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SOLVABILITY AND NUMERICAL APPROXIMATION OF THE SHELL EQUATION DERIVED BY THE Γ-CONVERGENCE

Abstract. A mixed boundary value problem for the Lamé equation in a thin layer $\Omega^h = \mathcal{C} \times [-h, h]$ around a surface $\mathcal{C}$ with the Lipshitz boundary is investigated. The main goal is to find out what happens when the thickness of the layer tends to zero, $h \to 0$. To this end, we reformulate BVP into an equivalent variational problem and prove that the energy functional has the Γ-limit of the energy functional on the mid-surface $\mathcal{C}$. The corresponding BVP on $\mathcal{C}$, considered as the Γ-limit of the initial BVP, is written in terms of Günter's tangential derivatives on $\mathcal{C}$ and represents a new form of the shell equation. It is shown that the Neumann boundary condition from the initial BVP on the upper and lower surfaces transforms into the right-hand side of the basic equation of the limit BVP. The finite element method is established for the obtained BVP.

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Key words and phrases. Hypersurface, Günter's derivatives, Lamé equation, Γ-convergence, shell equation.

რეზიუმე. გამოკვლეულია ლიფშიცის საზღვრიანი $\mathcal{C}$ ზედაპირის მომცველ $\Omega^h = \mathcal{C} \times [-h, h]$ თხელ ფენაში განსაზღვრული ლამეს განტოლებისთვის დასმული შერეული სასაზღვრო ამოცანა. გამოკვლევის მთავარი მიზანია დადგინდეს რა ხდება, როდესაც ფენის h სისქე ნულისკენ მიისწრაფვის, $h \to 0$. ამისთვის, ჩვენ, დასმული სასაზღვრო ამოცანა შევცვალეთ ექვივალენტური ვარიაციული ამოცანით და ვაჩვენეთ, რომ ენერგიის ფუნქციონალი, Γ-კრებადობის აზრით, მიისწრაფვის $\mathcal{C}$ შუა ზედაპირზე განსაზღვრული ფუნქციონალისკენ. ამ ფუნქციონალის შესაბამისი სასაზღვრო ამოცანა განიხილება როგორც თავდაპირველი სასაზღვრო ამოცანის Γ-ზღვარი. ის ჩაწერილია $\mathcal{C}$ ზედაპირზე განსაზღვრულ გიუნტერის მხებ წარმოებულებში და წარმოადგენს გარსის განტოლების ახალ ფორმას. ნაჩვენებია, რომ თავდაპირველი ამოცანის ნეიმანის სასაზღვრო პირობები, ზედა და ქვედა საზღვარზე, გარდაიქმნება ზღვრული განტოლების მარჯვენა მხარედ. მიღებული სასაზღვრო ამოცანისთვის დაფუძნებულია სასრულ ელემენტთა მეთოდი.

1 Introduction

In the present paper, we study a mixed boundary value problem for the Lamé equation in a thin layer $\Omega^h := \mathcal{C} \times [-h, h]$ of thickness $2h$ around a smooth mid-hypersurface $\mathcal{C} \subset \mathbb{R}^3$ written in terms of Günter's derivatives and the energy functional associated to it. We show that when thickness of the layer tends to zero, $h \to 0$, the corresponding energy functional, scaled properly, converges in the Γ-limit sense to some functional defined on mid-surface $\mathcal{C}$ of the layer, which corresponds to the two-dimensional boundary value problem for associated Euler–Lagrange equation in terms of Günter's derivatives. The obtained equations together with boundary conditions can be considered as a boundary value problem defined on a shell model. We employ Galerkin's method to establish numerical approximation for solutions of the obtained BVP.

The equations of three-dimensional linearized elasticity have been studied mostly in Cartesian coordinates. The linear shell theory justified in the present paper is based on the natural curvilinear coordinates, defined on the mid-surface $\mathcal{C}$ extended by the normal vector field of this surface, which "follow the geometry" of the shell in a most natural way. Accordingly, the purpose of the present preliminary section is to provide a thorough derivation and a mathematical treatment of the equations of linearized three-dimensional elasticity in terms of special curvilinear coordinates.

Let $\mathcal{C} \subset \mathbb{R}^3$ be an open surface with the boundary $\Gamma = \partial\mathcal{C}$ in the Euclidean space $\mathbb{R}^3$, represented by a single coordinate function $\theta : \omega \to \mathcal{C}$ (the case of multiple coordinate function is similar and we skip this case for the simplicity). Let $\boldsymbol{\nu}(\mathcal{X}) = (\nu_1(\mathcal{X}), \nu_2(\mathcal{X}), \nu_3(\mathcal{X}))^\top$, $\mathcal{X} \in \mathcal{C}$, be the normal vector field on $\mathcal{C}$ and $\nu(x) = (\mathcal{N}_1(x), \mathcal{N}_2(x), \mathcal{N}_3(x))^\top$ be its extension in the neighbourhood $U_{\mathcal{C}}$ of the surface $\mathcal{C}$. It is known that such extension is unique under the assumption that the extension, as the field on the surface itself, is a gradient vector field $\partial_j\mathcal{N}_k = \partial_k\mathcal{N}_j$ for all $j, k = 1, 2, 3$ and is called the proper extension (see [6] for details).

The 3-tuple of tangential vector fields to the surface $\mathbf{g}_1 := \partial_1\Theta, \mathbf{g}_2 := \partial_2\Theta$ (the covariant basis) together with the proper extension $\mathbf{g}_3 := \mathcal{N}$ of normal vector field $\boldsymbol{\nu}$ from the surface $\mathcal{C}$ into the neighborhood Ω^h depend only on the variable $x' \in \mathcal{C}$ and constitute a basis in Ω^h. That means that an arbitrary vector field $\mathbf{U} = \sum\limits_{j=1}^{3} U_j\mathbf{e}^j$ can also be represented with this basis in "curvilinear coordinates". Along with the covariant basis, the use is made of the contravariant basis $\mathbf{g}^1$, $\mathbf{g}^2$ which is the bi-orthogonal system to the covariant basis $\langle \mathbf{g}_j, \mathbf{g}^k \rangle = \delta_{jk}$, where δ_{jk} denotes Kroneker's symbol, $j, k = 1, 2$ (see, e.g., [3, 4]). In the classical geometry, the covariant $\{\langle \mathbf{g}_i, \mathbf{g}_k \rangle\}_{j,k=1,2}$ and contravariant $\{\langle \mathbf{g}^i, \mathbf{g}^k \rangle\}_{j,k=1,2}$ metric tensors together with the Christofell symbols $\Gamma^i_{jk} := \langle \mathbf{g}^i, \partial_j\mathbf{g}_k \rangle$ are the main tools of the calculus. For example, the covariant derivatives on the surface $\mathcal{C}$ are defined by $v_{i\|j} := \partial_j v_i - \sum\limits_{k=1}^{2} \Gamma^k_{ij} v_k$.

Our calculus on the surface $\mathcal{C}$ is based on a different curvilinear system of coordinates than the covariant and contravariant vector fields used usually by mathematicians and mechanists to derive the shell equations (see, e.g., P. Ciarlet [3,4]). Moreover, the system of curvilinear coordinates introduced below is linearly dependent but, surprisingly, many partial differential equations are written in this system in a simple form, including Laplace–Beltramy and shell equations on a hypersurface (see [5].

From now on, if not stated otherwise, we stick to the following notation: the terms with repeated indices are implicitly summed from 1 to 3 if indices are Greek $(\alpha, \beta, \gamma, \dots)$ and are summed from 1 to 4 if indices are Latin $(j, k, l, \dots)$, as shown in the following examples:

$$a_\alpha b_\alpha := \sum_{\alpha=1}^{3} a_\alpha b_\alpha, \quad b_\alpha^2 := \sum_{\alpha=1}^{3} b_\alpha^2, \quad c_j d_j := \sum_{j=1}^{4} c_j d_j, \quad c_j^2 := \sum_{j=1}^{4} c_j^2.$$

We consider a deformation of an isotropic layer domain $\Omega^h := \mathcal{C} \times (-h, h)$ of thickness $2h$ around the mid-surface $\mathcal{C}$ which has the nonempty Lipschitz boundary $\partial\mathcal{C}$. The deformation is governed by the Lamé equation with the classical mixed boundary conditions, Dirichlet conditions on the lateral surface $\Gamma^h_L := \partial\mathcal{C}\times(-h,h)$ and Neumann conditions on the upper and lower surfaces $\Gamma^{\pm} := \mathcal{C}\times\{\pm h\}$:

$$\begin{gathered}\mathcal{L}_{\Omega^h}\mathbf{U}(x)=\mathbf{F}(x),\ \ x\in\Omega^h:=\mathcal{C}\times(-h,h),\\ \mathbf{U}^+(t)=\mathbf{G}(t),\ \ t\in\Gamma^h_L:=\partial\mathcal{C}\times(-h,h),\\ (\mathfrak{T}(\mathcal{x},\nabla)\mathbf{U})^+(\mathcal{x})=\mathbf{H}(\mathcal{x},\pm h),\ \ (\mathcal{x},t)\in\Gamma^{\pm}\ =\mathcal{C}\times\{\pm h\}.\end{gathered}\tag{1.1}$$

Here $\mathbf{U}(x)=(U_1(x),U_2(x),U_3(x))^\top$ is the displacement vector, $\mathcal{L}_{\Omega^h}$ is the Lamé differential operator and $\mathfrak{T}(\mathcal{x},\nabla)$ is the traction operator

$$\begin{gathered}\mathcal{L}_{\Omega^h}\mathbf{U}=-\mu\,\boldsymbol{\Delta}\mathbf{U}-(\lambda+\mu)\,\nabla\operatorname{div}\mathbf{U},\\ [\mathfrak{T}(\mathcal{x},\nabla)\mathbf{U}]_\beta=\lambda\,\nu_\beta\partial_\gamma\mathbf{U}_\gamma+\mu\,\nu_\gamma\partial_\beta\mathbf{U}_\gamma+\mu\,\partial_{\boldsymbol{\nu}}\mathbf{U}_\beta,\ \ \beta=1,2,3.\end{gathered}\tag{1.2}$$

The BVP (1.1) we consider in the following weak classical setting:

$$\mathbf{U}\in\mathbb{H}^1(\Omega^h),\ \ \mathbf{F}\in\widetilde{\mathbb{H}}^{-1}(\Omega^h),\ \ \mathbf{G}\in\mathbb{H}^{1/2}(\Gamma^h_L),\ \ \mathbf{H}(\,\cdot\,,\pm h)\in\mathbb{H}^{-1/2}(\mathcal{C}).\tag{1.3}$$

For definitions of Bessel potential spaces $\mathbb{H}^s$, $\widetilde{\mathbb{H}}^s$ see, e.g., [8].

