and the output coefficients is less than 10^{-5}.

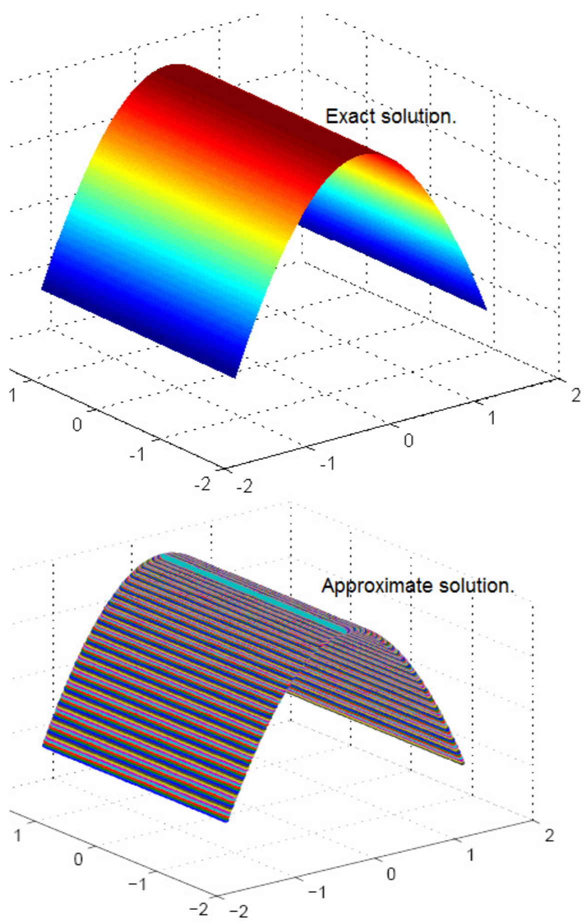

Figure 1. Exact and Approximate solution.

Table 1 shows, for different values of the discretization parameter h, the error between the bicubic spline collocation $\widetilde{S}_\varepsilon$ and the true solution u. We note that the convergence of the solution $\widetilde{S}_\varepsilon$ to the function u depends on the discretization parameter h and the penalty parameter ε. Theorem 3.1 implies that for a fixed h, this convergence is guaranteed only if there exists $\varepsilon_h > 0$ such that $\varepsilon \geq \varepsilon_h$. Some experimental values of ε_h are given in Table 1.

Theorem 3.3 implies that we have the error estimate between the exact solution and the discrete penalty solution given by $\|u(x,y)-u_{bc}(x,y)\|_\infty \le \epsilon + kh^2$. The obtained results show the convergence of the discrete penalty solution to the solution of the original obstacle problem as the parameters h and ε get smaller provided they satisfy relation (3.1). Moreover, the numerical error estimates behave like $\varepsilon + kh^2$ which confirms what we were expecting.

Table 1. Numerical results

ϵ	10^{-2}	10^{-3}	5×10^{-4}	$2\times10^{-4}=\varepsilon_h$
For $h=0.05$				
$\|u-\widetilde{S}_\varepsilon\|_\infty$	5×10^{-3}	10.61×10^{-4}	10.12×10^{-4}	9.84×10^{-4}
For $h=0.02$				
$\|u-\widetilde{S}_\varepsilon\|_\infty$	4.7×10^{-3}	7.21×10^{-4}	2.34×10^{-4}	2.03×10^{-4}
For $h=0.01$				
$\|u-\widetilde{S}_\varepsilon\|_\infty$	4.63×10^{-4}	7.03×10^{-5}	3.15×10^{-6}	1.84×10^{-6}

5 Concluding remarks

In this paper, we have considered an approximation of a bidimensional unilateral obstacle problem by a sequence of penalty problems, which are nonsmooth equation problems, presented in [14, 16]. Then we have developed a numerical method for solving each nonsmooth equation, based on a bicubic collocation spline method and the generalized Newton method. We have shown the convergence of the method provided that the penalty and discret parameters satisfy relation (3.1). Moreover, we have provided an error estimate of order $O(h^2)$ with respect to the norm $\|\cdot\|_\infty$. The obtained numerical results show the convergence of the approximate penalty solutions to the exact one and confirm the error estimates provided in this paper.

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(Received 04.06.2020)

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