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STUDY OF STABILITY IN NONLINEAR NEUTRAL DYNAMIC EQUATIONS USING KRASNOSELSKII–BURTON'S FIXED POINT THEOREM

Abstract. Let $\mathbb{T}$ be an unbounded above and below time scale such that $0 \in \mathbb{T}$. Let $id - \tau : [0, \infty) \cap \mathbb{T}$ be such that $(id - \tau)([0, \infty) \cap \mathbb{T})$ is a time scale. We use Krasnoselskii–Burton's fixed point theorem to obtain stability results about the zero solution for the following nonlinear neutral dynamic equation with a variable delay:

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + Q(t, x(t - \tau(t)))^{\Delta} + G\big(t, x(t), x(t - \tau(t))\big).$$

The stability of the zero solution of this equation is provided by $h(0) = Q(t, 0) = G(t, 0, 0) = 0$. The Carathéodory condition is used for the functions Q and G. The results obtained here extend the work of Mesmouli, Ardjouni and Djoudi [21].

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რეზიუმე. ვთქვათ, $\mathbb{T}$ ზემოდან და ქვემოდან შემოუსაზღვრელი ისეთი დროის სკალაა, რომ $0 \in \mathbb{T}$, ხოლო $id - \tau : [0, \infty) \cap \mathbb{T}$ ისეთია, რომ $(id - \tau)([0, \infty) \cap \mathbb{T})$ წარმოადგენს დროის სკალას. განხილულია შემდეგი არაწრფივი ნეიტრალური დინამიკური განტოლება ცვლადი დაგვიანებით

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + Q(t, x(t - \tau(t)))^{\Delta} + G\big(t, x(t), x(t - \tau(t))\big)$$

ამ განტოლების ნულოვანი ამონახსნის მდგრადობის დასადგენად გამოყენებულია კრასნოსელ-სკი-ბარტონის უძრავი წერტილის თეორემა. ნულოვანი ამოხსნის მდგრადობას უზრუნველყოფს პირობა $h(0) = Q(t, 0) = G(t, 0, 0) = 0$. Q და G ფუნქციებისთვის გამოიყენება კარათეოდორის პირობა. მიღებული შედეგები ანზოგადებს მესმულის, არჯაუნისა და ჯაუდის [21] ნაშრომში მიღებულ შედეგებს.

1 Introduction

The concept of time scales analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [17]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area by Bohner and Peterson [9] and [10], more and more researchers were getting involved in this fast-growing field of mathematics. The study of dynamic equations brings together the traditional research areas of differential and difference equations. It allows one to handle these two research areas simultaneously, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [1, 4–6, 18] and the references therein).

There is no doubt that the Lyapunov method have been used successfully to investigate stability properties of wide variety of ordinary, functional and partial equations. Nevertheless, the application of this method to the problem of stability in differential equations with a delay has encountered serious difficulties if the delay is unbounded or if the equation has an unbounded term. It has been noticed that some of theses difficulties vanish by using the fixed point technic. Other advantages of fixed point theory over Lyapunov's method is that the conditions of the former are average, while those of the latter are pointwise (see [2–4, 6–8, 12–15, 18–22] and the references therein).

In this paper, we consider the nonlinear neutral dynamic equations with a variable delay given by

$$x^{\Delta}(t) = -a(t)h(x^{\sigma}(t)) + (Q(t, x(t-\tau(t))))^{\Delta} + G\big(t, x(t), x(t-\tau(t))\big), \tag{1.1}$$

with an assumed initial function

$$x(t) = \psi(t), \ \ t \in [m_0, 0] \cap \mathbb{T},$$

where $\mathbb{T}$ is an unbounded above and below time scale such that $0 \in \mathbb{T}$.

Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due to Burton (see [12, Theorem 3]) to show the asymptotic stability and the stability of the zero solution for equation (1.1). Clearly, the present problem is totally nonlinear so that the variation of parameters cannot be applied directly. Then we resort to the idea of adding and subtracting a linear term. As is noted by T. A. Burton in [12], the added term destroys a contraction already present in part of the equation but it replaces it with the so-called large contraction mapping which is suitable for the fixed point theory. During the process we have to transform (1.1) into an integral equation written as a sum of two mappings; one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii's fixed point theorem to show the asymptotic stability and the stability of the zero solution for equation (1.1). In the special case $\mathbb{T} = \mathbb{R}$, Mesmouli, Ardjouni and Djoudi [21] show that the zero solution of (1.1) is asymptotically stable by using Krasnoselskii–Burton's fixed point theorem. Then the results presented in this paper extend the main results obtained in [21].

The paper is organized as follows. In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale. In Section 3, we present the inversion of (1.1) and state the modification of Krasnoselskii's fixed point theorem established by Burton (see [10, Theorem 3] and [14]). For details on Krasnoselskii's theorem, we refer the reader to [23]. We present our main results on the stability in Section 4.

In this paper, we give the assumptions below that will be used in the main results.

(H1) $\tau : [0, \infty) \cap \mathbb{T} \to \mathbb{T}$ is a positive right dense continuous (rd-continuous) function, $id - \tau : [0, \infty) \cap \mathbb{T} \to \mathbb{T}$ is an increasing mapping such that $(id - \tau)([0, \infty) \cap \mathbb{T})$ is closed, where id is the identity function. Moreover, there exists a constant $l_2 > 0$ such that for $0 \le t_1 < t_2$

$$|\tau(t_2) - \tau(t_1)| \le l_2 |t_2 - t_1|.$$

(H2) $\psi : [m_0, 0] \cap T \to \mathbb{R}$ is a rd-continuous function with $m_0 = -\tau(0)$.