Let us consider the following subspace of $\mathbb{H}^1(\Omega^h)$:

$$\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L):=\Big\{\mathbf{V}\in\mathbb{H}^1(\Omega^h):\ \ \mathbf{V}^+(t)=0\ \text{ for all }\ t\in\Gamma^h_L\Big\}.\tag{1.4}$$

Theorem 1.1. *The BVP* (1.1) *in the weak classical setting* (1.3) *has a unique solution.*

Proof. The Lamé operator $\mathcal{L}_{\Omega^h}$ is strictly positive on the subspace $\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$,

$$\langle\mathcal{L}_{\Omega^h}\mathbf{V},\mathbf{V}\rangle\geqslant M\|\mathbf{V}\|^2\quad\forall\,\mathbf{V}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L),$$

and the proof follows easily from the Lax-Milgram Lemma (a similar proof see, e.g., in [7]). □

To find what happens with the BVP (1.1), (1.3) as $h\to0$, we first reformulate this BVP into the equivalent variational problem: Find the vector $\mathbf{U}$ which minimizes the energy functional $\mathcal{E}_{\Omega^h}(\mathbf{U})$ (see (3.4)) under the same constraints (1.3). It is proved that if the weak limits

$$\lim_{h\to0}\mathbf{F}(\mathcal{x},h\tau)=\mathbf{F}(\mathcal{x}),\quad \lim_{h\to0}\frac{1}{2h}\big[\mathbf{H}(\mathcal{x},+h)-\mathbf{H}(\mathcal{x},-h)\big]=\mathbf{H}^{(1)}(\mathcal{x}),\ \ \mathbf{F},\mathbf{H}^{(1)}\in\mathbb{L}_2(\mathcal{C}),$$

exist in $\mathbb{L}_2(\Omega^h)$ and $\mathbb{L}_2(\mathcal{C})$, respectively, then there exists the Γ-limit of the energy functional $\lim\limits_{h\to0}\mathcal{E}_{\Omega^h}(\mathbf{U})=\mathcal{E}^3_{\mathcal{C}}(\overline{U})$ (cf. (4.2)), and the equivalent BVP on the surface $\mathcal{C}$, using Einstein's convention, is written as follows:

$$\begin{cases}\mu\Big[\Delta_{\mathcal{C}}\overline{U}_\alpha+\mathcal{D}_\beta\mathcal{D}_\alpha\overline{U}_\beta-2\mathcal{H}_{\mathcal{C}}\nu_\beta\mathcal{D}_\alpha\overline{U}_\beta-\mathcal{D}_\gamma(\nu_\alpha\nu_\beta\mathcal{D}_\gamma\overline{U}_\beta)\Big]\\ \qquad\qquad+\dfrac{4\lambda\mu}{\lambda+2\mu}\Big[\mathcal{D}_\alpha\mathcal{D}_\beta\overline{U}_\beta-2\mathcal{H}_{\mathcal{C}}\nu_\alpha\mathcal{D}_\beta\overline{U}_\beta\Big]=\dfrac{1}{2}\,F_\alpha+H^{(1)}_\alpha\ \text{ on }\ \mathcal{C}, \qquad \alpha=1,2,3.\\ \overline{U}_\alpha(t)=0\ \text{ on }\ \Gamma=\partial\mathcal{C},\end{cases}\tag{1.5}$$

In (1.5), $\boldsymbol{\nu}:=(\nu_1,\nu_2,\nu_3)^\top$ is the unit normal vector filed on $\mathcal{C}$, $\mathcal{H}_{\mathcal{C}}$ is the mean curvature of $\mathcal{C}$, $\mathcal{D}_\alpha:=\partial_\alpha-\nu_\alpha\partial_{\boldsymbol{\nu}}$, $\alpha=1,2,3$, are Günter's tangential derivatives on $\mathcal{C}$ (see Section 2) and $\overline{\mathbf{U}}:=(U_1(\mathcal{x},0),U_2(\mathcal{x},0),U_3(\mathcal{x},0))^\top$, $\mathcal{x}\in\mathcal{C}$, is the trace of the displacement vector field

$$\mathbf{U}(\mathcal{x},t):=\big(U_1(\mathcal{x},t),U_2(\mathcal{x},t),U_3(\mathcal{x},t)\big)^\top,\ \ (\mathcal{x},t)\in\Omega^h:=\mathcal{C}\times(-h,h),$$

on the mid-surface $\mathcal{C}$ (see Theorem 4.3).

The BVP (1.5) represents a new 2D shell equation in terms of Günter's tangential derivatives on the mid-surface $\mathcal{C}$.

2 Auxiliaries

We commence with the definition of a new system of coordinates: the system of 4-vectors

$$\mathbf{d}^j := \mathbf{e}^j - \mathcal{N}_j\mathcal{N}, \quad j = 1, 2, 3, \quad \text{and} \quad \mathbf{d}^4 := \mathcal{N}, \tag{2.1}$$

where $\mathbf{e}^1 = (1, 0, 0)^\top$, $\mathbf{e}^2 = (0, 1, 0)^\top$, $\mathbf{e}^3 = (0, 0, 1)^\top$ is the Cartesian basis in $\mathbb{R}^3$; the first 3 vectors $\mathbf{d}^1$, $\mathbf{d}^2$, $\mathbf{d}^3$ are projections of the Cartesian vectors and are tangential to the surface $\mathcal{C}$, while the last one $\mathbf{d}^4 = \mathcal{N}$ is orthogonal to it and, thus, to $\mathbf{d}^1$, $\mathbf{d}^2$, $\mathbf{d}^3$. The system is linearly dependent, but full, and any vector field $\mathbf{U} = U_\alpha\mathbf{e}^\alpha$ in Ω_h can be written in the following form:

$$\begin{aligned} \mathbf{U} = U_\alpha\mathbf{e}^\alpha = U_j^0\mathbf{d}^j = \mathbf{U}^0 = \mathbf{U}_0 + U_4^0\mathcal{N}, \\ \mathbf{U}_0 := \mathbf{U} - \langle\mathcal{N}, \mathbf{U}\rangle\mathcal{N}, \quad U_4^0 := \langle\mathcal{N}, \mathbf{U}\rangle = \mathcal{N}_\alpha U_\alpha, \end{aligned} \tag{2.2}$$

and the vector $\mathbf{U}_0 := (U_1^0, U_2^0, U_3^0)^\top$ is chosen to be tangential to the surface $\langle\mathcal{N}, \mathbf{U}_0\rangle = 0$.

Since the proper extension depends only on the surface variable $\mathcal{N}(x, t) = \mathcal{N}(x)$ (see [6]), the same is true for the entire basis $\mathbf{d}^j(x, t) = \mathbf{d}^j(x)$, $j = 1, 2, 3, 4$.

Note that

$$\mathcal{N}_4 := \langle\mathcal{N}, \mathcal{N}\rangle = 1.$$

Although the system $\{\mathbf{d}^j\}_{j=1}^4$ is linearly dependent, the following holds.

In [2, Lemma 1], it is proved that representation (2.2) is unique, that is,

$$\text{if} \;\; \mathbf{U}^0 = U_j^0\mathbf{d}^j = 0, \;\; \text{then} \;\; U_1^0 = U_2^0 = U_3^0 = U_4^0 = 0.$$

Moreover, the scalar product and, consequently, the distance between two vectors in the Cartesian and new coordinate systems coincide:

$$\langle\mathbf{U}^0, \mathbf{V}^0\rangle = U_j^0V_j^0 = U_\alpha V_\alpha = \langle\mathbf{U}, \mathbf{V}\rangle, \;\; \|\mathbf{U}^0 - \mathbf{V}^0\| = \|\mathbf{U} - \mathbf{V}\|$$

for arbitrary vectors $\mathbf{U} = (U_1, U_2, U_3)^\top$, $\mathbf{V} = (V_1, V_2, V_3)^\top \in \mathbb{R}^3$.

Günter's derivatives

$$\mathcal{D}_\alpha\varphi := \partial_\alpha\varphi - \nu_\alpha\partial_\nu\varphi, \quad \alpha = 1, 2, 3, \tag{2.3}$$

represent tangential differential operators on the surface $\mathcal{C}$ (orthogonal projections of the coordinate derivatives ∂_1, ∂_2, ∂_3) and have the extensions

$$\mathcal{D}_\alpha\varphi := \partial_\alpha\varphi - \mathcal{N}_\alpha\partial_\mathcal{N}\varphi$$

in the neighbourhood of the surface $\mathcal{C}$. The system $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ is, obviously, linearly dependent, but full: any tangential linear differential operator on the surface $\mathbf{A}(D)$ is written in the following form:

$$\mathbf{A}(D) = a_\alpha(x)\partial_\alpha = a_\alpha(x)\mathcal{D}_\alpha, \;\; \text{provided} \;\; a_\alpha(x)\nu_\alpha(x) \equiv 0, \quad x \in \mathcal{C}.$$

In particular,

$$\partial_\mathbf{U} = U_\alpha\partial_\alpha = U_j^0\mathcal{D}_j.$$

The adjoint operator to $\mathcal{D}_j$, $j = 1, 2, 3$, is

$$\mathcal{D}_j^*\varphi = -\mathcal{D}_j\varphi + 2\nu_j\mathcal{H}_\mathcal{C}\varphi, \;\; \varphi \in \mathbb{C}^1(\mathcal{C}),$$

where

$$\mathcal{H}_\mathcal{C}(x) := \frac{1}{2}\,\mathcal{D}_\alpha\nu_\alpha(x) = \frac{1}{2}\,\mathcal{D}_\alpha\mathcal{N}_\alpha(x), \quad x \in \mathcal{C}, \tag{2.4}$$

is the mean curvature of the surface $\mathcal{C}$.

Definition 2.1. For a function $\varphi \in \mathbb{W}^1(\Omega^h)$, we define the extended gradient

$$\nabla_{\Omega^h}\varphi = \big\{\mathcal{D}_1\varphi, \mathcal{D}_2\varphi, \mathcal{D}_3\varphi, \mathcal{D}_4\varphi\big\}^\top, \ \ \mathcal{D}_4\varphi := \partial_{\mathcal{N}}\varphi, \tag{2.5}$$

and, for a vector field $\mathbf{U} = U_\alpha \mathbf{e}^\alpha = U_j^0 \mathbf{d}^j \in \mathbb{W}^1(\Omega^h)$, we define the extended divergence

$$\operatorname{div}_{\Omega^h} \mathbf{U} := \mathcal{D}_j U_j^0 + 2\mathcal{H}_{\mathcal{C}} U_4^0 = -\nabla^*_{\Omega^h}\mathbf{U}, \tag{2.6}$$

where $\nabla^*_{\Omega^h}$ denotes the formally adjoint operator to the gradient ∇_{Ω^h}, $\mathcal{H}_{\mathcal{C}}$ is the mean curvature (cf. (2.4)) and

$$\mathcal{D}_4 U_4^0 := \partial_{\mathcal{N}} U_4^0 = \langle \mathcal{N}, \partial_{\mathcal{N}}\mathbf{U}\rangle = (\mathcal{D}_4\mathbf{U})_4^0.$$

Caution: While defining the extended divergence in (2.6), we have to use only the representation $\mathbf{U} = U_j^0 \mathbf{d}^j$ (cf. (2.2)), because any other representation differs from the indicated one by the vector $c\mathcal{N}$, where $c(\mathcal{x})$ is an arbitrary function. Then the extended divergences will differ by the summand $\operatorname{div}_{\Omega^h}(c(\mathcal{x})\mathcal{N}(\mathcal{x})) = \partial_{\mathcal{N}} c(\mathcal{x}) + 2c(\mathcal{x})\mathcal{H}_{\mathcal{C}}(\mathcal{x})$.

