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Aurelian Cernea

ON SOME FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSIONS WITH ERDÉLYI–KOBER FRACTIONAL INTEGRAL BOUNDARY CONDITIONS

Abstract. We study two classes of fractional integro-differential inclusions with Erdélyi–Kober fractional integral boundary conditions and we obtain existence results in the case of the set-valued map has nonconvex values.

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1 Introduction

In recent years, the systems defined by fractional order derivatives have attracted increasing interest mainly due to their applications in different fields of science and engineering. The main reason is that a lot of phenomena in nature can be better explained using fractional-order systems (see, e.g., [5, 10, 13, 15, 16], etc.).

The present paper is concerned with the following boundary value problems. First, we consider a fractional integro-differential inclusion defined by the Caputo fractional derivative

$$D_c^q x(t) \in F(t, x(t), V(x)(t)) \quad a.e.\ ([0,T]) \tag{1.1}$$

with the boundary conditions of the form

$$\begin{aligned} x(0) &= \alpha \frac{1}{\Gamma(p)} \int_0^\zeta (\zeta - s)^{p-1} x(s)\,ds = \alpha J^p x(\zeta), \\ x(T) &= \beta \frac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^\xi \frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta - s^\eta)^{1-\delta}}\, x(s)\,ds = \beta I_\eta^{\gamma,\delta} x(\xi), \end{aligned} \tag{1.2}$$

where $q \in (1,2]$, D_c^q is the Caputo fractional derivative of order q, $0 < \zeta, \xi < T$, $\alpha, \beta, \gamma \in \mathbb{R}$, $p, \delta, \eta > 0$, J^p is the Riemann–Liouville fractional integral of order p, $I_\eta^{\gamma,\delta}$ is the Erdélyi–Kober fractional integral of order $\delta > 0$ with $\eta > 0$ and $\gamma \in \mathbb{R}$, $F : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map and $V : C([0,T], \mathbb{R}) \to C([0,T], \mathbb{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t, s, x(s))\,ds$ with $k(\cdot,\cdot,\cdot) : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a given function. We note that the fractional derivative introduced by Caputo in [6] and afterwards adopted in the theory of linear visco-elasticity allows to use Cauchy conditions with physical meanings.

Next, we consider the problem

$$D^q x(t) \in F(t, x(t), V(x)(t)) \ \ a.e.\ ([0,T]) \tag{1.3}$$

with the boundary conditions of the form

$$x(0) = 0, \quad \alpha x(T) = \sum_{i=1}^m \beta_i I_{\eta_i}^{\gamma_i,\delta_i} x(\xi_i), \tag{1.4}$$

where D^q is the Riemann–Liouville fractional derivative of order $q \in (1,2]$, $0 < \xi_i < T$, $\alpha, \beta_i, \gamma_i \in \mathbb{R}$, $\delta_i, \eta_i > 0$, $i = 1, 2, \dots, m$, F and V are as above.

Our aim is to obtain the existence of solutions for problems (1.1), (1.2) and (1.3), (1.4) in case where the set-valued map F has nonconvex values, but is assumed to be Lipschitz in the second and third variable. Our results use Filippov's techniques (see [12]); namely, the existence of solutions is obtained by starting from a given "quasi" solution. In addition, the result provides an estimate between the "quasi" solution and the solution obtained.

Note that in the case when F does not depend on the last variable and is single-valued, the existence results for problem (1.1), (1.2) may be found in [2], and in the situation when F does not depend on the last variable, the existence results for problem (1.3), (1.4) are given in [1]. All the results in [1, 2] are proved by using several suitable theorems from fixed point theory.

Our results improve some existence theorems in [1] and, respectively, in [2] in the case where the right-hand side is Lipschitz in the second variable. Moreover, these results may be regarded as generalizations to the case where the right-hand side contains a nonlinear Volterra integral operator. It should be also mentioned that the method used in our approach is known in the theory of differential inclusions; similar results for other classes of fractional differential inclusions have been obtained in our previous papers (see [7–9], etc.). However, the exposition of this method in the framework of problems (1.1), (1.2) and (1.3), (1.4) is new.

The paper is organized as follows. In Section 2, we recall some preliminary results that we need in the sequel and in Section 3, we prove our main results.