(H3) $a : [0, \infty) \cap \mathbb{T} \to (0, \infty)$ is a bounded rd-continuous function and there exists a constant $l_3 > 0$ such that for $0 \le t_1 < t_2$,

$$\left| \int_{t_1}^{t_2} a(u)\, \Delta u \right| \le l_3 |t_2 - t_1|.$$

(H4) $Q : \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function and $Q(t,0)=0$, that is, for $t_1,t_2\geq 0$ and $x,y\in[-R,R]$, where $R\in(0,1]$, there exist the constants $l_0, E_Q>0$ such that

$$|Q(t_1,x)-Q(t_2,y)|\leq l_0|t_1-t_2|+E_Q|x-y|.$$

Also, Q is a bounded function satisfying the Carathéodory condition with respect to $L^1_\Delta([0,\infty)\cap\mathbb{T})$ such that

$$\big|Q(t,\varphi(t-\tau(t)))\big|\leq q_R(t)\leq\frac{\alpha_1}{2}R,$$

where α_1 is a positive constant.

(H5) The function $G:\mathbb{T}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the Carathéodory condition with respect to $L^1_\Delta([0,\infty)\cap\mathbb{T})$, G/a is a bounded function and $G(t,0,0)=0$ such that for $t\geq 0$,

$$\big|G(t,\varphi(t),\varphi(t-\tau(t)))\big|\leq g_{\sqrt{2}\,R}(t)\leq\alpha_2 a(t)R,$$

where α_2 is a positive constant.

(H6) There exists a constant $J>3$ such that

$$J(\alpha_1+\alpha_2)\leq 1$$

and

$$(E_Q+E_Q l_2)l_1+l_0+3R\Big(\frac{\alpha_1}{2}+\alpha_2+\frac{2}{J}\Big)l_3<l_1,$$

where l_1 is a positive constant.

(H7) $h:\mathbb{R}\to\mathbb{R}$ is continuous and strictly increasing on $[-R,R]$, $h(0)=0$, h is differentiable on $(-R,R)$ with $h'(x)\leq 1$ for $x\in(-R,R)$.

(H8) For $\gamma>0$ small enough,

$$[1+E_Q]\gamma+(E_Q+E_Q l_2)l_1+l_0+3R\Big(\frac{\alpha_1}{2}+\alpha_2+\frac{2}{J}\Big)l_3\leq l_1$$

and

$$[1+E_Q]\gamma e_{\ominus a}(t,0)+\frac{3R}{J}\leq R.$$

Also,

$$\max\big\{|H(-R)|,|H(R)|\big\}\leq\frac{2R}{J},$$

where $H(x)=x^\sigma-h(x^\sigma)$.

(H9) $t-\tau(t)\to\infty$, $e_{\ominus a}(t,0)\to 0$, $q_R(t)\to 0$ and $\frac{g_{\sqrt{2}R}(t)}{a(t)}\to 0$ as $t\to\infty$.

2 Preliminaries

In this section, we consider some advanced topics in the theory of dynamic equations on a time scales. Again, we remind that for a review of this topic we direct the reader to the monographs of Bohner and Peterson [9] and [10].

A time scale $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t\in\mathbb{T}$, the forward jump operator σ and the backward jump operator ρ, respectively, are defined as $\sigma(t)=\inf\{s\in\mathbb{T}: s>t\}$ and $\rho(t)=\sup\{s\in\mathbb{T}: s<t\}$. These operators allow the elements in the time scale to be classified as follows. We say t is right scattered if $\sigma(t)>t$ and right dense if $\sigma(t)=t$. We say t is left scattered if $\rho(t)<t$ and left dense if $\rho(t)=t$. The graininess function $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t)=\sigma(t)-t$ and gives the distance between an element and its successor. We set $\inf\varnothing=\sup\mathbb{T}$ and $\sup\varnothing=\inf\mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, we define $\mathbb{T}^k=\mathbb{T}\setminus\{M\}$. Otherwise, we define $\mathbb{T}^k=\mathbb{T}$. If $\mathbb{T}$ has a right scattered minimum m, we define $\mathbb{T}_k=\mathbb{T}\setminus\{m\}$. Otherwise, we define $\mathbb{T}_k=\mathbb{T}$.

Let $t \in \mathbb{T}^k$ and let $f : \mathbb{T} \to \mathbb{R}$. The delta derivative of $f(t)$, denoted by $f^\Delta(t)$, is defined to be the number (if any) with the property that for each $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \le \varepsilon |\sigma(t) - s|$$

for all $s \in U$. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f'(t)$ is the usual derivative. If $\mathbb{T} = \mathbb{Z}$, then $f^\Delta(t) = \Delta f(t) = f(t+1) - f(t)$ is the forward difference of f at t.

A function f is rd-continuous, $f \in C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$, if it is continuous at every right dense point $t \in \mathbb{T}$ and its left-hand limits exist at each left dense point $t \in \mathbb{T}$. The function $f : \mathbb{T} \to \mathbb{R}$ is differentiable on $\mathbb{T}^k$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$.

We are now ready to state some properties of the delta-derivative of f. Note that $f^\sigma(t) = f(\sigma(t))$.

Theorem 2.1 ([9, Theorem 1.20]). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$ and let α be a scalar.*

(i) $(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t)$.

(ii) $(\alpha f)^\Delta(t) = \alpha f^\Delta(t)$.

(iii) *The product rules*

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t),$$
$$(fg)^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t).$$

(iv) *If $g(t)g^\sigma(t) \neq 0$, then*

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}.$$

The next theorem is the chain rule on time scales (see [9, Theorem 1.93]).

Theorem 2.2 (Chain Rule). *Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $v^\Delta(t)$ and $w^{\widetilde{\Delta}}(v(t))$ exist for $t \in \mathbb{T}^k$, then*

$$(w \circ v)^\Delta = (w^{\widetilde{\Delta}} \circ v)v^\Delta.$$

In the sequel, we will need to differentiate and integrate functions of the form $f(t - \tau(t)) = f(v(t))$, where $v(t) := t - \tau(t)$. Our next theorem is the substitution rule (see [9, Theorem 1.98]).