Lemma 2.2. *The classical gradient $\nabla\varphi := \{\partial_1\varphi, \partial_2\varphi, \partial_3\varphi\}^\top$, written in the full system of vectors $\{\mathbf{d}^j\}_{j=1}^4$ in* (2.1)*, coincides with the extended gradient $\nabla\varphi = \nabla_{\Omega^h}\varphi$ in* (2.5)*.*

The classical divergence $\operatorname{div}\mathbf{U} := \partial_\alpha U_\alpha$ *of a vector field $\mathbf{U} := U_\alpha \mathbf{e}^\alpha$, written in the full system* (2.1)*, coincides with the extended divergence* $\operatorname{div}\mathbf{U} = \operatorname{div}_{\Omega^h} \mathbf{U}^0$ *in* (2.6)*.*

The gradient and the negative divergence are the adjoint operators, $\nabla^*_{\Omega^h} = -\operatorname{div}_{\Omega^h}$ with respect to the scalar product induced from the ambient Euclidean space $\mathbb{R}^n$.

In the domain Ω_h, the classical Laplace operator

$$\Delta_{\Omega^h}\varphi(x) := (\operatorname{div}_{\Omega^h}\nabla_{\Omega^h}\varphi)(x) = -\big(\nabla^*_{\Omega^h}(\nabla_{\Omega^h}\varphi)\big)(x), \ \ x \in \Omega^h,$$

written in the full system (2.1)*, acquires the following form:*

$$\Delta_{\Omega^h}\varphi = \mathcal{D}_j^2\varphi + 2\mathcal{H}_{\mathcal{C}}\mathcal{D}_4\varphi, \ \ \varphi \in \mathbb{W}^2(\Omega^h).$$

Proof see in [2, Lemma 2].

The Lamé operator

$$\begin{aligned}\mathcal{L}\mathbf{U} = -\mu\,\boldsymbol{\Delta}\mathbf{U} - (\lambda+\mu)\,\nabla\operatorname{div}\mathbf{U} &= -\big[\mu\delta_{\alpha\beta}\partial_k^2 + (\lambda+\mu)\partial_\alpha\partial_\beta\big]_{3\times3}\mathbf{U}\\ &= -\big[c_{\alpha\gamma\beta\omega}\partial_\gamma\partial_\omega\big]_{3\times3}\mathbf{U}, \ \ c_{\alpha\gamma\beta\omega} = \lambda\delta_{\alpha\gamma}\delta_{\beta\omega} + \mu(\delta_{\alpha\beta}\delta_{\gamma\omega} + \delta_{\alpha\omega}\delta_{\beta\gamma})\end{aligned}$$

is formally the self-adjoint differential operator of the second order and, written in the full system (2.1), acquires the form

$$\mathcal{L}_{\Omega^h}\mathbf{U}^0 = -\mu\,\boldsymbol{\Delta}_{\Omega^h}\mathbf{U}^0 - (\lambda+\mu)\,\nabla_{\Omega^h}\operatorname{div}_{\Omega^h}\mathbf{U}^0.$$

To reformulate the BVP (1.1) in curvilinear coordinates we introduce the traction operator (cf. (1.2))

$$\mathfrak{T}(x,\partial)\mathbf{U} = (\mathfrak{T}_{\alpha\beta}(x,\partial)\mathbf{U}_\beta)e^\alpha = \Big(\big\{\lambda\nu_\alpha\partial_\beta + \mu\nu_\beta\partial_\alpha + \delta_{\alpha\beta}\mu\,\partial_{\boldsymbol{\nu}}\big\}U_\beta\Big)\mathbf{e}^\alpha, \ \ \mathbf{U} = (U_1, U_2, U_3)^\top = U_\alpha\mathbf{e}^\alpha$$

and Gunter's derivatives (see [2, (25)])

$$\begin{aligned}\mathfrak{T}(\mathcal{x},\mathcal{D}) &= \mathbf{e}^\alpha \otimes \mathbf{e}^\beta\big\{\lambda\nu_\alpha\partial_\beta + \mu\nu_\beta\partial_\alpha + \delta_{\alpha\beta}\mu\,\partial_{\boldsymbol{\nu}}\big\}\\ &= \lambda\mathbf{d}^4 \otimes (\mathbf{d}^\beta + \nu_\beta\mathbf{d}^4)(\mathcal{D}_\beta + \nu_\beta\mathcal{D}_4)\\ &\quad + \mu(\mathbf{d}^\beta + \nu_\alpha\mathbf{d}^4) \otimes (\mathbf{d}^\beta + \nu_\beta\mathbf{d}^4)\mathcal{D}_4 + \mu(\mathbf{d}^\beta + \nu_\beta\mathbf{d}^4) \otimes \mathbf{d}^4(\mathcal{D}_\beta + \nu_\beta\mathcal{D}_4)\\ &= \begin{bmatrix}\mu\mathcal{D}_4 & 0 & 0 & \mu\mathcal{D}_1\\ 0 & \mu\mathcal{D}_4 & 0 & \mu\mathcal{D}_2\\ 0 & 0 & \mu\mathcal{D}_4 & \mu\mathcal{D}_3\\ \lambda\mathcal{D}_1 & \lambda\mathcal{D}_2 & \lambda\mathcal{D}_3 & (\lambda+2\mu)\mathcal{D}_4\end{bmatrix}.\end{aligned}$$

Let us recall some results related to the uniqueness of solutions to an arbitrary elliptic equation.

Definition 2.3. Let Ω be an open subset with the Lipschitz boundary $\partial\Omega \neq \varnothing$ either on a Lipschitz hypersurface $\mathcal{C} \subset \mathbb{R}^n$, or in the Euclidean space $\mathbb{R}^{n-1}$.

We say that a class of functions $\mathcal{U}(\Omega)$, defined in a domain Ω in $\mathbb{R}^n$, has the **strong unique continuation property** if every $u \in \mathcal{U}(\Omega)$ in this class which vanishes to infinite order at one point must vanish identically.

If a surface $\mathcal{C}$ is C^∞-smooth, any elliptic operator on $\mathcal{C}$ has the strong unique continuation property due to Holmgren's theorem. But we can have more.

Lemma 2.4. *Let $\mathcal{C}$ be a C^2-smooth hypersurface in $\mathbb{R}^n$. The class of solutions to a second order elliptic equation $\mathbb{A}(\mathcal{x},\mathcal{D})u = 0$ with the Lipschitz continuous top order coefficients on a surface $\mathcal{C}$ has the strong unique continuation property.*

In particular, if the solution $u(\mathcal{x}) = 0$ vanishes in any open subset of $\mathcal{C}$, it vanishes identically on entire $\mathcal{C}$.

Proof see in [1, Lemma 1.7.2].

Lemma 2.5. *Let $\mathcal{C}$ be a C^2-smooth hypersurface in $\mathbb{R}^n$ with the Lipschitz boundary $\Gamma := \partial\mathcal{C}$ and $\gamma \subset \Gamma$ be an open part of the boundary Γ. Let $\mathbb{A}(\mathcal{x},\mathcal{D})$ be a second order elliptic system with the Lipschitz continuous top order matrix coefficients on a surface $\mathcal{C}$.*

The Cauchy problem

$$\begin{cases} \mathbb{A}(\mathcal{x},\mathcal{D})u = 0 & \text{on } \mathcal{C}, \quad u \in \mathbb{H}^1(\Omega), \\ u(\mathfrak{s}) = 0 & \text{for all } \mathfrak{s} \in \gamma, \\ (\partial_{\mathbf{V}}u)(\mathfrak{s}) = 0 & \text{for all } \mathfrak{s} \in \gamma, \end{cases}$$

where $\mathbf{V}$ is a non-tangential vector to Γ, but tangent to $\mathcal{C}$, has only a trivial solution $u(\mathcal{x}) = 0$ on entire $\mathcal{C}$.

Proof see in [1, Lemma 1.7.3].

3 Variational reformulation of the problem

To apply the method of Γ-convergence, we have to reformulate the BVP (1.1) into an equivalent variational problem for the energy functional. To this end, we have to consider the BVP with the vanishing Dirichlet condition on the lateral surface:

$$\begin{gathered} \mathcal{L}_{\Omega^h}\mathbf{U}_0(x) = \mathbf{F}_0(x), \quad x \in \Omega^h := \mathcal{C} \times (-h,h), \\ \mathbf{U}_0^+(t) = 0, \quad t \in \Gamma_L^h := \partial\mathcal{C} \times (-h,h), \\ (\mathfrak{T}(\mathcal{x},\nabla)\mathbf{U}_0)^+(\mathcal{x},\pm h) = \mathbf{H}_0(\mathcal{x},\pm h), \quad \mathcal{x} \in \mathcal{C}. \end{gathered}$$

It is possible to rewrite the BVP (1.1) in the equivalent BVP (3.2). Indeed, consider the BVP

$$\begin{gathered} \mathcal{L}_{\Omega^h}\mathbf{V}(x) = 0, \quad x \in \Omega^h := \mathcal{C} \times (-h,h), \\ \mathbf{V}^+(t) = \mathbf{G}(t), \quad t \in \Gamma_L^h, \\ (\mathfrak{T}((\mathcal{x},\nabla)\mathbf{V})^+(\mathcal{x},\pm h) = 0, \quad (\mathcal{x},\pm h) \in \Gamma^\pm \ = \mathcal{C} \times \{\pm h\}, \end{gathered} \tag{3.1}$$

which has a unique solution $\mathbf{V} \in \mathbb{W}^1(\Omega^h)$ (see Theorem 1.1) and note that the difference $\mathbf{U}_0 := \mathbf{U} - \mathbf{V}$ of solutions to BVPs (1.1) and (3.1) is a solution to the BVP (3.2), where $\mathbf{F}_0(\mathcal{x}) = \mathbf{F}(\mathcal{x}) - \mathcal{L}_{\Omega^h}\mathbf{V}(\mathcal{x})$, $= \mathbf{H}_0(\mathcal{x},\pm h) == \mathbf{H}(\mathcal{x},\pm h) - (\mathfrak{T}((\mathcal{x},\nabla)\mathbf{V})^+(\mathcal{x},\pm h)$. Vice versa, a solution to the BVP (1.1) is recovered as the sum of solutions $\mathbf{U} = \mathbf{U}_0 + \mathbf{V}$ of the BVPs (3.2) and (3.1).

