Memoirs on Differential Equations and Mathematical Physics

Volume 83, 2021, 31–41

F. Bouzeffour, M. Garayev

THE HARTLEY TRANSFORM VIA SUSY QUANTUM MECHANICS

Abstract. We present the connection between Hartley transform (HT) and a one-dimensional realization by difference-differential operator of $N = \frac{1}{2}$-supersymmetric quantum mechanics elaborated by S. Post, L. Vinet and A. Zhedanov. The key feature of our approach is that the Hartley transform commutes with the supercharge and provides the overcomplete bases of the HT eigenvectors.

2010 Mathematics Subject Classification. 37K20, 81Q60.

Key words and phrases. Special functions, supersymmetry.

რეზიუმე. წარმოგიდგენთ ჰარტლის გარდაქმნის (HT) კავშირს $N = \frac{1}{2}$ სუპერსიმეტრიული კვანტური მექანიკის დიფერენციალურ-სხვაობიანი ოპერატორის ერთგანზომილებიან რეალიზაციასთან, რომელიც შემუშავებულია ს. პოსტის, ლ. ვინეტის და ა. ჟედანოვის მიერ. ჰარტლის გარდაქმნა კომუტირებს სუპერმუხტთან და წარმოქმნის HT საკუთრივი ვექტორების ზესრულ ბაზისს - სწორედ ეს არის ჩვენი მიდგომის მთავარი დამახასიათებელი თვისება.

1 Preliminaries

The Fourier transform of a suitable function f is defined by the formula

$$(\mathcal{F}f)(\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} f(x) e^{i\lambda t}\, dx.$$

Recently, the one-dimensional harmonic oscillator has been approached by the Fourier transform method (see [9, 13, 15, 17]). Let us recall some remarks related to the Fourier transform and harmonic oscillator. In one-dimension coordinates, the representation of the creation and annihilation operators $a^\dagger$, a and the harmonic oscillator H are given by

$$a = \frac{1}{\sqrt{2}}\,(x + ip_x), \quad a^\dagger = \frac{1}{\sqrt{2}}\,(x - ip_x), \quad H = -\frac{1}{2}\,p_x^2 + \frac{1}{2}\,x^2, \quad p_x = -i\,\frac{d}{dx}\,. \tag{1.1}$$

They satisfy

$$[a, a^\dagger] = 1, \quad [H, a] = -a, \quad [H, a^\dagger] = a^\dagger,$$

where $[A, B] = AB - BA$ denotes the usual commutator of A and B.

The wave functions $\psi_n(x)$ of the linear harmonic oscillator,

$$\int\limits_{-\infty}^{\infty} \psi_n(x)\, \psi_m(x)\, dx = \delta_{n\,m}, \quad n, m = 0, 1, 2, \ldots,$$

are explicitly given as

$$\psi_n(x) = \Big(\sqrt{\pi}\, n! 2^n\Big)^{-\frac{1}{2}} e^{-x^{\frac{2}{2}}}\, H_n(x),$$

where $H_n(x)$ is the Hermite polynomial of degree n, which is orthogonal over the real line $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$ [14]. In quantum mechanics, the wave functions emerge as eigenfunctions of the Hamiltonian H,

$$H\psi_n(x) = \Big(n + \frac{1}{2}\Big)\psi_n(x), \quad n = 0, 1, 2, \ldots. \tag{1.2}$$

The Fourier transform simply changes the basis from the coordinate basis x to the momentum basis p_x and, consequently, commutes with the harmonic oscillator H. Namely, we have

$$\mathcal{F}H = H\mathcal{F}. \tag{1.3}$$

Form (1.3) in the standard algebraic way expresses the fact that the Hamiltonian H and the Fourier transform $\mathcal{F}$ have a common set of eigenfunctions $\psi_n(x)$. More precisely, the wave functions $\psi_n(x)$ are eigenfunctions of the Fourier transform associated with the eigenvalues i^n, that is,

$$\mathcal{F}(\psi_n)(x) = i^n\, \psi_n(x).$$

The one-dimensional harmonic oscillator was also studied by Schrödinger via Laplace transform when discussing the radial eigenfunction of the hydrogen atom [19], and later, Englefield approached the Schrödinger equation with Coulomb, oscillator, exponential, and Yamaguchi potentials [10].