2 Preliminaries

Let (X, d) be a metric space. Recall that the Pompeiu–Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\left\{d^*(A, B), d^*(B, A)\right\}, \quad d^*(A, B) = \sup\left\{d(a, B);\ a \in A\right\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

Let $I = [0, T]$, we denote by $C(I, \mathbb{R})$ the Banach space of all continuous functions from I to $\mathbb{R}$ with the norm $\|x(\cdot)\|_C = \sup_{t \in I} |x(t)|$, and $L^1(I, \mathbb{R})$ is the Banach space of integrable functions $u(\cdot) : I \to \mathbb{R}$ endowed with the norm $\|u(\cdot)\|_1 = \int_0^T |u(t)|\, dt$.

The fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$J^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\, ds,$$

provided the right-hand side is defined pointwise on $(0, \infty)$, and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, \mathrm{d}t$.

The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a Lebesgue integrable function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{-\alpha+n-1} f(s)\, ds,$$

where $n = [\alpha] + 1$, provided the right-hand side is defined pointwise on $(0, \infty)$.

The Caputo fractional derivative of order $\alpha > 0$ of a function $f : [0, \infty) \to \mathbb{R}$ is defined by

$$D_c^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s)\, \mathrm{d}s,$$

where $n = [\alpha] + 1$. It is assumed implicitly that f is n times differentiable whose n-th derivative is absolutely continuous.

The Erdélyi–Kober fractional integral of order $\delta > 0$ *with* $\eta > 0$ *and* $\gamma \in \mathbb{R}$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by

$$I_\eta^{\gamma,\delta} f(t) = \frac{\eta t^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^{\eta\gamma+\eta-1}}{(t^\eta - s^\eta)^{1-\delta}} f(s)\, ds,$$

provided the right-hand side is defined pointwise on $(0, \infty)$.

We recall that for $\eta = 1$,

$$I_1^{\gamma,\delta} f(t) = \frac{t^{-(\delta+\gamma)}}{\Gamma(\delta)} \int_0^t \frac{s^\gamma}{(t-s)^{1-\delta}} f(s)\, ds$$

is the Kober operator introduced by Kober in [14]. If $\gamma = 0$, the Kober operator reduces to the Riemann–Liouville fractional integral with a power weight

$$I_1^{0,\delta} f(t) = \frac{t^{-\delta}}{\Gamma(\delta)} \int_0^t \frac{f(s)}{(t-s)^{1-\delta}}\, ds.$$

Lemma 2.1 ([2]). *Let $\delta, \eta > 0$ and $\gamma, q \in \mathbb{R}$. Then*

$$I_\eta^{\gamma,\delta}(t^q) = \frac{t^q \Gamma(\gamma + \frac{q}{\eta} + 1)}{\Gamma(\gamma + \frac{q}{\eta} + \delta + 1)}\,.$$

By definition, a function $x(\cdot) \in C^2(I, \mathbb{R})$ is called a solution of problem (1.1), (1.2) *if there exists $f(\cdot) \in L^1(I, \mathbb{R})$ such that $f(t) \in F(t, x(t), V(x)(t))$ a.e. (I), $D_c^q x(t) = f(t)$ a.e. (I) and conditions* (1.2) *are satisfied.*

Lemma 2.2 ([2]). *For $f(\cdot) \in AC(I, \mathbb{R})$, $x(\cdot) \in C^2(I, \mathbb{R})$ is a solution of the problem*

$$D_c^q x(t) = f(t) \;\; \text{a.e. } (I),$$

with the boundary conditions (1.2) *if and only if*

$$x(t) = J^q f(t) + \frac{\alpha}{\Lambda}\,(v_4 - t v_3) J^{p+q} f(\zeta) + \frac{1}{\Lambda}\,(v_2 + t v_1)\big(\beta I_\eta^{\gamma,\delta} J^q f(\xi) - J^q f(T)\big),$$

where

$$\Lambda = v_1 v_4 + v_2 v_3 \neq 0, \quad v_1 = 1 - \alpha\, \frac{\zeta^p}{\Gamma(p+1)}\,, \quad v_2 = \alpha\, \frac{\zeta^{p+1}}{\Gamma(p+2)}\,,$$
$$v_3 = 1 - \beta\, \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\delta+1)}\,, \quad v_4 = T - \beta\zeta\, \frac{\Gamma(\gamma + \frac{1}{\eta} + 1)}{\Gamma(\gamma + \frac{1}{\eta} + \delta + 1)}\,.$$