Theorem 2.3 (Substitution). *Assume $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and v is differentiable with an rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)v^\Delta(t)\,\Delta t = \int_{v(a)}^{v(b)} (f \circ v^{-1})(s)\,\widetilde{\Delta} s.$$

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$. The set of all positively regressive functions $\mathcal{R}^+$ is given by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}$.

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log(1 + \mu(z)p(z))\,\Delta z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^\Delta = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given by the following lemma.

Lemma 2.1 ([9, Theorem 2.36]). *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$,

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$,

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$,

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$,

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$,

(vi) $e_p^{\Delta}(\,\cdot\,,s) = pe_p(\,\cdot\,,s)$ *and* $(\frac{1}{e_p(\,\cdot\,,s)})^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\,\cdot\,,s)}$.

Lemma 2.2 ([1]). *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \le \exp\Big(\int_s^t p(u)\,\Delta u\Big), \quad \forall\, t \in \mathbb{T}.$$

Corollary 2.1 ([1]). *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \le t$ we have*

$$0 < e_p(t,s) \le \exp\Big(\int_s^t p(u)\,\Delta u\Big) < 1.$$

3 The inversion and the fixed point theorem

We begin this section with the following

Lemma 3.1. *x is a solution of equation* (1.1) *if and only if*

$$\begin{aligned} x(t) = \big[\psi(0) - Q(0, \psi(-\tau(0)))\big]e_{\ominus a}(t,0) + \int_0^t a(s)e_{\ominus a}(t,s)H(x(s))\,\Delta s + Q(t, x(t-\tau(t))) \\ + \int_0^t e_{\ominus a}(t,s)\Big[-a(s)Q^{\sigma}(s, x(s-\tau(s))) + G\big(s, x(s), x(s-\tau(s))\big)\Big]\,\Delta s, \end{aligned} \tag{3.1}$$

where

$$H(x) = x^{\sigma} - h(x^{\sigma}). \tag{3.2}$$

Proof. Let x be a solution of (1.1). Rewrite equation (1.1) as

$$\begin{aligned} \big(x(t) - Q(t, x(t-\tau(t)))\big)^{\Delta} + a(t)\big[x^{\sigma}(t) - Q^{\sigma}(t, x(t-\tau(t)))\big] \\ = a(t)\big[x^{\sigma}(t) - h(x^{\sigma}(t))\big] - a(t)Q^{\sigma}(t, x(t-\tau(t))) + G\big(t, x(t), x(t-\tau(t))\big). \end{aligned}$$

Multiplying both sides of the above equation by $e_a(t,0)$ and then integrating from 0 to t, we obtain

$$\begin{aligned} \int_0^t \big((x(s) - Q(s, x(s-\tau(s))))e_a(s,0)\big)^{\Delta}\,\Delta s = \int_0^t a(s)\big[x^{\sigma}(s) - h(x^{\sigma}(s))\big]e_a(s,0)\,\Delta s \\ + \int_0^t \Big[-a(s)Q^{\sigma}(s, x(s-\tau(s))) + G\big(s, x(s), x(s-\tau(s))\big)\Big]e_a(s,0)\,\Delta s. \end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}\big[x(t)-Q(t,x(t-\tau(t)))\big]e_a(t,0)-\psi(0)+Q(0,\psi(-\tau(0)))\\ &=\int_0^t a(s)[x^\sigma(s)-h(x^\sigma(s))]e_a(s,0)\,\Delta s\\ &\quad+\int_0^t\Big[-a(s)Q^\sigma(s,x(s-\tau(s)))+G\big(s,x(s),x(s-\tau(s))\big)\Big]e_a(s,0)\,,\Delta s.\end{aligned}$$

By dividing both sides of the above equation by $e_a(t,0)$, we obtain

$$\begin{aligned}x(t)-Q(t,x(t-\tau(t)))-\big[\psi(0)-Q(0,\psi(-\tau(0)))\big]e_{\ominus a}(t,0)\\ &=\int_0^t a(s)[x^\sigma(s)-h(x^\sigma(s))]e_{\ominus a}(t,s)\,\Delta s\\ &\quad+\int_0^t\Big[-a(s)Q^\sigma(s,x(s-\tau(s)))+G\big(s,x(s),x(s-\tau(s))\big)\Big]e_{\ominus a}(t,s)\,\Delta s.\end{aligned}\tag{3.3}$$

The converse implication is easily obtained and the proof is complete. □

Now, we give some definitions which will be used in this paper.

Definition 3.1. The map $f:[0,\infty)\cap\mathbb{T}\times\mathbb{R}^n\to\mathbb{R}$ is an L^1_Δ-Carathéodory function if it satisfies the following conditions:

(i) for each $z\in\mathbb{R}^n$, the mapping $t\mapsto f(t,z)$ is Δ-measurable,

(ii) for almost all $t\in[0,\infty)\cap\mathbb{T}$, the mapping $z\mapsto f(t,z)$ is continuous on $\mathbb{R}^n$,

(iii) for each $r>0$, there exists $\alpha_r\in L^1_\Delta([0,\infty)\cap\mathbb{T},\mathbb{R}^+)$ such that for almost all $t\in[0,\infty)\cap\mathbb{T}$ and for all z such that $|z|<r$, we have $|f(t,z)|\le\alpha_r(t)$.

T. A. Burton studied the theorem of Krasnoselskii (see [14] and [23]) and observed (see [11]) that Krasnoselskii's result may be more interesting in applications with certain changes, and formulated Theorem 3.1 below (see [11] for its proof).