The fundamental purpose of the present work is to extended the integral approach of the harmonic oscillator to the setting of supersymmetric quantum mechanics "SUSY QM". Let us first recall some mathematical aspects of the supersymmetric quantum mechanics. The "SUSY QM", introduced by Witten [23], may be generated by three operators Q_-, Q_+ and H satisfying

$$Q_\pm^2 = 0, \quad [Q_\pm, H] = 0, \quad \{Q_-, Q_+\} = H,$$

with $\{A, B\} = AB + BA$ denoting the anti-commutator of A and B.

For a complete correspondence with the quantum mechanical oscillator problem, the supersymmetric quantum mechanics models need an analogue of the Fourier transformation. In the present work we fill this gap. Indeed, we propose the Hartley transform as an alternative of the Fourier transform approach to the SUSY quantum mechanics.

Recall that the Hartley transform of a suitable function $f(x)$ is defined by

$$(\mathcal{H}f)(\lambda) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x) \operatorname{cas}(\lambda x)\, dx,$$

where the kernel of the integral, known as cas function, is defined as $\operatorname{cas}(x) = \cos(x) - \sin(x)$. The relation between the Hartley transform and the Fourier transform is given by

$$(\mathcal{H}f)(\lambda) = \sqrt{2}\left(\Re((\mathcal{F}f)(\lambda)) - \Im((\mathcal{F}f)(\lambda))\right),$$

where $\Re$ and $\Im$ denote, respectively, the real and imaginary parts of the Fourier transform. Compared to the Fourier transform, the Hartley transform has the advantages of transforming real functions to real functions (as opposed to requiring complex numbers), also this transform has complementary symmetry properties with respect to their real and imaginary axis and of being its own inverse.

The paper is organized as follows. In Section 2, we recall general properties of the supersymmetric quantum mechanics with reflection. In Section 3, we give some details related to the Hartley transform and difference-differential operator. Finally, in Section 4, we develop the connection between HT and SUSY Quantum Mechanics and exploit it to obtain overcomplete bases for Hartley transform eigenvectors.

2 The Hartley transform

Our first observation in this section is the following representation of the function $\operatorname{cas}(x)$ defined in (2.2) by the power series:

$$\operatorname{cas}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{\binom{n+1}{2}}}{n!}\, x^n, \tag{2.1}$$

where $\binom{n}{2}$ is the binomial coefficient given by

$$\binom{n}{2} = \frac{n(n-1)}{2}\,.$$

Theorem 2.1. *For $\lambda \in \mathbb{C}$, the function $\operatorname{cas}(\lambda x)$ is the unique analytic solution of the problem*

$$\begin{cases} (\partial_x R)u(x) = \lambda u(x), \\ u(0) = 0. \end{cases}$$

Proof. From the well known identity for binomial coefficients

$$\binom{n+1}{2} = \binom{n}{1} + \binom{n}{2} = n + \binom{n}{2},$$

we have

$$\partial_x \operatorname{cas}(\lambda x) = \lambda \sum_{n=1}^{\infty} \frac{(-1)^{\binom{n+1}{2}}}{(n-1)!}(\lambda x)^{n-1} = \lambda \sum_{n=0}^{\infty} \frac{(-1)^{\binom{n+2}{2}}}{n!}(\lambda x)^{n} = -\lambda \operatorname{cas}(-\lambda x).$$