Remark 2.3. The solution $x(\cdot)$ in Lemma 2.2 can be written as

$$\begin{aligned}
x(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\, f(s)\, ds + \frac{\alpha}{\Lambda}\, \frac{(v_4 - t v_3)}{\Gamma(q)} \int_0^\zeta (\zeta - s)^{p+q-1} f(s)\, ds \\
&\quad + \frac{\beta(v_2 + t v_1)}{\Lambda}\, \frac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^\xi \frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta - s^\eta)^{1-\delta}} \Big(\frac{1}{\Gamma(q)} \int_0^s (s-u)^{q-1} f(u)\, du\Big)\, ds \\
&\quad - \frac{1}{\Lambda}\,(v_2 + t v_1) \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)} f(s)\, ds \\
&= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s)\, ds + \frac{\alpha}{\Lambda}\, \frac{(v_4 - t v_3)}{\Gamma(q)} \int_0^\zeta (\zeta - s)^{p+q-1} f(s)\, ds \\
&\quad + \frac{\beta(v_2 + t v_1)}{\Lambda\Gamma(q)}\, \frac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_0^\xi \bigg(\int_u^\xi \frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta - s^\eta)^{1-\delta}}\, (s-u)^{q-1}\, ds\bigg) f(u)\, du \\
&\quad - \frac{1}{\Lambda}\,(v_2 + t v_1) \int_0^T \frac{(T-s)^{q-1}}{\Gamma(q)}\, f(s)\, ds \\
&= \int_0^T G_1(t,s) f(s)\, ds,
\end{aligned}$$

where

$$\begin{aligned}
G_1(t,u) &= \frac{(t-u)^{q-1}}{\Gamma(q)}\, \chi_{[0,t]}(u) + \frac{\alpha}{\Lambda}\, \frac{(v_4 - t v_3)}{\Gamma(q)}\, (\zeta - u)^{p+q-1} \chi_{[0,\zeta]}(u) \\
&\quad + \frac{\beta(v_2 + t v_1)}{\Lambda\Gamma(q)}\, \frac{\eta \xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)} \int_u^\xi \frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta - s^\eta)^{1-\delta}}\, (s-u)^{q-1}\, ds \chi_{[0,\xi]}(u) - \frac{v_2 + t v_1}{\Lambda\Gamma(q)}\, (T-u)^{q-1},
\end{aligned}$$

$\chi_S(\cdot)$ denotes the characteristic function of the set S.

Using the fact that $q>1$ and taking into account Lemma 2.1, one has

$$\frac{\eta\xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int\limits_u^{\xi}\frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta-s^\eta)^{1-\delta}}(s-u)^{q-1}\,ds \le \frac{\eta\xi^{-\eta(\delta+\gamma)}}{\Gamma(\delta)}\int\limits_0^{\xi}\frac{s^{\eta\gamma+\eta-1}}{(\xi^\eta-s^\eta)^{1-\delta}}s^{q-1}\,ds = \frac{\xi^{q-1}\Gamma(\gamma+\frac{q-1}{\eta}+1)}{\Gamma(\gamma+\frac{q-1}{\eta}+\delta+1)}.$$

Therefore, for any $t,u\in I$,

$$|G_1(t,u)|\le\frac{T^{q-1}}{\Gamma(q)}+\frac{|\alpha|(|v_4|+T|v_3|)\zeta^{p+q-1}}{|\Lambda|\Gamma(q)} + \frac{|\beta|(|v_2|+T|v_1|)}{|\Lambda|\Gamma(q)}\frac{\xi^{q-1}\Gamma(\gamma+\frac{q-1}{\eta}+1)}{\Gamma(\gamma+\frac{q-1}{\eta}+\delta+1)}+\frac{(|v_2|+T|v_1|)T^{q-1}}{|\Lambda|\Gamma(q)}=:K_1.$$

By definition, a function $x(\cdot)\in C^2(I,\mathbb{R})$ is called a solution of problem (1.3), (1.4) if there exists $f(\cdot)\in L^1(I,\mathbb{R})$ such that $f(t)\in F(t,x(t),V(x)(t))$ a.e. (I), $D_c^q x(t)=f(t)$ a.e. (I) and conditions (1.4) are satisfied.