Definition 3.2. Let $(\mathcal{M},d)$ be a metric space and assume that $B:\mathcal{M}\to\mathcal{M}$. B is said to be a large contraction if for $\varphi,\phi\in\mathcal{M}$, with $\varphi\ne\phi$, we have $d(B\varphi,B\phi)<d(\varphi,\phi)$, and if $\forall\,\varepsilon>0$, $\exists\,\delta<1$ such that

$$\big[\varphi,\phi\in\mathcal{M},\ d(\varphi,\phi)\ge\varepsilon\big]\Longrightarrow d(B\varphi,B\phi)<\delta d(\varphi,\phi).$$

It is proved in [11] that a large contraction defined on a closed bounded and complete metric space has a unique fixed point.

Theorem 3.1 (Krasnoselskii–Burton)**.** *Let $\mathcal{M}$ be a closed bounded convex nonempty subset of a Banach space $(\chi,\|\cdot\|)$. Suppose that A and B map $\mathcal{M}$ into $\mathcal{M}$ such that*

(i) *A is continuous and $A\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$,*

(ii) *B is large contraction,*

(iii) *$x,y\in\mathcal{M}$, implies $Ax+By\in\mathcal{M}$.*

Then there exists $z\in\mathcal{M}$ with $z=Az+Bz$.

Here we manipulate the function spaces defined on infinite t-intervals. So, for the compactness, we need an extension of Arzela–Ascoli's theorem. This extension is taken from [14, Theorem 1.2.2, p. 20] and is presented as follows.

Theorem 3.2. *Let $q : [0,\infty) \cap \mathbb{T} \to \mathbb{R}^+$ be an rd-continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\varphi_n(t)\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $[0,\infty) \cap \mathbb{T}$ with $|\varphi_n(t)| \le q(t)$ for $t \in [0,\infty) \cap \mathbb{T}$, then there is a subsequence that converges uniformly on $[0,\infty) \cap \mathbb{T}$ to an rd-continuous function $\varphi(t)$ with $|\varphi(t)| \le q(t)$ for $[0,\infty) \cap \mathbb{T}$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

4 The stability by Krasnoselskii–Burton's theorem

From the existence theory, which can be found in [14] or [16], we conclude that for each rd-continuous initial function $\psi \in C_{rd}([m_0,0] \cap \mathbb{T}, \mathbb{R})$, there exists an rd-continuous solution $x(t,0,\psi)$ which satisfies (1.1) on an interval $[0,\sigma) \cap \mathbb{T}$ for some $\sigma > 0$ and $x(t,0,\psi) = \psi(t)$, $t \in [m_0,0] \cap \mathbb{T}$. We refer the reader to [14] for the stability definitions.

Definition 4.1. The zero solution of (1.1) is said to be stable at $t = 0$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\psi : [m_0,0] \cap \mathbb{T} \to (-\delta,\delta)$ implies that $|x(t)| < \varepsilon$ for $t \ge m_0$.

Definition 4.2. The zero solution of (1.1) is said to be asymptotically stable if it is stable at $t = 0$ and there exists $\delta > 0$ such that for any rd-continuous function $\psi : [m_0,0] \cap \mathbb{T} \to (-\delta,\delta)$, the solution x with $x(t) = \psi(t)$ on $[m_0,0] \cap \mathbb{T}$ tends to zero as $t \to \infty$.

To apply Theorem 3.1, we need to define a Banach space χ, a closed bounded convex subset $\mathcal{M}$ of χ and construct two mappings; one large contraction and the other a compact operator. So, let $\omega : [m_0,\infty) \cap \mathbb{T} \to [1,\infty)$ be any strictly increasing and rd-continuous function with $\omega(m_0) = 1$, $\omega(t) \to \infty$ as $t \to \infty$. Let $(S, |\cdot|_\omega)$ be the Banach space of rd-continuous $\varphi : [m_0,\infty) \cap \mathbb{T} \to \mathbb{R}$ for which

$$|\varphi|_\omega = \sup_{t \ge m_0} \left| \frac{\varphi(t)}{\omega(t)} \right| < \infty.$$

Let $R \in (0,1]$ and define the set

$$\begin{aligned}\mathcal{M} := \Big\{ \varphi \in S :\ & \varphi \text{ is } l_1\text{-Lipschitzian}, \\ & |\varphi(t)| \le R,\ t \in [m_0,\infty) \cap \mathbb{T} \text{ and } \varphi(t) = \psi(t) \text{ if } t \in [m_0,0] \cap \mathbb{T} \Big\}.\end{aligned}$$

Clearly, if $\{\varphi_n\}$ is a sequence of l_1-Lipschitzian functions converging to some function φ, then

$$\begin{aligned}|\varphi(t) - \varphi(s)| &= \big|\varphi(t) - \varphi_n(t) + \varphi_n(t) - \varphi_n(s) + \varphi_n(s) - \varphi(s)\big| \\ &\le |\varphi(t) - \varphi_n(t)| + |\varphi_n(t) - \varphi_n(s)| + |\varphi_n(s) - \varphi(s)| \\ &\le l_1 |t - s|\end{aligned}$$

as $n \to \infty$, which implies that φ is l_1-Lipschitzian. It is clear that $\mathcal{M}$ is closed convex and bounded. For $\varphi \in \mathcal{M}$ and $t \ge 0$, we define by (3.1) the mapping $P : \mathcal{M} \to S$ as follows:

$$\begin{aligned}(P\varphi)(t) = \big[\psi(0) - Q(0, \psi(-\tau(0)))\big] e_{\ominus a}(t,0) + \int_0^t a(s) e_{\ominus a}(t,s) H(\varphi(s))\, \Delta s + Q(t, \varphi(t - \tau(t))) \\ + \int_0^t e_{\ominus a}(t,s) \Big[- a(s) Q^\sigma(s, \varphi(s - \tau(s))) + G\big(s, \varphi(s), \varphi(s - \tau(s))\big)\Big] \Delta s. \quad (4.1)\end{aligned}$$