Hence $(\partial_x R)u(x) = \lambda u(x)$. □

Since

$$(-1)^{\binom{2n}{2}} = (-1)^n, \quad (-1)^{\binom{2n+1}{2}} = (-1)^n,$$

the sum in (2.1) turns to be

$$\operatorname{cas}(x) = \cos(x) - \sin(x). \tag{2.2}$$

The Hartley transform pair for f in a suitable functions class is given by (see [4, 12])

$$\begin{cases} (\mathcal{H}f)(\lambda) = \dfrac{1}{\sqrt{\pi}} \displaystyle\int_{\mathbb{R}} f(x) \operatorname{cas}(\lambda x)\, dx, \\ f(x) = \dfrac{1}{\sqrt{\pi}} \displaystyle\int_{\mathbb{R}} (\mathcal{H}f)(\lambda) \operatorname{cas}(\lambda x)\, d\lambda. \end{cases}$$

Accordingly,

$$\mathcal{H}^2 = I.$$

The function $\operatorname{cas}(x)$ satisfies the product formula

$$\operatorname{cas}(x)\operatorname{cas}(y) = \frac{1}{2}\left((1-R)\operatorname{cas}\right)(x+y) + \frac{1}{2}\left((1+R)\operatorname{cas}\right)(x-y).$$

This allows us to define the generalized translation operator related to the differential-difference operator ∂R by

$$\tau_y f(x) = \frac{1}{2}\left((1-R)f\right)(x+y) + \frac{1}{2}\left((1+R)f\right)(x-y),$$

and the convolution product by

$$f * g(x) = \int_{\mathbb{R}} f(y) \tau_x g(y)\, dy.$$

The Hartley transform has the following properties:

$$\mathcal{H}(\tau_x f)(\lambda) = \operatorname{cas}(\lambda x)\mathcal{H}(f)(\lambda), \quad \mathcal{H}(f * g)(\lambda) = \mathcal{H}(f)(\lambda)\mathcal{H}(g)(\lambda).$$

3 SUSY QM with reflection

Let us first recall some mathematical aspects of the supersymmetric quantum mechanics. The "SUSY QM" introduced by Witten [23] can be generated by three operators Q_-, Q_+ and H satisfying

$$Q_\pm^2 = 0, \quad [Q_\pm, H] = 0, \quad \{Q_-, Q_+\} = H \tag{3.1}$$

(with $\{A, B\} = AB + BA$ denoting the anti-commutator of A and B) to facilitate the comparison with the usual harmonic oscillator. The minimal version of $N = 1$ supersymmetric quantum mechanics is achieved by taking the supercharges Q_+ and (Q_-) as product of the bosonic operator a $(a^\dagger)$ defined in (1.1) and the fermionic operator ψ $(\psi^\dagger)$. Namely, we have

$$Q = a\psi^\dagger, \quad Q^\dagger = a^\dagger \psi,$$

where the matrix fermionic creation and annihilation operators are defined via

$$\psi = \sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \psi^\dagger = \sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Thus, ψ and $\psi^\dagger$ obey the usual algebra of the fermionic creation and annihilation operators, namely,

$$\{\psi^\dagger, \psi\} = 1, \quad \{\psi^\dagger, \psi^\dagger\} = \{\psi, \psi\} = 0.$$

They also satisfy the commutation relation

$$[\psi^\dagger, \psi] = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The SUSY Hamiltonian can be rewritten in the form

$$H = QQ^\dagger + Q^\dagger Q = -\frac{d^2}{dx^2} + \frac{1}{4}\, x^2 - \frac{1}{2}\, [\psi, \psi^\dagger].$$

Note that if the supercharge Q in (3.1) is self-adjoint, i.e., $Q^\dagger = Q$. Then $H = 2Q^2$, and the model is said to be $N = \frac{1}{2}$ supersymmetric.

In [18], the authors developed several realizations of $N = \frac{1}{2}$ supersymmetric quantum mechanics in one-dimension by taking the supercharge as the following Dunkl-type difference-differential operator:

$$Q = \frac{1}{\sqrt{2}}\, (\partial_x R + UR + V),$$

where $U(x)$ is even, $V(x)$ is odd, and the operator R is the reflection operator which acts as $Rf(x) = f(-x)$. In this case, the SUSY Hamiltonian takes the form

$$\widehat{H} = Q^2 = -\frac{1}{2}\, \frac{d^2}{dx^2} + \frac{1}{2}\, (U^2 + V^2) + \frac{1}{2}\, \frac{dU}{dx} - \frac{1}{2}\, \frac{dV}{dx}\, R.$$

The wave functions for such systems have been obtained in [18], where it was shown that they define orthogonal polynomials, expressed in terms of Hermite and Jacobi polynomials.