Lemma 2.4 ([1]). *For $f(\cdot)\in AC(I,\mathbb{R})$, $x(\cdot)\in C^2(I,\mathbb{R})$ is a solution of the problem*

$$D_c x(t)=f(t)\ \text{ a.e. } (I),$$

with the boundary conditions (1.4) *if and only if*

$$x(t)=J^qf(t)-\frac{t^{q-1}}{\Lambda}\Big(\alpha J^qf(t)-\sum_{i=1}^m\beta_iI_{\eta_i}^{\gamma_i,\delta_i}J^qf(\xi_i)\Big),$$

where

$$\Lambda=\alpha T^{q-1}-\sum_{i=1}^m\frac{\beta_1\xi_i^{q-1}\Gamma(\gamma_i+\frac{q-1}{\eta_i}+1)}{\Gamma(\gamma_i+\frac{q-1}{\eta_i}+\delta_i+1)}\neq 0.$$

Remark 2.5. The solution $x(\cdot)$ in Lemma 2.4 can be written as $x(t)=\int\limits_0^T G_2(t,s)f(s)\,ds$, where

$$G_2(t,u)=\frac{(t-u)^{q-1}}{\Gamma(q)}\chi_{[0,t]}(u)-\frac{\alpha t^{q-1}}{\Lambda\Gamma(q)}(t-u)^{q-1}\chi_{[0,t]}(u) + \sum_{i=1}^m\frac{\beta_it^{q-1}}{\Lambda\Gamma(q)}\frac{\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\delta_i)}\int\limits_u^{\xi_i}\frac{s^{\eta_i\gamma_i+\eta_i-1}}{(\xi_i^{\eta_i}-s^{\eta_i})^{1-\delta_i}}(s-u)^{q-1}\,ds\,\chi_{[0,\xi_i]}(u).$$

As in Remark 2.3, for $i=1,2,\dots,m$, one has

$$\frac{\eta_i\xi_i^{-\eta_i(\delta_i+\gamma_i)}}{\Gamma(\delta_i)}\int\limits_u^{\xi_i}\frac{s^{\eta_i\gamma_i+\eta_i-1}}{(\xi_i^{\eta_i}-s^{\eta_i})^{1-\delta_i}}(s-u)^{q-1}\,ds\le\frac{\xi_i^{q-1}\Gamma(\gamma_i+\frac{q-1}{\eta_i}+1)}{\Gamma(\gamma_i+\frac{q-1}{\eta_i}+\delta_i+1)}$$

and thus, for any $t,u\in I$,

$$|G_2(t,u)|\le\frac{T^{q-1}}{\Gamma(q)}+\frac{T^{q-1}}{|\Lambda|\Gamma(q)}\left[|\alpha|T^{q-1}+\sum_{i=1}^m\frac{|\beta_i|\xi_i^{q-1}\Gamma(\gamma_i+\frac{q-1}{\eta_i}+1)}{\Gamma(\gamma_i+\frac{q-1}{\eta_i}+\delta_i+1)}\right]=:K_2.$$

3 The main results

First, we recall a selection result (see [4]) which is a version of the celebrated Kuratowski and Ryll–Nardzewski selection theorem.

Lemma 3.1. *Suppose X is a separable Banach space, B is the closed unit ball in X, $H : I \to \mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g : I \to X$, $L : I \to \mathbb{R}_+$ are measurable functions. If*

$$H(t) \cap (g(t) + L(t)B) \neq \varnothing \ \text{ a.e. } (I),$$

then the set-valued map $t \to H(t) \cap (g(t) + L(t)B)$ has a measurable selection.

In order to prove our results, we need the following hypotheses.

Hypothesis 3.2.

(i) $F(\,\cdot\,, \,\cdot\,) : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R} \times \mathbb{R})$ *measurable.*

(ii) *There exists $L(\,\cdot\,) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I$, $F(t, \,\cdot\,, \,\cdot\,)$ is $L(t)$-Lipschitz in the sense that*

$$d_H\big(F(t, x_1, y_1), F(t, x_2, y_2)\big) \leq L(t)\big(|x_1 - x_2| + |y_1 - y_2|\big) \ \ \forall\, x_1, x_2, y_1, y_2 \in \mathbb{R}.$$

(iii) $k(\,\cdot\,, \,\cdot\,, \,\cdot\,) : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is a function such that* $\forall\, x \in \mathbb{R}$, $(t, s) \to k(t, s, x)$ *is measurable.*

(iv) $|k(t, s, x) - k(t, s, y)| \leq L(t)|x - y|$ *a.e.* $(t, s) \in I \times I$, $\forall\, x, y \in \mathbb{R}$.