Therefore, we express mapping (4.1) as

$$P\varphi = A\varphi + B\varphi,$$

where $A, B : \mathcal{M} \to S$ are given by

$$(A\varphi)(t) = Q(t, \varphi(t - \tau(t))) + \int_0^t e_{\ominus a}(t, s)\Big[- a(s)Q^\sigma(s, \varphi(s - \tau(s))) + G\big(s, \varphi(s), \varphi(s - \tau(s))\big)\Big]\,\Delta s, \tag{4.2}$$

$$(B\varphi)(t) = \big[\psi(0) - Q(0, \psi(-\tau(0)))\big]e_{\ominus a}(t, 0) + \int_0^t a(s)e_{\ominus a}(t, s)H(\varphi(s))\,\Delta s. \tag{4.3}$$

By applying Theorem 3.1, we need to prove that P has a fixed point φ on the set $\mathcal{M}$, where $x(t, 0, \psi) = \varphi(t)$ for $t \geq 0$ and $x(t, 0, \psi) = \psi(t)$ on $[m_0, 0]\cap\mathbb{T}$, $x(t, 0, \psi)$ satisfies (1.1) and $|x(t, 0, \psi)| \leq R$ with $R \in (0, 1]$.

By a series of steps we will prove the fulfillment of (i), (ii) and (iii) of Theorem 3.1.

Lemma 4.1. *For A defined in* (4.2)*, suppose that* (H1)–(H6) *hold. Then $A : \mathcal{M} \to \mathcal{M}$ and A is continuous and $A\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$.*

Proof. Let A be defined by (4.2). Then for any $\varphi \in \mathcal{M}$, we have

$$\begin{aligned}|(A\varphi)(t)| &\leq |Q(t, \varphi(t - \tau(t)))| + \int_0^t e_{\ominus a}(t, s)\Big[a(s)|Q^\sigma(s, \varphi(s - \tau(s)))| + \big|G\big(s, \varphi(s), \varphi(s - \tau(s))\big)\big|\Big]\,\Delta s \\ &\leq q_R(t) + R\int_0^t e_{\ominus a}(t, s)\Big(a(s)\,\frac{q_R(s)}{R} + \frac{g_{\sqrt{2}R}(s)}{R}\Big)\,\Delta s \leq \frac{\alpha_1}{2}\,R + \frac{\alpha_1}{2}\,R + \alpha_2 R \leq \frac{R}{J} < R.\end{aligned}$$

That is, $|(A\varphi)(t)| < R$. Second, we show that for any $\varphi \in \mathcal{M}$, the function $A\varphi$ is l_1-Lipschitzian. Let $\varphi \in \mathcal{M}$, and let $0 \leq t_1 < t_2$, then

$$\begin{aligned}\big|(A\varphi)(t_2) - (A\varphi)(t_1)\big| &\leq \Big|Q(t_2, \varphi(t_2 - \tau(t_2))) - Q(t_1, \varphi(t_1 - \tau(t_1)))\Big| \\ &\quad + \Big|\int_0^{t_2} e_{\ominus a}(t_2, s)\Big[- a(s)Q^\sigma(s, \varphi(s - \tau(s))) + G\big(s, \varphi(s), \varphi(s - \tau(s))\big)\Big]\,\Delta s \\ &\quad - \int_0^{t_1} e_{\ominus a}(t_1, s)\Big[- a(s)Q^\sigma(s, \varphi(s - \tau(s))) + G\big(s, \varphi(s), \varphi(s - \tau(s))\big)\Big]\,\Delta s\Big|.\end{aligned} \tag{4.4}$$

By hypotheses (H1), (H3) and (H4), we have

$$\begin{aligned}&\Big|Q(t_2, \varphi(t_2 - \tau(t_2))) - Q(t_1, \varphi(t_1 - \tau(t_1)))\Big| \\ &\quad\leq l_0|t_2 - t_1| + E_Q l_1\big|(t_2 - t_1) - (\tau(t_2) - \tau(t_1))\big| \leq (l_0 + E_Q l_1 + E_Q l_1 l_2)|t_2 - t_1|,\end{aligned} \tag{4.5}$$

where l_1 is the Lipschitz constant of φ. In the same way, by (H3)–(H5), we have

$$\begin{aligned}&\Big|\int_0^{t_2} e_{\ominus a}(t_2, s)\Big[- a(s)Q^\sigma(s, \varphi(s - \tau(s))) + G\big(s, \varphi(s), \varphi(s - \tau(s))\big)\Big]\,\Delta s \\ &\quad - \int_0^{t_1} e_{\ominus a}(t_1, s)\Big[- a(s)Q^\sigma(s, \varphi(s - \tau(s))) + G\big(s, \varphi(s), \varphi(s - \tau(s))\big)\Big]\,\Delta s\Big|\end{aligned}$$

$$\leq \Big| \int_0^{t_1} \Big[-a(s)Q^\sigma(s,\varphi(s-\tau(s))) + G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big]\,\Delta s \cdot e_{\ominus a}(t_1,s)(e_{\ominus a}(t_2,t_1)-1)\,\Delta s\Big|$$

$$+ \Big| \int_{t_1}^{t_2} e_{\ominus a}(t_2,s)\Big[-a(s)Q^\sigma(s,\varphi(s-\tau(s))) + G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big]\,\Delta s\Big|$$

$$\leq \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R|e_{\ominus a}(t_2,t_1)-1| \int_0^{t_1} a(s)e_{\ominus a}(t_1,s)\,\Delta s$$

$$+ \int_{t_1}^{t_2} e_{\ominus a}(t_2,s)\big(a(s)q_R(s)+g_{\sqrt{2}R}(s)\big)\,\Delta s$$