Consider the supercharge

$$Q = \frac{1}{\sqrt{2}}\, (\partial_x R + x). \tag{3.2}$$

Note that this supercharge corresponds to the case $U(x) = 0$ and $V(x) = x$ in (3). Upon computing Q^2, we readily find that

$$\widehat{H} = Q^2 = -\frac{1}{2}\, \frac{d^2}{dx^2} + \frac{1}{2}\, x^2 - \frac{1}{2}\, R.$$

The spectrum of the supersymmetric Hamiltonian $\widehat{H}$ is easily obtained by observing that

$$\widehat{H} = H - \frac{1}{2}\, R,$$

where

$$H = -\frac{1}{2}\, \frac{d^2}{dx^2} + \frac{1}{2}\, x^2.$$

Since

$$R\psi_n(x) = (-1)^n \psi_n(x),$$

it follows from (1.2) that

$$\widehat{H}\psi_n = E_n \psi_n,$$

where

$$E_n = n + \frac{1 - (-1)^n}{2}, \quad n = 0, 1, \dots .$$

Therefore, the spectrum will only consist of even numbers. Each level is degenerate, except for the ground state, which is unique.

4 Eigenfunctions of the Hartley transform

Now, we are interested in finding all eigenfunctions of the Hartley transform operator explicitly. Since mutually commuting operators have the same set of eigenfunctions, one can solve this problem by defining such a self-adjoint operator with a simple spectrum of distinct eigenvalues, which commutes with the Hartley transform.

In what follows, the following lemma is needed.

Lemma 4.1. *For $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq -\beta$, the problem*

$$\begin{cases} -u'(-x) + x(u(x) - u(-x)) + \alpha u(-x) = \beta u(x), \\ u(0) = 1. \end{cases} \tag{4.1}$$

has a unique analytic solution given by

$$u(x) = {}_1F_1\left(\begin{matrix} \frac{\alpha^2-\beta^2}{4} \\ \frac{1}{2} \end{matrix}; x^2\right) + (\alpha - \beta)x\ {}_1F_1\left(\begin{matrix} \frac{2+\alpha^2-\beta^2}{4} \\ \frac{3}{2} \end{matrix}; x^2\right).$$

where

$${}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}$$

is Kummer's confluent hypergeometric function (see [14]).

Proof. Note that one can always write u as the superposition $u = u_e + u_o$ of an even function u_e and of an odd function u_o by the formulae

$$u_e(x) = \frac{u(x) + u(-x)}{2}, \quad u_o(x) = \frac{u(x) - u(-x)}{2}.$$

Further, this decomposition is unique. This allows us to rewrite the eigenvalue equation (4.1) equivalently as a system of two linear differential equations of first order:

$$\begin{cases} u_e' = (\alpha + \beta)u_o, \\ u_o' - 2xu_o = -(\alpha - \beta)u_e. \end{cases} \tag{4.2}$$

We can eliminate the function $u_o(x)$ from system (4.2) and obtain for $u_e(x)$ a second-order differential equation

$$u_e''(x) - 2xu_e'(x) = -(\alpha^2 - \beta^2)u_e(x). \tag{4.3}$$

We choose $t = x^2$ as a new variable and reduce equation (4.3) to

$$tv'' + \left(\frac{1}{2} - t\right)v' = -\frac{\alpha^2 - \beta^2}{4} w,$$

so that the general solution of (4.3) can be written in the form

$$u_e(x) = A\ {}_1F_1\left(\begin{matrix} \frac{\alpha^2-\beta^2}{4} \\ \frac{1}{2} \end{matrix}; x^2\right) + Bx\ {}_1F_1\left(\begin{matrix} \frac{2+\alpha^2-\beta^2}{4} \\ \frac{3}{2} \end{matrix}; x^2\right),$$

where A and B are constants depending on λ, α and β. Since the function $u_e(x)$ is even, we have

$$u_e(x) = A\ {}_1F_1\left(\begin{matrix} \frac{\alpha^2-\beta^2}{4} \\ \frac{1}{2} \end{matrix}; x^2\right).$$