Thus, in the BVP (1.1) we can assume, without restricting generality, that $\mathbf{G} = 0$ and consider the BVP

$$\begin{gathered} \mathcal{L}_{\Omega^h}\mathbf{U}(x) = \mathbf{F}(x), \quad x \in \Omega^h := \mathcal{C} \times (-h,h), \\ \mathbf{U}^+(t) = 0, \quad t \in \Gamma_L^h := \partial\mathcal{C} \times (-h,h), \\ (\mathfrak{T}(\mathcal{x},\nabla)\mathbf{U})^+(\mathcal{x},\pm h) = \mathbf{H}(\mathcal{x},\pm h), \quad \mathcal{x} \in \mathcal{C}. \end{gathered} \tag{3.2}$$

Theorem 3.1. *Problem* (3.2) *with the constraints*

$$\mathbf{U}\in\mathbb{H}^1(\Omega^h,\Gamma^h_L),\quad \mathbf{F}\in\widetilde{\mathbb{H}}^{-1}(\Omega^h),\quad \mathbf{H}(\,\cdot\,,\pm h)\in\mathbb{H}^{-1/2}(\mathcal{C}) \tag{3.3}$$

is reformulated into the following equivalent variational problem: Under the same constraints (3.3)*, look for a displacement vector-function* $\mathbf{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$*, which is a stationary point of the following functional:*

$$\begin{aligned}
\mathcal{E}_{\Omega^h}(\mathbf{U}) &:= \frac{1}{2}\int\limits_{\Omega^h}\Big[\mu\partial_\beta\mathbf{U}_\alpha\cdot\partial_\beta\mathbf{U}_\alpha+\mu\partial_\beta\mathbf{U}_\alpha\cdot\partial_\alpha\mathbf{U}_\beta+\lambda\partial_\alpha\mathbf{U}_\alpha\cdot\partial_\gamma\mathbf{U}_\gamma+2\mathbf{F}_\beta\cdot\mathbf{U}_\beta\Big]\,dx\\
&\quad+\int\limits_{\mathcal{C}}\Big[\big\langle\mathbf{H}(x,+h),\mathbf{U}^+(x,+h)\big\rangle-\big\langle\mathbf{H}(x,-h),\mathbf{U}^+(x,-h)\big\rangle\Big]\,d\sigma\\
&=\frac{1}{2}\int\limits_{-h}^{h}\int\limits_{\mathcal{C}}\Big[\mu\partial_\beta\mathbf{U}_\alpha\cdot\partial_\beta\mathbf{U}_\alpha+\mu\partial_\beta\mathbf{U}_\alpha\cdot\partial_\alpha\mathbf{U}_\beta+\lambda\partial_\alpha\mathbf{U}_\alpha\cdot\partial_\gamma\mathbf{U}_\gamma+2\mathbf{F}_\beta\cdot\mathbf{U}_\beta\\
&\quad+\frac{1}{h}\Big[\big\langle\mathbf{H}(x,+h),\mathbf{U}^+(x,+h)\big\rangle-\big\langle\mathbf{H}(x,-h),\mathbf{U}^+(x,-h)\big\rangle\Big]\Big]\,d\sigma\,dt,
\end{aligned}\tag{3.4}$$

Proof see in [2, Theorem 2].

Remark 3.2. The integral on $\mathcal{C}$ in (3.4) is understood in the sense of duality between the spaces $\widetilde{\mathbb{H}}^{1/2}(\mathcal{C})$ and $\mathbb{H}^{-1/2}(\mathcal{C})$ because $\mathbf{H}(\,\cdot\,,\pm h)\in\mathbb{H}^{-1/2}(\mathcal{C}_N)$ and the condition $\mathbf{U}\in\widetilde{\mathbb{H}}^1(\Omega^h,\Gamma^h_L)$ implies the inclusion $\mathbf{U}^+(\,\cdot\,,\pm h)\in\widetilde{\mathbb{H}}^{1/2}(\mathcal{C}_N)$.

Let us prove the following auxiliary lemma.

Lemma 3.3. *Let* $\mu>0$ *and* $\mu+\lambda>0$*. Then the quantity* $n(\mathbf{E}):=2\mu\,|\mathbf{E}|^2+\lambda(\operatorname{Trace}\mathbf{E})^2$ *is non-negative,* $n(\mathbf{E})\geqslant 0$ *for an arbitrary matrix* $\mathbf{E}=[E_{\alpha\beta}]_{3\times3}$*.*

Proof. We proceed as follows:

$$\begin{aligned}
n(\mathbf{E}) &= 2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+2\mu\sum_{\alpha}E^2_{\alpha\alpha}+(\mu+\lambda)\sum_{\alpha,\beta}E_{\alpha\alpha}E_{\beta\beta}-\mu\sum_{\alpha,\beta}E_{\alpha\alpha}E_{\beta\beta}\\
&=2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+(\mu+\lambda)\Big(\sum_{\alpha}E_{\alpha\alpha}\Big)^2+\mu\Big[2\sum_{\alpha}E^2_{\alpha\alpha}-\sum_{\alpha\neq\beta}E_{\alpha\alpha}E_{\beta\beta}\Big]\\
&=2\mu\sum_{\alpha\neq\beta}E^2_{\alpha\beta}+(\mu+\lambda)\Big(\sum_{\alpha}E_{\alpha\alpha}\Big)^2+\mu\sum_{\alpha\neq\beta}(E_{\alpha\alpha}-E_{\beta\beta})^2\geq 0,
\end{aligned}$$

since $\mu>0$, $\mu+\lambda>\frac{2\mu+3\lambda}{3}>0$ (see (1.2)). □

4 Shell operator is non-negative

The main theorem of the present paper, Theorem 4.3, will be proved later. Here we recall the main results about Γ-limit of the energy functional $\mathcal{E}_{\Omega^h}(\mathbf{U})$ in (3.4).

Next, we perform the scaling of the variable $t=h\tau$, $-1<\tau<1$, in the modified kernel $Q_4(\nabla\mathbf{U})$ of the quadratic part of energy functional (3.4) and divide by h.

Lemma 4.1. *The scaled and divided by* h *energy functional*

$$\mathcal{E}^0_{\Omega^h}(\widetilde{\mathbf{U}}^h)=\frac{1}{h}\,\mathcal{E}_{\Omega^h}(\widetilde{\mathbf{U}}^h)=\frac{1}{2}\,\mathcal{Q}^0_4(\widetilde{\mathbf{U}}^h)-\mathcal{F}^0(\widetilde{\mathbf{U}}^0_h) \tag{4.1}$$

with the quadratic and linear parts

$$\mathcal{Q}^0_4(\widetilde{\mathbf{U}}^h) = \int\limits_{-1}^{1}\int\limits_{\mathcal{C}} Q^0_4(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h(x,\tau))\, d\sigma\, d\tau,$$

$$\mathcal{F}^0(\widetilde{\mathbf{U}}^0_h) = -\int\limits_{-h}^{h}\int\limits_{\mathcal{C}} \Big[\big\langle \widetilde{\mathbf{F}}^0_h, \mathbf{U}^0_h\big\rangle + \frac{1}{h}\Big[\big\langle \widetilde{\mathbf{H}}(x,+h), \widetilde{\mathbf{U}}^{0,+}(x,+h)\big\rangle - \big\langle \widetilde{\mathbf{H}}^0(x,-h), \widetilde{\mathbf{U}}^{0,+}(x,-h)\big\rangle\Big]\Big],$$

$$\mathbf{F}^0_h(x,\tau) := \big(F^0_1(x,h\tau), F^0_2(x,h\tau), F^0_3(x,h\tau), F^0_4(x,h\tau)\big)^\top, \quad F^0_4 = \mathcal{N}_\alpha F_\alpha,$$
$$\widetilde{\mathbf{H}}^0_h(x,\tau) := \big(H^0_1(x,h\tau), H^0_2(x,h\tau), H^0_3(x,h\tau), H^0_4(x,h\tau)\big)^\top, \quad H^0_4 = \mathcal{N}_\alpha H_\alpha,$$

is correctly defined on the space $\widetilde{\mathbb{H}}^1(\Omega^1,\Gamma^1_L)$ *(see* (1.4)*) and is convex:*

$$\mathcal{E}^0_{\Omega^h}(\theta\widetilde{\mathbf{U}}^h + (1-\theta)\widetilde{\mathbf{V}}^h) \leqslant \mathcal{E}^0_{\Omega^h}(\widetilde{\mathbf{U}}^h) + (1-\theta)\mathcal{E}^0_{\Omega^h}(\widetilde{\mathbf{V}}^h), \quad 0<\theta<1,$$

for arbitrary vector $\widetilde{\mathbf{V}}^h(x,\tau) := (V_1(x,h\tau), V_2(x,h\tau), V^0_3(x,h\tau), V_4(x,h\tau))^\top$, $\widetilde{\mathbf{V}}^h \in \widetilde{\mathbb{H}}^1(\Omega^1,\Gamma^1_L)$.

Moreover, if $\widetilde{\mathbf{F}}^0_h(x,\tau) := \mathbf{F}^0(x,h\tau)$ *are uniformly bounded in* $\mathbb{L}_2(\Omega^1)$*, i.e.,*

$$\sup_{h<h_0} \|\widetilde{\mathbf{F}}^0_h|\mathbb{L}_2(\Omega^1)\| < \infty$$

for some $h_0 > 0$*, the energy functional has the following quadratic estimate: there exist positive constants* C_1*,* C_2 *and* C_3 *independent of the parameter* h *such that*

$$C_1\int\limits_{\Omega^1}\Big[\big(\mathcal{D}_\alpha U^0_j(x,h\tau)\big)^2 + \Big(\frac{1}{h}\,\frac{\partial U^0_j(x,h\tau)}{\partial\tau}\Big)^2\Big]dx - C_2 \leqslant \mathcal{E}^0_{\Omega^h}(\widetilde{\mathbf{U}}^h)$$
$$\leqslant C_3\Big\{1 + \int\limits_{\Omega^1}\Big[\big(\mathcal{D}_\alpha U^0_j(x,h\tau)\big)^2 + \Big(\frac{1}{h}\,\frac{\partial U^0_j(x,h\tau)}{\partial\tau}\Big)^2\Big]dx\Big\}$$

for all $\widetilde{\mathbf{U}}^h \in \widetilde{\mathbb{H}}^1(\Omega^1,\Gamma^1_L)$.

Proof see in [2, Lemma 5].