$$\leq \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s + \int_{t_1}^{t_2} a(s)e_{\ominus a}(t_2,s)\Big(\int_{t_1}^{s}\big(a(r)q_R(r)+g_{\sqrt{2}R}(r)\big)\,\Delta r\Big)^{\Delta}\Delta s$$

$$\leq \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s + \Big[e_{\ominus a}(t_2,s)\int_{t_1}^{s}\big(a(r)q_R(r)+g_{\sqrt{2}R}(r)\big)\,\Delta r\Big]_{t_1}^{t_2}$$

$$+ \int_{t_1}^{t_2} a(s)e_{\ominus a}(t_2,s)\int_{t_1}^{s}\big(a(r)q_R(r)+g_{\sqrt{2}R}(r)\big)\,\Delta r\,\Delta s$$

$$\leq \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s + \int_{t_1}^{t_2}\big(a(s)q_R(s)+g_{\sqrt{2}R}(s)\big)\,\Delta s\Big(1+\int_{t_1}^{t_2} a(s)e_{\ominus a}(t_2,s)\,\Delta s\Big)$$

$$\leq \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s + 2\int_{t_1}^{t_2}\big(a(s)q_R(s)+g_{\sqrt{2}R}(s)\big)\,\Delta s$$

$$\leq \Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s + 2\Big(\frac{\alpha_1}{2}+\alpha_2\Big)R\int_{t_1}^{t_2} a(s)\,\Delta s \leq 3R\Big(\frac{\alpha_1}{2}+\alpha_2\Big)l_3|t_2-t_1|. \quad (4.6)$$

Thus, by substituting (4.5) and (4.6) into (4.4), we obtain

$$\big|(A\varphi)(t_2)-(A\varphi)(t_1)\big| \leq (l_0+E_Q l_1+E_Q l_1 l_2)|t_2-t_1| + 3R\Big(\frac{\alpha_1}{2}+\alpha_2\Big)l_3|t_2-t_1| \leq l_1|t_2-t_1|.$$

This shows that $A\varphi$ is l_1-Lipschitzian if φ is. This completes the proof that $A:\mathcal{M}\to\mathcal{M}$.

Since $A\varphi$ is l_1-Lipschitzian, we have that $A\mathcal{M}$ is equicontinuous, which implies that the set $A\mathcal{M}$ resides in a compact set in the space $(S,|\cdot|_\omega)$.

Now, we show that A is continuous in the weighted norm letting $\varphi_n \in \mathcal{M}$, where n is a positive integer such that $\varphi_n \to \varphi$ as $n\to\infty$. Then

$$\Big|\frac{(A\varphi_n)(t)-(A\varphi)(t)}{\omega(t)}\Big| \leq \Big|Q(t,\varphi_n(t-\tau(t))) - Q(t,\varphi(t-\tau(t)))\Big|_\omega$$

$$+ \int_0^t a(s)e_{\ominus a}(t,s)\Big|Q^\sigma(s,\varphi_n(s-\tau(s))) - Q^\sigma(s,\varphi(s-\tau(s)))\Big|_\omega\,\Delta s$$

$$+ \int_0^t e_{\ominus a}(t,s)\Big|G\big(s,\varphi_n(s),\varphi_n(s-\tau(s))\big) - G\big(s,\varphi(s),\varphi(s-\tau(s))\big)\Big|_\omega\,\Delta s.$$

By the dominated convergence theorem, $\lim_{n\to\infty} |(A\varphi_n)(t) - (A\varphi)(t)|_\omega = 0$. Then A is continuous. This completes the proof that $A : \mathcal{M} \to \mathcal{M}$ is continuous and $A\mathcal{M}$ is contained in a compact subset of $\mathcal{M}$. $\square$

Now, we state an important result implying that the mapping H given by (3.2) is a large contraction on the set $\mathcal{M}$. This result was already obtained in [1] and for convenience we present below its proof.

Theorem 4.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying* (H7). *Then the mapping H in* (3.2) *is a large contraction on the set $\mathcal{M}$.*

Proof. Let $\varphi^\sigma, \phi^\sigma \in \mathcal{M}$ with $\varphi^\sigma \neq \phi^\sigma$. Then $\varphi^\sigma(t) \neq \phi^\sigma(t)$ for some $t \in \mathbb{T}$. Let us denote the set of all such t by $D(\varphi, \phi)$, i.e.,

$$D(\varphi,\phi) = \big\{t \in \mathbb{T} :\ \varphi^\sigma(t) \neq \phi^\sigma(t)\big\}.$$

For all $t \in D(\varphi, \phi)$, we have

$$\begin{aligned} |(H\varphi)(t) - (H\phi)(t)| &\leq \big|\varphi^\sigma(t) - \phi^\sigma(t) - h(\varphi^\sigma(t)) + h(\phi^\sigma(t))\big| \\ &\leq |\varphi^\sigma(t) - \phi^\sigma(t)| \left|1 - \frac{h(\varphi^\sigma(t)) - h(\phi^\sigma(t))}{\varphi^\sigma(t) - \phi^\sigma(t)}\right|. \end{aligned} \tag{4.7}$$

Since h is a strictly increasing function, we have

$$\frac{h(\varphi^\sigma(t)) - h(\phi^\sigma(t))}{\varphi^\sigma(t) - \phi^\sigma(t)} > 0 \ \text{ for all } \ t \in D(\varphi,\phi). \tag{4.8}$$

For each fixed $t \in D(\varphi, \phi)$, we define the interval $I_t \subset [-R, R]$ by

$$I_t = \begin{cases} (\varphi^\sigma(t), \phi^\sigma(t)) & \text{if } \ \varphi^\sigma(t) < \phi^\sigma(t), \\ (\phi^\sigma(t), \varphi^\sigma(t)) & \text{if } \ \phi^\sigma(t) < \varphi^\sigma(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\varphi, \phi)$ there exists a real number $c_t \in I_t$ such that

$$\frac{h(\varphi^\sigma(t)) - h(\phi^\sigma(t))}{\varphi^\sigma(t) - \phi^\sigma(t)} = h'(c_t).$$