From (4.2), for the function $u_o(x)$ we obtain

$$u_o(x) = A\,\frac{\alpha-\beta}{2}x\;{}_1F_1\left(\begin{matrix}1+\dfrac{\alpha^2-\beta^2}{4}\\ \dfrac{3}{2}\end{matrix};x^2\right).$$

We have the general solution of (4.5)

$$u(x) = A\;{}_1F_1\left(\begin{matrix}\dfrac{\alpha^2-\beta^2}{4}\\ \dfrac{1}{2}\end{matrix};x^2\right) + A(\alpha-\beta)x\;{}_1F_1\left(\begin{matrix}1-\dfrac{\alpha^2-\beta^2}{4}\\ \dfrac{3}{2}\end{matrix};x^2\right).$$

From the initial condition in (4.1), we get $A = 1$. □

The following theorem states that the Hartley transform commutes with the supercharge Q defined in (3.2).

Theorem 4.2. *We have*

$$\mathcal{H}Q = Q\mathcal{H}.$$

Proof. Using integration by parts, we can show that the Hartley transform satisfies the following intertwining relations:

$$\mathcal{H}R = R\mathcal{H}, \quad \mathcal{H}\partial_x R = x\mathcal{H}, \quad \mathcal{H}x = \partial_x R\mathcal{H}.$$

The two last intertwining relations provide the proof of the theorem. □

The ground state wave function $\psi_0(x)$ is given by $\psi_0(x) = e^{-x^{\frac{2}{2}}}$ and satisfies $Q\psi_0 = 0$. Let us now carry out the gauge transformation of Q with the ground state ψ_0. Let

$$\widetilde{Q} = \psi_0^{-1}Q\psi_0. \tag{4.4}$$

It is not difficult to see that

$$\widetilde{Q} = \frac{1}{\sqrt{2}}\frac{d}{dx}R + \frac{1}{\sqrt{2}}x(1-R).$$

From Theorem 4.2, we see that the eigenfunctions of the Hartley transform can be obtained by finding the eigenvalues of the supercharge Q. So, in this way, one reduces the problem of funding the eigenfunctions of the Hartley transform into one of solving the following difference-differential equation

$$-u'(-x) + x(u(x) - u(-x)) = \sqrt{2}\,\lambda u(x). \tag{4.5}$$

From Lemma (4.1), the general solution of (4.5) is given by

$$u(x) = A\left({}_1F_1\left(\begin{matrix}-\dfrac{\lambda^2}{2}\\ \dfrac{1}{2}\end{matrix};x^2\right) - \sqrt{2}\,\lambda x\;{}_1F_1\left(\begin{matrix}1-\dfrac{\lambda^2}{2}\\ \dfrac{3}{2}\end{matrix};x^2\right)\right). \tag{4.6}$$

It can be is easily seen that polynomial solutions are possible only if $\lambda = \pm\sqrt{2n}$, $n = 0, 1, 2, \ldots$. If $\lambda = \pm\sqrt{2n}$, then the first term in (4.6) is a polynomial of degree $2n$ and the second term is a polynomial of degree $2n-1$.