Theorem 4.2. *Let the weak limits*

$$\lim_{h\to 0}\mathbf{F}(x,h\tau) = \mathbf{F}(x), \quad \lim_{h\to 0}\frac{1}{2h}\big[\mathbf{H}(x,+h) - \mathbf{H}(x,-h)\big] = \mathbf{H}^{(1)}(x), \quad \mathbf{F}, \mathbf{H}^{(1)} \in \mathbb{L}_2(\mathcal{C}),$$

in $\mathbb{L}_2(\Omega^h)$ *and* $\mathbb{L}_2(\mathcal{C})$*, respectively, exist. Then the* Γ*-limit of the energy functional* $\mathcal{E}^0_{\Omega^h}(\widetilde{\mathbf{U}}^h)$ *exists:*

$$\Gamma - \lim_{h\to 0}\mathcal{E}^0_{\Omega^h}(\widetilde{\mathbf{U}}^h) = \mathcal{E}^3_{\mathcal{C}}(\overline{\mathbf{U}}) := \int\limits_{\mathcal{C}} Q_3(\overline{\mathbf{U}}(x))\, d\sigma, \tag{4.2}$$

where

$$Q_3(\overline{\mathbf{U}}) = \frac{\mu}{2}\Big[\big[\mathcal{D}_\alpha\overline{U}_\beta + \mathcal{D}_\beta\overline{U}_\alpha\big]^2 - 2\nu_\beta\nu_\gamma\mathcal{D}_\alpha\overline{U}_\beta\mathcal{D}_\alpha\overline{U}_\gamma\Big]$$
$$+ \frac{2\lambda\mu}{\lambda+2\mu}(\mathcal{D}_\alpha\overline{U}_\alpha)^2 + \big\langle \mathbf{F}(x) + 2\mathbf{H}^{(1)}(x), \overline{U}(x)\big\rangle \tag{4.3}$$

and

$$\overline{\mathbf{U}}(x) := \big(\overline{U}_1(x), \overline{U}_2(x), \overline{U}_3(x)\big)^\top, \quad \overline{U}_\alpha(x) := U_\alpha(x,0), \quad \alpha = 1,2,3.$$

Proof see in [2, Theorem 3].

Theorem 4.3. *Let $\mathbf{F}, \mathbf{H}^{(1)} \in \mathbb{L}_2(\mathcal{C})$. The vector-function $\overline{\mathbf{U}} \in \widetilde{\mathbb{H}}^1(\mathcal{C})$ which minimizes the energy functional $\mathcal{E}^3_{\mathcal{C}}(\overline{\mathbf{U}})$ in (4.2), (4.3) is a solution to the following boundary value problem:*

$$\begin{cases} (\mathcal{L}_{\mathcal{C}}\overline{\mathbf{U}})_\alpha := \mu\Big[\Delta_{\mathcal{C}}\overline{U}_\alpha + \mathcal{D}_\beta\mathcal{D}_\alpha\overline{U}_\beta - 2\mathcal{H}_{\mathcal{C}}\nu_\beta\mathcal{D}_\alpha\overline{U}_\beta - \mathcal{D}_\gamma(\nu_\alpha\nu_\beta\mathcal{D}_\gamma\overline{U}_\beta)\Big] \\ \qquad\qquad + \dfrac{4\lambda\mu}{\lambda+2\mu}\left[\mathcal{D}_\alpha\mathcal{D}_\beta\overline{U}_\beta - 2\mathcal{H}_{\mathcal{C}}\nu_\alpha\mathcal{D}_\beta\overline{U}_\beta\right] = \dfrac{1}{2}\,F_\alpha + H^{(1)}_\alpha \ \ on\ \mathcal{C}, \qquad \alpha = 1,2,3. \\ \overline{U}_\alpha(t) = 0 \quad on \quad \Gamma = \partial\mathcal{C}, \end{cases} \tag{4.4}$$

Vice versa: on the solution $\overline{\mathbf{U}} \in \widetilde{\mathbb{H}}^1(\mathcal{C})$ to the boundary value problem (4.4) under the condition $\mathbf{F}, \mathbf{H}^{(1)} \in \mathbb{L}_2(\mathcal{C})$, the energy functional $\mathcal{E}^3_{\mathcal{C}}(\overline{\mathbf{U}})$ in (4.2), (4.3) attains the minimum.

Moreover, the operator $\mathcal{L}_{\mathcal{C}}$ in the left-hand side of the shell equation (4.4) is elliptic, positive definite and has finite dimensional kernel consisting of the solutions to the following system of equations:

$$\mathcal{D}_\alpha\overline{U}_\beta + \mathcal{D}_\beta\overline{U}_\alpha - \sum_\gamma \left[\nu_\alpha\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma) + \nu_\beta\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)\right] \equiv 0, \ \ \alpha,\beta = 1,2,3. \tag{4.5}$$

The boundary value problem (4.4) has a unique solution in the classical setting:

$$\overline{\mathbf{U}} := (\overline{U}_1, \overline{U}_2, \overline{U}_3)^\top \in \mathbb{H}^1(\mathcal{C}), \ \ \frac{1}{2}\,\mathbf{F} + \mathbf{H}^{(1)} \in \mathbb{L}_2(\mathcal{C}).$$

Proof. The first part of the theorem, that BVP (4.4) is the Γ-limit of the BVP (3.2) (i.e., the solution to the BVP (4.4) $\overline{\mathbf{U}} \in \widetilde{\mathbb{H}}^1(\mathcal{C})$ minimizes the energy functional $\mathcal{E}^3_{\mathcal{C}}(\overline{\mathbf{U}})$ in (4.2), (4.3)) is proved in [2, Theorem 4].

Ellipticity of the operator $\mathcal{L}_{\mathcal{C}}$ in the left-hand side of the shell equation (4.4) is checked directly and from the Lax–Milgram Lemma, it follows that it is the Fredholm operator in the setting $\mathcal{L}_{\mathcal{C}} : \mathbb{H}^{-1}(\mathcal{C}) \to \mathbb{H}^1(\mathcal{C})$ (see [7, Theorem 14]) for a similar proof). Therefore, $\mathcal{L}_{\mathcal{C}}$ has the finite dimensional kernel.

Let us start with the energy functional and recall the quadratic part of the energy functional (see (4.1) and formulae [2, (33)]):

$$\begin{gathered} \mathcal{Q}^0_4(\mathbf{U}) = \int\limits_{-h}^{h}\int\limits_{\mathcal{C}} Q^0_4(\nabla\mathbf{U}(\mathcal{x},t))\,d\sigma\,dt, \\ Q^0_4(\mathbf{F}) = 2\mu|\mathbf{E}|^2 + \lambda(\operatorname{Trace}\mathbf{E})^2, \ \ \mathbf{E} = \frac{1}{2}\,(\mathbf{F} + \mathbf{F}^\top), \end{gathered} \tag{4.6}$$

where $\mathbf{F} = [F_{\alpha\beta}]_{3\times3}$ and $\mathbf{E} = [E_{\alpha\beta}]_{3\times3}$ are the 3×3 matrices and $|\mathbf{E}|^2 = \operatorname{Trace}(\mathbf{E}^\top\mathbf{E}) = \sum\limits_{\alpha,\beta} E^2_{\alpha\beta}$. From Lemma 3.3 it follows that the kernel $Q^0_4(\mathbf{F})$ is non-negative:

$$Q^0_4(\mathbf{F}) = 2\mu\sum_{\alpha\neq\beta} E^2_{\alpha\beta} + (\mu+\lambda)\Big(\sum_\alpha E_{\alpha\alpha}\Big)^2 + \mu\sum_{\alpha\neq\beta}(E_{\alpha\alpha} - E_{\beta\beta})^2 \geq 0. \tag{4.7}$$

Let us rewrite the kernel $Q^0_4(\nabla\mathbf{U})$ of the quadratic part $\mathcal{Q}^0_4(\mathbf{U})$ of the energy functional in (4.1), (4.6), (4.7) by using the equalities

$$\mathbf{F} = \nabla\mathbf{U} = [\partial_\alpha U_\beta]_{3\times3}, \quad (\operatorname{Def}\mathbf{U}) := \frac{1}{2}\left((\nabla\mathbf{U}) + (\nabla\mathbf{U})^\top\right) = \Big[\frac{1}{2}\,(\partial_\alpha U_\beta + \partial_\beta U_\alpha)\Big]_{3\times3}$$

and (2.3) as follows:

$$\begin{aligned} Q_4(\nabla\mathbf{U}) &= 2\mu\sum_{\alpha\neq\beta}(\operatorname{Def}\mathbf{U})^2_{\alpha\beta} + (\mu+\lambda)\Big(\sum_\alpha \partial_\alpha U_\alpha\Big)^2 + \mu\sum_{\alpha\neq\beta}\left[\partial_\alpha U_\alpha - \partial_\beta U_\beta\right]^2 \\ &= 2\mu\sum_{\alpha\neq\beta}\Big[(\operatorname{Def}\mathbf{U})_{\alpha\beta} + \frac{\nu_\alpha\mathcal{D}_4 U_\alpha + \nu_\beta\mathcal{D}_4 U_\beta}{2}\Big]^2 + (\mu+\lambda)\Big(\sum_\alpha \mathcal{D}_\alpha U_\alpha + \mathcal{D}_4 U_4\Big)^2 \end{aligned}$$

$$
\begin{aligned}
&+\mu \sum_{\alpha\neq\beta}\big[\mathcal{D}_\alpha U_\alpha - \mathcal{D}_\beta U_\beta + \nu_\alpha \mathcal{D}_4 U_\alpha - \nu_\beta \mathcal{D}_4 U_\beta\big]^2 \\
&= 2\mu \sum_{\alpha\neq\beta}\Big[(\operatorname{Def}\mathbf{U})_{\alpha\beta} + \frac{\nu_\alpha \mathcal{D}_4 U_\alpha + \nu_\beta \mathcal{D}_4 U_\beta}{2}\Big]^2 + (\mu+\lambda)\Big(\sum_\alpha \mathcal{D}_\alpha U_\alpha + \mathcal{D}_4 U_4\Big)^2 \\
&+\mu \sum_{\alpha,\beta}\big[\mathcal{D}_\alpha U_\alpha - \mathcal{D}_\beta U_\beta + \nu_\alpha \mathcal{D}_4 U_\alpha - \nu_\beta \mathcal{D}_4 U_\beta\big]^2,
\end{aligned}
\tag{4.8}
$$

where

$$
\operatorname{Def}\mathbf{U})_{\alpha\beta} := \frac{\mathcal{D}_\alpha U_\beta + \mathcal{D}_\beta U_\alpha}{2}, \quad \alpha,\beta = 1,2,3.
$$