By (H7), we have

$$0 \leq \inf_{s\in(-R,R)} h'(s) \leq \inf_{s\in I_t} h'(s) \leq h'(c_t) \leq \sup_{s\in I_t} h'(s) \leq \sup_{s\in(-R,R)} h'(s) \leq 1. \tag{4.9}$$

Hence, by (4.7)–(4.9), we obtain

$$|(H\varphi)(t) - (H\phi)(t)| \leq |\varphi^\sigma(t) - \phi^\sigma(t)| \left|1 - \inf_{s\in(-R,R)} h'(s)\right| \tag{4.10}$$

for all $t \in D(\varphi, \phi)$. This implies a large contraction in the supremum norm. To see this, choose a fixed $\varepsilon \in (0, 1)$ and assume that φ and ϕ are two functions in $\mathcal{M}$ satisfying

$$\varepsilon \leq \sup_{t\in(-R,R)} |\varphi(t) - \phi(t)| = \|\varphi - \phi\|.$$

If $|\varphi^\sigma(t) - \phi^\sigma(t)| \leq \frac{\varepsilon}{2}$ for some $t \in D(\varphi, \phi)$, then we get by (4.9) and (4.10) that

$$\big|(H\varphi)(t) - (H\phi)(t)\big| \leq \frac{1}{2}\, |\varphi^\sigma(t) - \phi^\sigma(t)| \leq \frac{1}{2}\, \|\varphi - \phi\|. \tag{4.11}$$

Since h is continuous and strictly increasing, the function $h(s + \frac{\varepsilon}{2}) - h(s)$ attains its minimum on the closed and bounded interval $[-R, R]$. Thus, if $\frac{\varepsilon}{2} \leq |\varphi^\sigma(t) - \phi^\sigma(t)|$ for some $t \in D(\varphi, \phi)$, then by (H7) we conclude that

$$1 \geq \frac{h(\varphi^\sigma(t)) - h(\phi^\sigma(t))}{\varphi^\sigma(t) - \phi^\sigma(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2R} \min \Big\{ h\Big(s + \frac{\varepsilon}{2}\Big) - h(s) : \ s \in [-R, R] \Big\} > 0.$$

Hence, (4.7) implies

$$\big|(H\varphi)(t) - (H\phi)(t)\big| \le (1 - \lambda)\|\varphi - \phi\|. \tag{4.12}$$

Consequently, combining (4.11) and (4.12) we obtain

$$\big|(H\varphi)(t) - (H\phi)(t)\big| \le \delta\|\varphi - \phi\|,$$

where

$$\delta = \max\Big\{\frac{1}{2}, 1 - \lambda\Big\}.$$

□

The relations of (H8) will be used below in Lemma 4.2 and Theorem 4.2 to show that if $\varepsilon = R$ and $\|\psi\| < \gamma$, then the solution satisfies $|x(t, 0, \psi)| < \varepsilon$.

Lemma 4.2. *Let B be defined by* (4.3). *Suppose that* (H1)–(H3), (H7) *and* (H8) *hold. Then $B : \mathcal{M} \to \mathcal{M}$ and B is a large contraction.*

Proof. Let B be defined by (4.3). Obviously, B is continuous with the weighted norm. Let $\varphi \in \mathcal{M}$,

$$\begin{aligned} |(B\varphi)(t)| &\le \big|\psi(0) - Q(0, \psi(-\tau(0)))\big| e_{\ominus a}(t, 0) + \int_0^t a(s) e_{\ominus a}(t, s) |H(\varphi(s))| \, \Delta s \\ &\le [1 + E_Q]\gamma e_{\ominus a}(t, 0) + \int_0^t a(s) e_{\ominus a}(t, s) \max\big\{|H(-R)|, |H(R)|\big\} \, \Delta s \le R, \end{aligned}$$

and we use a method like in Lemma 4.1 and deduce that for any $\varphi \in \mathcal{M}$, the function $B\varphi$ is l_1-Lipschitzian, which implies $B : \mathcal{M} \to \mathcal{M}$.

By Theorem 4.1, H is a large contraction on $\mathcal{M}$, then for any $\varphi, \phi \in \mathcal{M}$ with $\varphi \ne \phi$ and for any $\varepsilon > 0$, from the proof of that theorem, we have found that $\delta < 1$ such that

$$\Big|\frac{B\varphi(t) - B\phi(t)}{\omega(t)}\Big| \le \int_0^t a(s) e_{\ominus a}(t, s) \big|H(\varphi(s)) - H(\phi(s))\big|_\omega \, \Delta s \le \delta |\varphi - \phi|_\omega.$$

□

Theorem 4.2. *Assume that* (H1)–(H8) *hold. Then the zero solution of* (1.1) *is stable.*

Proof. By Lemmas 4.1 and 4.3, $A : \mathcal{M} \to \mathcal{M}$ is continuous and $A\mathcal{M}$ is contained in a compact set. Also, from Lemma 4.2, the mapping $B : \mathcal{M} \to \mathcal{M}$ is a large contraction. First, we show that if $\varphi, \phi \in \mathcal{M}$, we have $\|A\varphi + B\phi\| \le R$. Let $\varphi, \phi \in \mathcal{M}$ with $\|\varphi\|, \|\phi\| \le R$, then

$$\|A\varphi + B\phi\| \le (1 + E_Q)\gamma e_{\ominus a}(t, 0) + (\alpha_1 + \alpha_2)R + \frac{2R}{J} \le (1 + E_Q)\gamma e_{\ominus a}(t, 0) + \frac{R}{J} + \frac{2R}{J} \le R.$$