Next, we use the notation

$$M(t) := L(t)(1 + \int_0^t L(u)\, du), \ \ t \in I, \ \ K_0 = \int_0^T M(t)\, dt.$$

Theorem 3.3. *Assume that Hypothesis 3.2 is satisfied and $K_1 K_0 < 1$. Let $y(\,\cdot\,) \in C^2(I, \mathbb{R})$ be such that $y(0) = \alpha J^p y(\zeta)$, $y(T) = \beta I_\eta^{\gamma,\delta} y(\xi)$ and there exist $p(\,\cdot\,) \in L^1(I, \mathbb{R}_+)$ with*

$$d\big(D_c^q y(t), F(t, y(t), V(y)(t))\big) \leq p(t) \ \text{ a.e. } (I).$$

Then there exists a solution $x(\,\cdot\,) : I \to \mathbb{R}$ of problem (1.1), (1.2) satisfying for all $t \in I$ the inequality

$$|x(t) - y(t)| \leq \frac{K_1}{1 - K_1 K_0}\, \|p(\,\cdot\,)\|_1.$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \big\{D_c^q y(t) + p(t)[-1, 1]\big\} \neq \varnothing \ \text{ a.e. } (I).$$

It follows from Lemma 3.1 that there exists a measurable selection $f_1(t) \in F(t, y(t), V(y)(t))$ a.e. (I) such that

$$|f_1(t) - D_c^q y(t)| \leq p(t) \ \text{ a.e. } (I). \tag{3.1}$$

Define $x_1(t) = \int\limits_0^T G_1(t, s) f_1(s)\, ds$. One has

$$|x_1(t) - y(t)| \leq M_1 \int_0^T p(t)\, dt.$$

We construct two sequences $x_n(\cdot)\in C(I,\mathbb{R})$, $f_n(\cdot)\in L^1(I,\mathbb{R})$, $n\geq 1$, with the following properties:

$$x_n(t)=\int\limits_0^T G_1(t,s)f_n(s)\,ds,\quad t\in I, \tag{3.2}$$

$$f_n(t)\in F\big(t,x_{n-1}(t),V(x_{n-1})(t)\big)\ \text{ a.e. } (I), \tag{3.3}$$

$$|f_{n+1}(t)-f_n(t)|\leq L(t)\bigg(|x_n(t)-x_{n-1}(t)|+\int\limits_0^t L(s)|x_n(s)-x_{n-1}(s)|\,ds\bigg)\ \text{ a.e. } (I). \tag{3.4}$$

If this is done, then from (3.1)–(3.4) for almost all $t\in I$ we have

$$|x_{n+1}(t)-x_n(t)|\leq K_1(K_1K_0)^n\int\limits_0^T p(t)\,dt\ \ \forall\, n\in\mathbf{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$\begin{aligned}|x_{n+1}(t)-x_n(t)|&\leq\int\limits_0^T|G_1(t,t_1)|\,|f_{n+1}(t_1)-f_n(t_1)|\,dt_1\\&\leq K_1\int\limits_0^T L(t_1)\bigg[|x_n(t_1)-x_{n-1}(t_1)|+\int\limits_0^{t_1}L(s)|x_n(s)-x_{n-1}(s)|\,ds\bigg]\,dt_1\\&\leq K_1\int\limits_0^T L(t_1)\bigg(1+\int\limits_0^{t_1}L(s)\,ds\bigg)\,dt_1\cdot K_1^nK_0^{n-1}\int\limits_0^T p(t)\,dt\\&=K_1(K_1K_0)^n\int\limits_0^T p(t)\,dt.\end{aligned}$$

Therefore, $\{x_n(\cdot)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbb{R})$ converging uniformly to some $x(\cdot)\in C(I,\mathbb{R})$. Hence, by (3.4), for almost all $t\in I$, the sequence $\{f_n(t)\}$ is Cauchy sequence in $\mathbb{R}$. Let $f(\cdot)$ be the pointwise limit of $f_n(\cdot)$.