Next, we perform the scaling of the variable $t = h\tau$, $-1 < \tau < 1$, in the modified kernel $Q_4(\nabla\mathbf{U})$ of the quadratic part of energy functional (4.8), divide by h and study the following kernel in the scaled domain $\Omega^1 = \mathcal{C}\times(1,1)$:

$$
\begin{aligned}
Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h(x,\tau)) &= \frac{1}{h}\, Q_4(\nabla\mathbf{U}(x,h\tau)) \\
&= \frac{\mu}{2}\sum_{\alpha\neq\beta}\bigg[\mathcal{D}_\alpha U_\beta(x,h\tau) + \mathcal{D}_\beta U_\alpha(x,h\tau) + \frac{\nu_\alpha}{h}\,\frac{\partial U_\beta(x,h\tau)}{\partial\tau} + \frac{\nu_\beta}{h}\,\frac{\partial U_\alpha(x,h\tau)}{\partial\tau}\bigg]^2 \\
&\quad + (\mu+\lambda)\Big(\sum_\alpha \mathcal{D}_\alpha U_\alpha(x,h\tau) + \frac{1}{h}\,\frac{\partial U_4(x,h\tau)}{\partial\tau}\Big)^2 \\
&\quad + \mu\sum_{\alpha,\beta}\bigg[\mathcal{D}_\alpha U_\alpha(x,h\tau) - \mathcal{D}_\beta U_\beta(x,h\tau) + \frac{\nu_\alpha}{h}\,\frac{\partial U_\alpha(x,h\tau)}{\partial\tau} - \frac{\nu_\beta}{h}\,\frac{\partial U_\beta(x,h\tau)}{\partial\tau}\bigg]^2,
\end{aligned}
\tag{4.9}
$$

where

$$
\widetilde{\mathbf{U}}^h(x,\tau) := \big(U_1^0(x,h\tau), U_2^0(x,h\tau), U_3^0(x,h\tau), U_4^0(x,h\tau)\big)^\top, \quad U_4^0 = \mathcal{N}_\alpha U_\alpha.
$$

For this, let us rewrite Q_4^0 in (4.9) in the form

$$
\begin{aligned}
Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h(x,\tau)) &= \frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathcal{D}_\alpha U_\beta(x,h\tau) + \mathcal{D}_\beta U_\alpha(x,h\tau) + \mathcal{N}_\alpha\xi_\beta + \mathcal{N}_\beta\xi_\alpha\Big]^2 \\
&+ (\mu+\lambda)\Big(\sum_\alpha \mathcal{D}_\alpha U_\alpha(x,h\tau) + \xi_4\Big)^2 + \mu\sum_{\alpha,\beta}\Big[\mathcal{D}_\alpha U_\alpha(x,h\tau) - \mathcal{D}_\beta U_\beta(x,h\tau) + \mathcal{N}_\alpha\xi_\alpha - \mathcal{N}_\beta\xi_\beta\Big]^2 \\
&= \frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathcal{D}_\alpha U_\beta(x,h\tau) + \mathcal{D}_\beta U_\alpha(x,h\tau) + \mathcal{N}_\alpha\xi_\beta + \mathcal{N}_\beta\xi_\alpha\Big]^2 \\
&+ (\mu+\lambda)(\mathcal{D}iv\,\mathbf{U}(x,h\tau) + \xi_4)^2 + \mu\sum_{\alpha,\beta}\Big[\mathcal{D}_\alpha U_\alpha(x,h\tau) - \mathcal{D}_\beta U_\beta(x,h\tau) + \mathcal{N}_\alpha\xi_\alpha - \mathcal{N}_\beta\xi_\beta\Big]^2,
\end{aligned}
\tag{4.10}
$$

where the variables

$$
\xi_\alpha = \xi_\alpha(x,h\tau) := \frac{1}{h}\,\frac{\partial U_\alpha(x,h\tau)}{\partial\tau}, \quad \alpha = 1,2,3, \quad \xi_4 = \mathcal{N}_\alpha\xi_\alpha
$$

depend on h and we find minimum of the kernel $Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}(x,\tau))$ with respect to the variables ξ_1,ξ_2,ξ_3. It was shown in [2] that by $Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h(x,\tau))$ the Γ-limit is attained on the following values of the variables:

$$
\xi_4 = -\frac{\lambda}{\lambda+2\mu}\,\mathcal{D}_\beta U_\beta = -\frac{\lambda}{\lambda+2\mu}\,\mathcal{D}iv\,\mathbf{U}, \tag{4.11}
$$

$$
\xi_\alpha = -\mathcal{N}_\gamma(\mathcal{D}_\alpha U_\gamma) - \frac{\lambda}{\lambda+2\mu}\,\mathcal{N}_\alpha\mathcal{D}iv\,\mathbf{U}, \quad \alpha = 1,2,3, \tag{4.12}
$$

where we remind that $\mathcal{D}iv\,\mathbf{U} = \mathcal{D}_\alpha U_\alpha$. From (4.11), (4.12) and (4.10) we find the Γ-limit $Q_3^0(\overline{\mathbf{U}})$ (the same as in [2], but written in a different form):

$$\begin{aligned}
Q_3^0(\overline{\mathbf{U}}) &= \min_{\xi_1,\xi_2,\xi_3} Q_4^0(\nabla_{\Omega^h}\widetilde{\mathbf{U}}^h) \\
&= \frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathcal{D}_\alpha\overline{U}_\beta + \mathcal{D}_\beta\overline{U}_\alpha - \sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma) + \nu_\beta\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)\big] - \frac{2\lambda}{\lambda+2\mu}\,\nu_\alpha\nu_\beta\mathcal{D}iv\,\overline{\mathbf{U}}\Big]^2 \\
&\qquad + (\mu+\lambda)\Big(\mathcal{D}iv\,\overline{\mathbf{U}} - \frac{\lambda}{\lambda+2\mu}\,\mathcal{D}iv\,\overline{\mathbf{U}}\Big)^2 \\
&\qquad + \mu\sum_{\alpha,\beta}\Big[\mathcal{D}_\alpha\overline{U}_\alpha - \mathcal{D}_\beta\overline{U}_\beta - \sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma) - \nu_\beta\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)\big] \\
&\qquad - \frac{\lambda}{\lambda+2\mu}\,\nu_\alpha^2\mathcal{D}iv\,\overline{\mathbf{U}} + \frac{\lambda}{\lambda+2\mu}\,\nu_\beta^2\mathcal{D}iv\,\overline{\mathbf{U}}\Big]^2 \\
&= \frac{\mu}{2}\sum_{\alpha\neq\beta}\Big[\mathcal{D}_\alpha\overline{U}_\beta + \mathcal{D}_\beta\overline{U}_\alpha - \sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma) + \nu_\beta\nu_\gamma\mathcal{D}_\alpha\overline{U}_\gamma\big] - \frac{2\lambda}{\lambda+2\mu}\,\nu_\alpha\nu_\beta\mathcal{D}iv\,\overline{\mathbf{U}}\Big]^2 \\
&\quad + \frac{4\mu^2(\mu+\lambda)}{(\lambda+2\mu)^2}\,[\mathcal{D}iv\,\overline{\mathbf{U}}]^2 + \mu\sum_{\alpha,\beta}\Big[\mathcal{D}_\alpha\overline{U}_\alpha - \mathcal{D}_\beta\overline{U}_\beta - \sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma) - \nu_\beta\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)\big]\Big]^2. \qquad (4.13)
\end{aligned}$$

From (4.13) it follows that $Q_3^0(\overline{\mathbf{U}})$ is a nonnegative quadratic form $Q_3^0(\overline{\mathbf{U}}) \geqslant 0$ for all $\mathbf{U} \in \mathbb{H}^1(\mathcal{C},\Gamma)$, $\Gamma := \partial\mathcal{C}$. □

5 Shell operator is positive definite

If $Q_3^0(\overline{\mathbf{U}}) \equiv 0$, from (4.13) we get

$$\begin{aligned}
&\mathcal{D}iv\,\overline{\mathbf{U}} \equiv 0, \\
&\mathcal{D}_\alpha\overline{U}_\alpha - \mathcal{D}_\beta\overline{U}_\beta - \sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma) - \nu_\beta\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma)\big] \equiv 0, \quad \alpha\neq\beta = 1,2,3, \\
&\mathcal{D}_\alpha\overline{U}_\beta + \mathcal{D}_\beta\overline{U}_\alpha - \sum_\gamma\big[\nu_\alpha\nu_\gamma(\mathcal{D}_\beta\overline{U}_\gamma) + \nu_\beta\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma)\big] \equiv 0, \quad \alpha\neq\beta = 1,2,3.
\end{aligned} \qquad (5.1)$$

By taking the sum with respect to β in the second equality in (5.1), we get

$$\mathcal{D}_\alpha\overline{U}_\alpha = \sum_\gamma \nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma), \quad \alpha = 1,2,3.$$

Note that the obtained equality implies both, the first and the second equalities from (5.1). Moreover, it coincides with the third equality in (5.1) if we allow there $\alpha = \beta = 1,2,3$. Thus, equation (4.5) implies all three equalities in (5.1) and describes the kernel $\operatorname{Ker}\mathcal{L}_\mathcal{C}$ of the shell equation $\mathcal{L}_\mathcal{C}$ in (4.4).

Now we rewrite the obtained equation in the following form:

$$\begin{aligned}
\mathcal{D}_\alpha\overline{U}_\alpha &= \sum_\gamma \nu_\alpha\nu_\gamma(\mathcal{D}_\alpha\overline{U}_\gamma) = \nu_\alpha\mathcal{D}_\alpha\Big(\sum_\gamma \nu_\gamma\overline{U}_\gamma\Big) - \sum_\gamma \nu_\alpha(\mathcal{D}_\alpha\nu_\gamma)\overline{U}_\gamma \\
&= \nu_\alpha(\mathcal{D}_\alpha\overline{U}_4) - \sum_\gamma \nu_\alpha(\mathcal{D}_\alpha\nu_\gamma)\overline{U}_\gamma, \quad \overline{U}_4 = \sum_\gamma \nu_\gamma\overline{U}_\gamma, \quad \alpha = 1,2,3. \qquad (5.2)
\end{aligned}$$

Similarly to (5.2), from equality (4.5) (see the third equality in (5.1) we derive

$$\mathcal{D}_\alpha\overline{U}_\beta + \mathcal{D}_\beta\overline{U}_\alpha = \nu_\alpha\mathcal{D}_\beta\overline{U}_4 + \nu_\beta\mathcal{D}_\alpha\overline{U}_4 - \sum_\gamma\big[\nu_\alpha(\mathcal{D}_\beta\nu_\gamma) + \nu_\beta(\mathcal{D}_\alpha\nu_\gamma)\big]\overline{U}_\gamma, \quad \alpha,\beta = 1,2,3. \qquad (5.3)$$

Besides the equalities (4.5), (5.2), (5.3) we have the following equality

$$\begin{aligned}\sum_{\alpha,\beta}\Big[\big[\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha\big]^2-2\sum_\gamma\nu_\beta\nu_\gamma\mathcal{D}_\alpha\overline{U}_\beta\mathcal{D}_\alpha\overline{U}_\gamma\Big]\\ =\sum_{\alpha,\beta}\Big[\big[\mathcal{D}_\alpha\overline{U}_\beta+\mathcal{D}_\beta\overline{U}_\alpha\big]^2\Big]-2\sum_\alpha(\mathcal{D}_\alpha\overline{U}_4)^2-2\sum_{\alpha,\beta,\gamma}(\mathcal{D}_\alpha\nu_\beta)(\mathcal{D}_\alpha\nu_\gamma)\overline{U}_\beta\overline{U}_\gamma\\ +2\sum_{\alpha,\beta}(\mathcal{D}_\alpha\nu_\beta)(\mathcal{D}_\alpha\overline{U}_4)\overline{U}_\beta-2\sum_{\alpha,\gamma}(\mathcal{D}_\alpha\nu_\gamma)(\mathcal{D}_\alpha\overline{U}_4)\overline{U}_\gamma\equiv 0,\end{aligned}\tag{5.4}$$

which follows from (4.3) if we apply the first equality from (5.1) and recall that $Q_3^0(\overline{\mathbf{U}})=0$.