Next, we prove that for any $\varphi, \phi \in \mathcal{M}$, the function $A\varphi + B\phi$ is l_1-Lipschitzian. Let $\varphi, \phi \in \mathcal{M}$, and let $0 \le t_1 < t_2$, then

$$\begin{aligned} &\big|(A\varphi + B\phi)(t_2) - (A\varphi + B\phi)(t_1)\big| \\ &\qquad \le \Big([1 + E_Q]\gamma + (E_Q + E_Q l_2) l_1 + l_0 + 3R\Big(\frac{\alpha_1}{2} + \alpha_2 + \frac{2}{J}\Big) l_3\Big)|t_2 - t_1| \le l_1 |t_2 - t_1|. \end{aligned}$$

Clearly, all the hypotheses of the Krasnoselskii–Burton theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = Az + Bz$. By Lemma 3.1, this fixed point is a solution of (1.1). Hence, the zero solution of (1.1) is stable. □

Remark 1. When $\mathbb{T} = \mathbb{R}$, Theorem 4.2 reduces to Theorem 4 of [21]. Therefore, Theorem 4.2 is a generalization of Theorem 4 of [21].

Now, for the asymptotic stability, define $\mathcal{M}_0$ by

$$\mathcal{M}_0 := \Big\{\varphi \in S : \ \varphi \text{ is } l_1\text{-Lipschitzian}, \ |\varphi(t)| \le R, \ t \in [m_0, \infty) \cap \mathbb{T}, \\ \varphi(t) = \psi(t) \text{ if } t \in [m_0, 0] \cap \mathbb{T} \text{ and } |\varphi(t)| \to 0 \text{ as } t \to \infty\Big\}.$$

All calculations in the proof of Theorem 4.2 hold with $\omega(t) = 1$, when $|\cdot|_\omega$ is replaced by the supremum norm $\|\cdot\|$.

Lemma 4.3. *Let* (H1)–(H6) *and* (H9) *hold. Then the operator A maps $\mathcal{M}$ into a compact subset of $\mathcal{M}$.*

Proof. First, we deduce by Lemma 4.1 that $A\mathcal{M}$ is equicontinuous. Next, we notice that for an arbitrary $\varphi \in \mathcal{M}$, we have

$$|(A\varphi)(t)| \le q_R(t) + \int_0^t e_{\ominus a}(t,s)a(s)\Big(q_R(s) + \frac{g_{\sqrt{2}R}(s)}{a(s)}\Big)\,\Delta s := q(t).$$

We see that $q(t) \to 0$ as $t \to \infty$ which implies that the set $A\mathcal{M}$ resides in a compact set in the space $(S, \|\cdot\|)$ by Theorem 3.2. □

Theorem 4.3. *Assume that* (H1)–(H9) *hold. Then the zero solution of* (1.1) *is asymptotically stable.*

Proof. Note that all of the steps in the proof of Theorem 4.2 hold with $\omega(t) = 1$ when $|\cdot|_\omega$ is replaced by the supremum norm $\|\cdot\|$. It suffices to show that for $\varphi \in \mathcal{M}_0$ we have $A\varphi \to 0$ and $B\varphi \to 0$. Let $\varphi \in \mathcal{M}_0$ be fixed, we will prove that $|(A\varphi)(t)| \to 0$ as $t \to \infty$. As above, we get

$$|(A\varphi)(t)| \le \big|Q(t, \varphi(t-\tau(t)))\big| \\ + \int_0^t e_{\ominus a}(t,s)\Big[a(s)\big|Q^\sigma(s, \varphi(s-\tau(s)))\big| + \big|G\big(s, \varphi(s), \varphi(s-\tau(s))\big)\big|\Big]\,\Delta s.$$

First of all, we have

$$\big|Q(t, \varphi(t-\tau(t)))\big| \le q_R(t) \to 0 \ \text{ as } \ t \to \infty.$$

Second, let $\varepsilon > 0$ be given. Find T such that $|\varphi(t-\tau(t))|, |\varphi(t)| < \varepsilon$ for $t \ge T$. Then we have

$$\int_0^t e_{\ominus a}(t,s)\Big[a(s)\big|Q^\sigma(s, \varphi(s-\tau(s)))\big| + \big|G\big(s, \varphi(s), \varphi(s-\tau(s))\big)\big|\Big]\,\Delta s \\ = e_{\ominus a}(t,T)\int_0^T e_{\ominus a}(T,s)\Big[a(s)\big|Q^\sigma(s, \varphi(s-\tau(s)))\big| + \big|G\big(s, \varphi(s), \varphi(s-\tau(s))\big)\big|\Big]\,\Delta s \\ + \int_T^t e_{\ominus a}(t,s)\Big[a(s)\big|Q^\sigma(s, \varphi(s-\tau(s)))\big| + \big|G\big(s, \varphi(s), \varphi(s-\tau(s))\big)\big|\Big]\,\Delta s \\ \le e_{\ominus a}(t,T)\Big(\frac{\alpha_1}{2} + \alpha_2\Big)R + \Big(\frac{\alpha_1}{2} + \alpha_2\Big)\varepsilon.$$

By (H9), the term $e_{\ominus a}(t,T)(\frac{\alpha_1}{2} + \alpha_2)R$ is arbitrarily small as $t \to \infty$. In the same way, we obtain $B\varphi \to 0$. Then, by the Krasnoselskii–Burton theorem, there exists a fixed point $z \in \mathcal{M}_0$ such that $z = Az + Bz$. By Lemma 3.1, this fixed point is a solution of (1.1). Hence, the zero solution of (1.1) is asymptotically stable. □

Remark 2.

1) When $\mathbb{T} = \mathbb{R}$, Theorem 4.3 reduces to Theorem 5 of [21]. Therefore, Theorem 4.3 is a generalization of Theorem 5 of [21].
2) The sufficient conditions (H1)–(H9) of Theorem 4.3 are essential for applying Theorems 3.1 and 3.2.

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