At the same time, one has

$$\begin{aligned}|x_n(t)-y(t)|&\leq|x_1(t)-y(t)|+\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)|\\&\leq M_1\int\limits_0^T p(t)\,dt+\sum_{i=1}^{n-1}\bigg(K_1\int\limits_0^T p(t)\,dt\bigg)(K_1K_0)^i=\frac{K_1\int_0^T p(t)\,dt}{1-K_1K_0}\,.\end{aligned} \tag{3.5}$$

On the other hand, from (3.1), (3.4) and (3.5) for almost all $t\in I$ we obtain

$$|f_n(t)-D_c^q y(t)|\leq\sum_{i=1}^{n-1}|f_{i+1}(t)-f_i(t)|+|f_1(t)-D_c^q y(t)|\leq L(t)\,\frac{K_1\int_0^T p(t)\,dt}{1-K_1K_0}+p(t).$$

Hence the sequence $f_n(\cdot)$ is integrably bounded and therefore $f(\cdot)\in L^1(I,\mathbb{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (3.2), (3.3), we deduce that $x(\cdot)$ is a solution of (1.1), (1.2). Finally, passing to the limit in (3.5), we obtain the desired estimate on $x(\cdot)$.

It remains to construct the sequences $x_n(\cdot)$, $f_n(\cdot)$ with the properties in (3.2)–(3.4). The construction will be done by induction.

Since the first step is already realized, assume that for some $N \geq 1$ we have already constructed $x_n(\cdot) \in C(I,\mathbb{R})$ and $f_n(\cdot) \in L^1(I,\mathbb{R})$, $n = 1,2,\dots,N$, satisfying (3.2), (3.4) for $n = 1,2,\dots,N$ and (3.3) for $n = 1,2,\dots,N-1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map

$$t \longrightarrow L(t)\bigg(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|\,ds\bigg)$$

is measurable. By the lipschitzianity of $F(t,\cdot)$ for almost all $t \in I$ we have

$$F\big(t, x_N(t), V(x_N)(t)\big) \cap \bigg\{ f_N(t) + L(t)\bigg(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|\,ds\bigg)[-1,1]\bigg\} \neq \varnothing.$$

Lemma 3.1 yields that there exists a measurable selection $f_{N+1}(\cdot)$ of $F(\cdot, x_N(\cdot), V(x_N)(\cdot))$ such that for almost all $t \in I$,

$$|f_{N+1}(t) - f_N(t)| \leq L(t)\bigg(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|\,ds\bigg).$$

We define $x_{N+1}(\cdot)$ as in (3.2) with $n = N+1$. Thus $f_{N+1}(\cdot)$ satisfies (3.3) and (3.4) and the proof is complete. □

The assumption in Theorem 3.3 is satisfied, in particular, for $y(\cdot) = 0$ and therefore with $p(\cdot) = L(\cdot)$. We obtain the following consequence of Theorem 3.3.

Corollary 3.4. *Assume that Hypothesis* 3.2 *is satisfied,* $d(0, F(t,0,0) \leq L(t)$ *a.e.* (I) *and* $K_1 K_0 < 1$. *Then there exists a solution* $x(\cdot)$ *of problem* (1.1), (1.2) *satisfying for all* $t \in I$, *the inequality*

$$|x(t)| \leq \frac{K_1}{1 - K_1 K_0}\, \|L(\cdot)\|_1.$$

Example 3.5. Consider

$$q = \frac{3}{2}\,, \quad T = 1, \quad \alpha = \frac{6}{13}\,, \quad p = \frac{1}{2}\,, \quad \zeta = \frac{1}{4}\,,$$
$$\beta = \frac{\sqrt{7}}{9}\,, \quad \gamma = \frac{3}{4}\,, \quad \delta = \frac{\sqrt{7}}{5}\,, \quad \eta = \frac{1}{6}\,, \quad \xi = \frac{3}{4}\,.$$

Denote by K_1^0 the corresponding estimate of $G_1(\cdot,\cdot)$ in Remark 2.3 and take $a \in \big(0, -1 + \sqrt{1 + \frac{2}{K_1^0}}\big)$.

Define $F(\cdot,\cdot) : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$F(t,x,y) = \Big[-a\,\frac{|x|}{1+|x|}\,, 0\Big] \cup \Big[0, a\,\frac{|y|}{1+|y|}\Big]$$

and $k(\cdot,\cdot,\cdot) : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $k(t,s,x) = ax$.

Since

$$\sup\big\{|u| : \ u \in F(t,x,y)\big\} \leq a \ \ \forall\, t \in [0,1], \ \ x,y \in \mathbb{R},$$
$$\mathrm{d}_H\big(F(t,x_1,y_1), F(t,x_2,y_2)\big) \leq a|x_1 - x_2| + a|y_1 - y_2| \ \ \forall\, x_1,x_2,y_1,y_2 \in \mathbb{R},$$

in this case $p(t) \equiv L(t) \equiv a$, $M(t) = a(1+at)$ and $K_0 = a + \frac{a^2}{2}$.