If $\overline{U}_\alpha(\mathfrak{s})=0$, $\alpha=1,2,3$, equalities (5.2)–(5.4) simplify:

$$\begin{gathered}\mathcal{D}_\alpha(\mathfrak{s})\overline{U}_\alpha(\mathfrak{s})=\nu_\alpha(\mathfrak{s})\mathcal{D}_\alpha\overline{U}_4(\mathfrak{s}),\\ \mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathcal{D}_\beta\overline{U}_\alpha(\mathfrak{s})=\nu_\alpha(\mathfrak{s})\mathcal{D}_\beta\overline{U}_4(\mathfrak{s})+\nu_\beta(\mathfrak{s})\mathcal{D}_\alpha\overline{U}_4(\mathfrak{s}),\quad \alpha,\beta=1,2,3,\\ \sum_{\alpha,\beta}\Big[\big[\mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathcal{D}_\beta\overline{U}_\alpha(\mathfrak{s})\big]^2\Big]=2\sum_\alpha(\mathcal{D}_\alpha\overline{U}_4(\mathfrak{s}))^2,\quad \mathfrak{s}\in\partial\mathcal{C}.\end{gathered}\tag{5.5}$$

We can see that not only the first equality in (5.5) is the consequence of the second one (by taking $\alpha=\beta$), but also the third equality follows from the second one if we take into account that $\sum\limits_\alpha\nu_\alpha^2=1$ and $\sum\limits_\alpha\nu_\alpha\mathcal{D}_\alpha=0$.

By inserting the first equality from (5.5) into the second one we get

$$\mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathcal{D}_\beta\overline{U}_\alpha(\mathfrak{s})=\frac{\nu_\alpha(\mathfrak{s})}{\nu_\beta(\mathfrak{s})}\,\mathcal{D}_\beta\overline{U}_\beta(\mathfrak{s})+\frac{\nu_\beta(\mathfrak{s})}{\nu_\alpha(\mathfrak{s})}\,\mathcal{D}_\alpha\overline{U}_\alpha(\mathfrak{s}),\quad \alpha,\beta=1,2,3.$$

If we succeed in proving that

$$\mathcal{D}_\alpha\overline{U}_4(\mathfrak{s})\equiv 0,\quad \mathfrak{s}\in\partial\mathcal{C},\quad \alpha=1,2,3,\tag{5.6}$$

then from (5.5) and (5.6) will follow

$$\mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})+\mathcal{D}_\beta\overline{U}_\alpha(\mathfrak{s})\equiv 0,\quad \mathfrak{s}\in\partial\mathcal{C},\quad \alpha,\beta=1,2,3.\tag{5.7}$$

The latter implies that

$$\mathcal{D}_\alpha\overline{U}_\beta(\mathfrak{s})\equiv 0\qquad \forall\,\alpha,\beta=1,2,3,\quad \forall\,\mathfrak{s}\in\partial\mathcal{C}.\tag{5.8}$$

Indeed (cf. [1, Lemma 1.7.4]), among directing tangential vector fields $\{\mathbf{d}^k(\mathfrak{s})\}_{k=1}^3$ generating Günter's derivatives $\mathcal{D}_k=\partial_{\mathbf{d}^k}$, $k=1,2,3$, only 2 are linearly independent (one of these vectors might even collapse at a point $\mathbf{d}^k(\mathfrak{s})=0$ if the corresponding basis vector $\mathbf{e}^k$ is orthogonal to the surface at $\mathfrak{s}\in\mathcal{C}$). One of these vectors might be tangential to the boundary curve $\partial\mathcal{C}$ and, at least one, say $\mathbf{d}^3(\mathfrak{s})$, is non-tangential to $\partial\mathcal{C}$. The vector $\mathbf{d}^\alpha$ for $\alpha=1,2,3$, is a linear combination $\mathbf{d}^\alpha(\mathfrak{s})=c_1(\mathfrak{s})\mathbf{d}^3(\mathfrak{s})+c_2(\mathfrak{s})\boldsymbol{\tau}^\alpha(\mathfrak{s})$ of the non-tangential vector $\mathbf{d}^3(\mathfrak{s})$ and of the projection $\boldsymbol{\tau}^\alpha(\mathfrak{s}):=\pi_{\partial\mathcal{C}}\mathbf{d}^\alpha(\mathfrak{s})$ of the vector $\mathbf{d}^\alpha(\mathfrak{s})$ to the boundary curve $\partial\mathcal{C}$ at the point $\mathfrak{s}\in\partial\mathcal{C}$. Then

$$(\mathcal{D}_\alpha U_3)(\mathfrak{s})=c_1(\mathfrak{s})(\partial_{\mathbf{d}^3}U_3)(\mathfrak{s})+c_2(\mathfrak{s})(\partial_{\boldsymbol{\tau}^\alpha}U_3)(\mathfrak{s})=c_1(\mathfrak{s})(\mathcal{D}_3U_3)(\mathfrak{s})\tag{5.9}$$

for all $\mathfrak{s}\in\gamma$ and all $\alpha=1,2,3$, since $(\mathcal{D}_{\mathbf{d}^3}U_3)(\mathfrak{s})=(\mathcal{D}_3U_3)(\mathfrak{s})U_3$, U_3 vanishes identically on $\partial\mathcal{C}$ and the derivative $(\partial_{\boldsymbol{\tau}^j}U_3^0)(\mathfrak{s})=0$ vanishes, as well.

On the other hand, from (5.7) for $\beta=\alpha=3$ follows $2\mathcal{D}_3U_3(\mathfrak{s})=0$ and, together with (5.9), gives $(\mathcal{D}_\alpha U_3)(\mathfrak{s})=0$ for all $\mathfrak{s}\in\gamma$, $\beta=1,2,3$. Then, due to (5.7), $(\mathcal{D}_3U_\alpha)(\mathfrak{s})=(\mathcal{D}_\alpha U_3)(\mathfrak{s})=0$ and, due to (5.7), $(\mathcal{D}_\alpha U_\alpha)(\mathfrak{s})=0$ for all $\mathfrak{s}\in\gamma$, $\alpha=1,2,3$. Applying again the above arguments, exposed for U_3, we prove equalities (5.8).

6 Numerical approximation of the shell equation

Consider the boundary value problem (4.4)

$$\begin{cases}(\mathcal{L}_{\mathcal{C}}\overline{\mathbf{U}})_\alpha := \mu\Big[\Delta_{\mathcal{C}}\overline{U}_\alpha + \mathcal{D}_\beta\mathcal{D}_\alpha\overline{U}_\beta - 2\mathcal{H}_{\mathcal{C}}\nu_\beta\mathcal{D}_\alpha\overline{U}_\beta - \mathcal{D}_\gamma(\nu_\alpha\nu_\beta\mathcal{D}_\gamma\overline{U}_\beta)\Big] \\ \qquad\qquad + \dfrac{4\lambda\mu}{\lambda+2\mu}\left[\mathcal{D}_\alpha\mathcal{D}_\beta\overline{U}_\beta - 2\mathcal{H}_{\mathcal{C}}\nu_\alpha\mathcal{D}_\beta\overline{U}_\beta\right] = \dfrac{1}{2}\,G_\alpha \ \text{ on } \mathcal{C}, \\ \overline{U}_\alpha(t) = 0 \ \text{ on } \Gamma = \partial\mathcal{C}, \ \ \alpha = 1,2,3,\end{cases}$$

where $G_\alpha = F_\alpha + 2H_\alpha^{(1)} \in [\mathbb{L}_2(\mathcal{C})]$, $\alpha = 1,2,3$.

In [2, Theorem 4], it is proved that if $\overline{\mathbf{U}} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3$ is a solution of BVP (4.4) and $\overline{\mathbf{V}} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3$, then

$$\int\limits_{\mathcal{C}} \left\{2\mu\Big[\mathcal{D}_\beta\overline{U}_\alpha\mathcal{D}_\beta\overline{V}_\alpha + \mathcal{D}_\alpha\overline{U}_\beta\mathcal{D}_\beta\overline{V}_\alpha - \nu_\alpha\nu_\beta\mathcal{D}_\gamma\overline{U}_\beta\mathcal{D}_\gamma\overline{V}_\alpha\Big] + \frac{4\lambda\mu}{\lambda+2\mu}\,\mathcal{D}_\beta\overline{U}_\beta\mathcal{D}_\alpha\overline{V}_\alpha\right\} d\sigma$$
$$= \int\limits_{\mathcal{C}} \langle\overline{G}_\alpha, \overline{V}_\alpha\rangle\, d\sigma. \tag{6.1}$$

Therefore, the BVP (4.4) can be reformulated in the following way.

Find a vector $\overline{\mathbf{U}} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3$ satisfying equation (6.1) for any $\overline{\mathbf{V}} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3$:

$$(c_{\alpha\beta\gamma\zeta}(x)\mathcal{D}_\beta U_\alpha, \mathcal{D}_\zeta V_\gamma) = (G_\alpha, V_\alpha) \ \ \forall\,\mathbf{V} \in [\widetilde{\mathbb{H}}^1(\mathcal{C})]^3, \tag{6.2}$$

where

$$c_{\alpha\beta\gamma\zeta}(x) = \frac{4\lambda\mu}{\lambda+2\mu}\,\delta_{\alpha\beta} + 2\mu(\delta_{\alpha\gamma}\delta_{\beta\zeta} + \delta_{\alpha\zeta}\delta_{\beta\gamma} - \nu_\alpha\nu_\gamma\delta_{\beta\zeta})$$

and $(\,\cdot\,,\,\cdot\,)$ denotes an inner product

$$(f,g) = \int\limits_{\mathcal{C}} \langle f, g\rangle\, d\sigma.$$

Due to (4.13), the sesquilinear form

$$a(U,V) := (c_{\alpha\beta\gamma\zeta}\mathcal{D}_\beta U_\alpha, \mathcal{D}_\zeta V_\gamma)$$

is bounded and coercive in $\mathbb{H}^1_0(\mathcal{C})$,

$$M_1\big\|U \mid \mathbb{H}^1(\mathcal{C})\big\|^2 \geq a(U,U) \geq M\big\|U \mid \mathbb{H}^1(\mathcal{C})\big\|^2 \ \ \forall\, U \in [\mathbb{H}^1_0(\mathcal{C})]^3$$

for some $M > 0$, $M_1 > 0$. Therefore, by the Lax–Milgram Theorem problem (6.2) possesses a unique solution.