Let us by $\widehat{\psi}_{\pm,n}(x)$ denote the eigenfunction of Q corresponding to the eigenvalue $\lambda_n = \pm\sqrt{2n}$. Then we have the following explicit expressions:

$$\widehat{\psi}_{\pm n}(x) = \kappa_n^{\pm}e^{-x^{\frac{2}{2}}}\left({}_1F_1\left(\begin{matrix}-n\\ \dfrac{1}{2}\end{matrix};x^2\right) \pm 2\sqrt{n}\,x\;{}_1F_1\left(\begin{matrix}1-n\\ \dfrac{3}{2}\end{matrix};x^2\right)\right).$$

The normalized constants $\kappa_{\pm n}$ are also chosen so that

$$\int_{-\infty}^{\infty} |\widehat{\psi}_{\pm n}| \, ds = 1.$$

A simple computation shows that $\kappa_n^{-1} = \kappa_{-n}^{-1} = \pi^{\frac{1}{4}} 2^{n+\frac{1}{2}} (2n!)^{-\frac{1}{2}} n!$, $n = 0, 1, 2, \ldots$. We denote by $\widehat{H}_n(x)$, $n \in \mathbb{Z}$, the orthogonal polynomial extracts that form the orthogonal function $\widehat{\psi}_{\pm n}(x)$. That is,

$$\widehat{\psi}_n(x) = \kappa_n e^{-x^{\frac{2}{2}}} \widehat{H}_n(x).$$

Using the well known explicit expressions of Hermite polynomials in terms of the Confluent hypergeometric series

$$H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} \; {}_1F_1 \begin{pmatrix} -n \\ \frac{1}{2} \end{pmatrix} ; x^2 ,$$

$$H_{2n+1}(x) = (-1)^n \frac{(2n+1)!}{n!} 2x \; {}_1F_1 \begin{pmatrix} -n \\ \frac{3}{2} \end{pmatrix} ; x^2 ,$$

we obtain

$$\widehat{H}_{\pm n}(x) = \frac{(-1)^n n!}{(2n)!} \Big(H_{2n}(x) \mp 2\sqrt{n} \, H_{2n-1}(x) \Big), \quad n = 0, 1, 2, \ldots .$$

They satisfy the orthogonality relations

$$\int_{\mathbb{R}} \widehat{H}_n(x) \widehat{H}_m(x) e^{-x^2} \, dx = \sqrt{\pi} \, 2^{2|n|+1} \frac{(|n|!)^2}{(2|n|)!} \delta_{nm}, \quad n, m \in \mathbb{Z}.$$

The system $\{\widehat{\psi}_{\pm n}(x)\}_{n\in\mathbb{Z}}$ is an orthonormal set in $L^2(\mathbb{R}, dx)$ and it is complete by the same argument which was used to prove that the classical Hermite functions form a complete orthogonal set in $L^2(\mathbb{R}, dx)$. Further, the operator Q with domain $D(Q) = \mathcal{S}(\mathbb{R})$ ($\mathcal{S}(\mathbb{R})$ is the Schwartz space) is essentially self-adjoint; the spectrum of its closure is discrete and, by (4.4), we easily obtain that

$$Q\widehat{\psi}_{\pm n}(x) = \pm\sqrt{2n} \, \widehat{\psi}_{\pm n}(x), \quad n = 0, 1, 2, \ldots .$$

Theorem 4.3. *For $n \in \mathbb{Z}$, we have*

$$\int_{-\infty}^{\infty} \operatorname{cas}(xy) \widehat{H}_n(x) e^{-x^{\frac{2}{2}}} \, dx = (-1)^n \widehat{H}_n(x) e^{-x^{\frac{2}{2}}}.$$

Acknowledgement

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for funding this research through the Research Group No. RGP-VPP-323.

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(Received 9.06.2020)

Authors’ addresses:

F. Bouzeffour

1. Department of Mathematics, College of Sciences, King Saud University, P. O Box 2455 Riyadh 11451, Saudi Arabia.

2. College of Sciences, University of Carthage, B. O. Box 64 Jarzouna 7021, Bizerte, Tunisia.

E-mail: `fbouzaffour@ksu.edu.sa`

M. Garayev

1. Department of Mathematics, College of Sciences, King Saud University, P. O Box 2455 Riyadh 11451, Saudi Arabia.

2. College of Sciences, University of Carthage, B. O. Box 64 Jarzouna 7021, Bizerte, Tunisia.

E-mail: `mgarayev@ksu.edu.sa`