According to the choice of a, we are able to apply Corollary 3.4 in order to deduce the existence of a solution of the problem

$$D_c^{\frac{3}{2}} x(t) \in \Big[-a\,\frac{|x(t)|}{1+|x(t)|}\,, 0\Big] \cup \Big[0, a^2\,\frac{|\int_0^t x(s)\,ds|}{1+a|\int_0^t x(s)\,ds|}\Big],$$

$$x(0) = \frac{6}{13}\, J^{\frac{1}{2}} x\Big(\frac{1}{4}\Big), \quad x(1) = \frac{\sqrt{7}}{9}\, I_{\frac{1}{6}}^{\frac{3}{4}, \frac{\sqrt{7}}{5}} x\Big(\frac{3}{4}\Big)$$

that satisfies

$$|x(t)| \le \frac{K_1^0 a}{1-(a+\frac{a^2}{2})K_1^0} \quad \forall\, t \in [0,1].$$

If F does not depend on the last variable, Hypothesis 3.2 becames

Hypothesis 3.6.

(i) $F(\,\cdot\,,\,\cdot\,) : I \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbb{R})$ *measurable.*

(ii) *There exists* $L(\,\cdot\,) \in L^1(I,(0,\infty))$ *such that for almost all* $t \in I$, $F(t,\,\cdot\,)$ *is* $L(t)$*-Lipschitz in the sense that*

$$d_H\big(F(t,x_1), F(t,x_2)\big) \le L(t)|x_1 - x_2| \;\; \forall\, x_1, x_2 \in \mathbb{R}.$$

Denote $L_0 = \int\limits_0^T L(t)\,dt$.

Corollary 3.7. *Assume that Hypothesis* 3.6 *is satisfied,* $d(0, F(t,0) \le L(t)$ *a.e.* (I) *and* $K_1 L_0 < 1$. *Then there exists a solution* $x(\,\cdot\,)$ *of the fractional differential inclusion*

$$D_c^q x(t) \in F(t, x(t)) \;\; \text{a.e. } (I),$$

with the boundary conditions (1.2) *satisfying for all* $t \in I$

$$|x(t)| \le \frac{K_1 L_0}{1 - K_1 L_0}\,. \tag{3.6}$$

Remark 3.8. If $F(\,\cdot\,,\,\cdot\,)$ is a single-valued map, the fractional differential inclusion reduces to the fractional differential equation

$$D_c^q x(t) = f(t, x(t)) \;\; \text{a.e. } (I).$$

In this case, a similar result to the one in Corollary 3.7 may be found in [2], namely, Theorem 3.1. It is assumed that the Lipschitz constant of $f(t,\,\cdot\,)$ does not depend on t and its proof is done by using the Banach fixed point theorem. Therefore, our Corollary 3.7 extends Theorem 3.1 in [2] to the situation when the Lipschitz constant of $f(t,\,\cdot\,)$ depends on t and to the set-valued framework. Moreover, Corollary 3.7 provides a priori bounds for the solution, as in (3.6).

The proof of the next theorem is similar to that of Theorem 3.3.

Theorem 3.9. *Assume that Hypothesis* 3.2 *is satisfied and* $K_2 K_0 < 1$. *Let* $y(\,\cdot\,) \in C^2(I,\mathbb{R})$ *be such that* $y(0) = 0$, $\alpha y(T) = \sum\limits_{i=1}^m \beta_i I_{\eta_i}^{\gamma_i,\delta_i} y(\xi_i)$ *and let there exist* $p(\,\cdot\,) \in L^1(I,\mathbb{R})$ *with*

$$d\big(D^q y(t), F(t, y(t, V(y)(t)))\big) \le p(t) \;\; \text{a.e. } (I).$$

Then there exists a solution $x(\,\cdot\,) : I \to \mathbb{R}$ *of problem* $(1.3), (1.4)$ *satisfying for all* $t \in I$