Now, let us consider the discrete counterpart of the problem.

Let X_h be a family of finite-dimensional subspaces approximating $[\mathbb{H}^1(\mathcal{C})]^3$, i.e., such that $\bigcup\limits_h X_h$ is dense in $[\mathbb{H}^1(\mathcal{C})]^3$.

Consider equation (6.2) in the finite-dimensional space X_h,

$$a(U_h, V_h) = \widetilde{g}(V_h) \ \ \forall\, V \in X_h, \tag{6.3}$$

where $\widetilde{g}(V_h) = -(G, V_h)_{\mathcal{C}}$.

Theorem 6.1. *Equation* (6.3) *has the unique solution $U_h \in X_h$ for all $h > 0$. This solution converges in $[\mathbb{H}^1(\mathcal{C})]^3$ to the solution U of* (6.2) *as $h \to 0$.*

Proof. Immediately follows from the coercivity of sesquilinear form a:

$$c_1 \left\| U_h \mid [\mathbb{H}^1(\mathcal{C})]^3 \right\|^2 \leq a(U_h, U_h) = |\widetilde{f}(U_h)| \leq c_2 \left\| U_h \mid [\mathbb{H}^1(\mathcal{C})]^3 \right\| \ \text{ for all } \ h. \tag{6.4}$$

Let U_h be the unique solution of the homogeneous equation

$$a(U_h, \psi_h) = 0 \ \text{ for all } \ \psi_h \in X_h.$$

Then (6.4) implies $\|U_h \mid [\mathbb{H}^1(\mathcal{C})]^3\| = 0$ and, consequently, $U_h = 0$. Therefore, equation (6.3) has a unique solution. From (6.4) it also follows that

$$\left\| U_h \mid [\mathbb{H}^1(\mathcal{C})]^3 \right\|^2 \leq \frac{c_2}{c_1} \left\| U_h \mid [\mathbb{H}^1(\mathcal{C})]^3 \right\|.$$

Hence, the sequence $\{\|U_h \mid [\mathbb{H}^1(\mathcal{C})]^3\|\}$ is bounded and we can extract a subsequence $\{U_{h_k}\}$ which converges weakly to some $U \in \mathbb{H}^1(\mathcal{C})$.

Let us take an arbitrary $V \in [\mathbb{H}^1(\mathcal{C})]^3$ and for each $h > 0$ choose $V_h \in X_h$ such that $V_h \to V$ in $[\mathbb{H}^1(\mathcal{C})]^3$. Then from (6.3) we have

$$a(U, V) = \widetilde{g}(V) \ \ \forall\, V \in [\mathbb{H}^1(\mathcal{C})]^3.$$

Hence, U solves (6.2). Note that since (6.2) is uniquely solvable, each subsequence $\{U_{h_k}\}$ converges weakly to the same solution U and, consequently, the whole sequence $\{U_h\}$ also converges weakly to U.

Now, let us prove that it converges in the space $[\mathbb{H}^1(\mathcal{C})]^3$.

Indeed, due to (6.4), we have

$$\begin{aligned} c_1\|U_h - U\|^2 &\leq |a(U_h - U, U_h - U)| \leq |a(U_h, U_h - U) - a(U, U_h - U)| \\ &= c_1|\widetilde{g}(U_h) - a(U_h, U) - \widetilde{g}(U_h - U)| \longrightarrow c_1|\widetilde{g}(U) - a(U, U)| = 0, \end{aligned}$$

which completes the proof. □

We can choose spaces X_h in different ways.

In particular, consider a case where $\omega = U_\alpha$ in the above parametrization is a square part of $\mathbb{R}^2$:

$$\omega = \big\{(x_1, x_2) :\ 0 < x_1 < 1,\ 0 < x_2 < 1\big\}, \ \ \zeta(\omega) = \mathcal{C}.$$

Allocate N^2 nodes $P_{ij} = (i/(N+1),\, j/(N+1))$, $i, j = 1, \dots, N$, on ω.

Let α_k, $k = 1, \dots, N$, be piecewise linear functions defined on the segment $[0, 1]$ as follows:

$$\alpha_k(x) = \begin{cases} 0, & 0 \leq x \leq \dfrac{k-1}{N+1}\,, \\ (N+1)(x - \dfrac{k-1}{N+1})\,, & \dfrac{k-1}{N+1} < x \leq \dfrac{k}{N+1}\,, \\ (N+1)(\dfrac{k+1}{N+1} - x)\,, & \dfrac{k}{N+1} < x \leq \dfrac{k+1}{N+1}\,, \\ 0, & \dfrac{k+1}{N+1} < x \leq 1, \\ j = k, \dots, N. \end{cases}$$

Denote by φ_{ij}, $i, j = 1, \dots, N$, the functions

$$\varphi_{ij}(x_1,\, x_2) = \alpha_i(x_1)\alpha_j(x_2), \ \ i, j = 1, \dots N, \ \ (x_1,\, x_2) \in \omega.$$

Evidently, φ_{ij} are continuous functions, which take their maximal value $\varphi_{ij}(P_{ij}) = 1$ at the point P_{ij} and vanish outside the set

$$\omega_{ij} = \omega \cap \left\{(x_1, x_2) :\ 0 \leq \left|x_1 - \frac{i}{N+1}\right| \leq 1,\ 0 \leq \left|x_2 - \frac{j}{N+1}\right| \leq 1\right\}.$$

Consequently, they belong to $\mathbb{H}^1(\omega)$ and are linearly independent.

Denote by X_N the linear span of the functions $\widehat{\varphi}_{ij} = \varphi_{ij} \circ \vartheta$, $i, j = 1, \dots, N$. The space X_N is N^2-dimensional space contained in $\mathbb{H}^1(\mathcal{C})$.

Let $\widetilde{\varphi}_{ij}^{(k)} = (\delta_{1k}, \delta_{2k}, \delta_{3k})\widehat{\varphi}_{ij} \in [X_N]^3$, $k = 1, 2, 3$, $i, j = 1, \dots, N$.

Consider equation (6.3) in the space $[X_N]^3$

$$a(U, V) = \widetilde{g}(V) \quad \forall V \in [X_N]^3. \tag{6.5}$$

We sought for the solution $U \in [X_N]^3$ of equation (6.5) in the form

$$U = \sum_{m=1}^{3} \sum_{i,j=1}^{N} C_{ij}^{(m)} \widetilde{\varphi}_{ij}^{(m)},$$

where $C_{ij}^{(m)}$ are unknown coefficients. Substituting U into (6.5) and replacing V successively by $\widetilde{\varphi}_{ij}^{(m)}$, $m = 1, 2, 3$, $i, j = 1, \dots, N$, we get the equivalent system of $3N^2$ linear algebraic equations

$$\sum_{m=1}^{3} \sum_{i,j=1}^{N} A_{ijkl}^{(m,n)} C_{ij}^{(m)} = g_{kl}^{(n)}, \quad n = 1, 2, 3, \quad k, l = 1, \dots, N, \tag{6.6}$$

where

$$A_{ijkl}^{(m,n)} = a\big(\widetilde{\varphi}_{ij}^{(m)}, \widetilde{\varphi}_{kl}^{(n)}\big), \quad g_{kl}^{(n)} = \widetilde{g}\big(\widetilde{\varphi}_{kl}^{(n)}\big).$$

The matrix $A = A_{(ijkl)}^{(m,n)}$ is Gram's matrix defined by the positive semidefinite bilinear form a attached to basis vectors $\widetilde{\varphi}_{ij}^{(m)}$, $m = 1, 2, 3$, $i, j = 1, \dots, N$, of $[X_N]^3$. Therefore, it is a nonsingular matrix and equation (6.6) has a unique solution

$$U = \sum_{i,j,k,l=1}^{N} (A^{-1})_{ijkl}^{(m,n)} \widetilde{\varphi}_{ij}^{(m)} g_{kl}^{(n)}.$$

To calculate explicitly $A_{ijkl}^{(m,n)}$ and $g_{kl}^{(n)}$ we note that

$$\begin{aligned}
\mathcal{D}_r \widetilde{\varphi}_{ij}^{(m)}(y) &= \partial_{y_r} \widetilde{\varphi}_{ij}^{(m)}(y) + \nu_r \partial_\nu \widetilde{\varphi}_{ij}^{(m)}(y) \\
&= \sum_{p=1}^{2} \partial_p \varphi_{ij}(\vartheta(y)) \big(\partial_r \vartheta_p(y) + \nu_r \nu_l \partial_l \vartheta_p(y)\big)(\delta_{m1}, \delta_{m2}, \delta_{m3}) \\
&= \sum_{p=1}^{2} \partial_p \varphi_{ij}(\vartheta(y)) \mathcal{D}_r \vartheta_p(y)(\delta_{m1}, \delta_{m2}, \delta_{m3}), \\
A_{ijkl}^{(m,n)} &= a\big(\widetilde{\varphi}_{ij}^{(m)}, \widetilde{\varphi}_{kl}^{(n)}\big) = (c_{qrst} \delta_{rm} \delta_{tn} \mathcal{D}_q \varphi_{ij}, \mathcal{D}_s \varphi_{kl}) \\
&= \sum_{\alpha,\beta=1}^{2} \int\limits_{\omega_{ij} \cap \omega_{kl}} c_{qmsn}(\vartheta(y)) \big(\partial_\alpha \varphi_{ij}(\vartheta(y))\big) \big(\partial_\beta \varphi_{kl}(\vartheta(y))\big) \mathcal{D}_q \vartheta_\alpha(y) \mathcal{D}_s \vartheta_\beta(y) |\sigma'(y)| \, dy, \\
g_{kl}^{(n)} &= -(g, \widetilde{\varphi}_{kl}^{(n)})_{\mathcal{C}} = -\int\limits_{\omega_{ij} \cap \omega_{kl}} g(\vartheta(y)) \, \varphi_{kl}^{(n)}(\vartheta(y)) |\sigma'(y)| \, dy,
\end{aligned}$$

where $|\sigma'(y)|$ is a surface element of $\mathcal{C}$

$$|\sigma'(y)| = |\partial_1 \vartheta(y) \times \partial_2 \vartheta(y)|.$$

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