$$|x(t) - y(t)| \le \frac{K_2}{1 - K_2 K_0}\, \|p(\,\cdot\,)\|_1.$$

Example 3.10. Consider

$$q=\frac{3}{2}\,,\ \ T=5,\ \ m=3,\ \ \alpha=\frac{2}{3}\,,\ \ \beta_1=\frac{e}{2}\,,\ \ \beta_2=\frac{\pi}{3}\,,\ \ \beta_3=\frac{\sqrt{\pi}}{6}\,,$$
$$\eta_1=\frac{\sqrt{3}}{5}\,,\ \ \eta_2=\frac{\sqrt{2}}{5}\,,\ \ \eta_3=\frac{e}{3}\,,\ \ \gamma_1=\frac{5}{3}\,,\ \ \gamma_2=\frac{2}{9}\,,\ \ \gamma_3=\frac{\sqrt{e}}{2}\,,$$
$$\delta_1=\frac{3}{7}\,,\ \ \delta_2=\frac{\sqrt{3}}{8}\,,\ \ \delta_3=\frac{e^2}{4}\,,\ \ \xi_1=\frac{4}{3}\,,\ \ \xi_2=\frac{3}{2}\,,\ \ \xi_3=\frac{2}{7}\,.$$

Denote by K_2^0 the corresponding estimate of $G_2(\,\cdot\,,\,\cdot\,)$ in Remark 2.5 and take $a\in\big(0,\frac{1}{5}\,(-1+\sqrt{1+\frac{2}{K_2^0}})\big)$.

Define $F(\,\cdot\,,\,\cdot\,):I\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ by

$$F(t,x,y)=\Big[-a\,\frac{|x|}{1+|x|},0\Big]\cup\Big[0,a\,\frac{|y|}{1+|y|}\Big]$$

and $k(\,\cdot\,,\,\cdot\,,\,\cdot\,):I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by $k(t,s,x)=ax$.

As above,

$$\sup\big\{|u|:\ u\in F(t,x,y)\big\}\le a\ \ \forall\,t\in[0,1],\ \ x,y\in\mathbb{R},$$
$$\mathrm{d}_H\big(F(t,x_1,y_1),F(t,x_2,y_2)\big)\le a|x_1-x_2|+a|y_1-y_2|\ \ \forall\,x_1,x_2,y_1,y_2\in\mathbb{R},$$

and, therefore, $p(t)\equiv L(t)\equiv a$, $M(t)=a(1+at)$ and $K_0=5a+\frac{25a^2}{2}$.

Taking into account the choice of a, we can apply Theorem 3.9 with $y(\,\cdot\,)=0$ and deduce the existence of a solution of the problem

$$D^{\frac{3}{2}}x(t)\in\Big[-a\,\frac{|x(t)|}{1+|x(t)|}\,,0\Big]\cup\Big[0,a^2\,\frac{\big|\int_0^t x(s)\,ds\big|}{1+a\big|\int_0^t x(s)\,ds\big|}\Big],$$
$$x(0)=0,\quad \frac{2}{3}\,x(5)=\frac{e}{2}\,I_{\frac{\sqrt{3}}{5}}^{\frac{5}{3},\frac{3}{7}}x\Big(\frac{4}{3}\Big)+\frac{\pi}{3}\,I_{\frac{\sqrt{2}}{5}}^{\frac{2}{9},\frac{\sqrt{3}}{8}}x\Big(\frac{3}{2}\Big)+\frac{\sqrt{\pi}}{6}\,I_{\frac{e}{3}}^{\frac{\sqrt{e}}{2},\frac{e^2}{4}}x\Big(\frac{2}{7}\Big)$$

that satisfies

$$|x(t)|\le\frac{5K_2^0a}{1-(5a+\frac{25a^2}{2})K_2^0}\ \ \forall\,t\in[0,5].$$

Remark 3.11. If $F(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ does not depend on the last variable and $y(\,\cdot\,)=0$, similar results to the one in Theorem 3.9 can be found in [1], namely, Theorem 3.1 and Theorem 4.2. Even if our hypothesis concerning the set-valued map is weaker than in [1] (in Theorem 3.1 of [1] it is assumed that F has the approximate end point property and in Theorem 4.2 of [1] it is assumed that F is a generalized contraction), our approach does not require for the values of F to be compact as in [1] and also provides a priori bounds for solutions.

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Author's addresses:

1. Faculty of Mathematics and Computer Science, University of Bucharest, Academiei 14, 010014 Bucharest, Romania.

2. Academy of Romanian Scientists, Splaiul Independenţei 54, 050094 Bucharest, Romania.

E-mail: `acernea@fmi.unibuc.ro`