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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

Abstract. The existence conditions and asymptotic representations as $t \uparrow \omega$ ($\omega \leq +\infty$) of one class of monotonous solutions of the n-th order differential equations containing on the right-hand side a sum of terms with regularly varying nonlinearities are established.

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1 Introduction

In the recent decades asymptotic properties of solutions of binomial essentially nonlinear second-order differential equations with a nonlinearity which differs from a power function have been actively studied (for the Emden–Fowler type not generalized equations see the monograph by I. T. Kiguradze and T. A. Chanturiya [13]). The case where the nonlinearity is a regularly varying function was investigated in [9, 12, 15, 16, 18], and the case where the nonlinearity is a rapidly varying function can be found in [1, 3–5, 8]. It should be noted here that the second-order equations containing in the right-hand side a sum of terms with nonlinearities that differ from power functions were considered only in the case when all nonlinearities are regularly varying functions (see, e.g., [6, 7]). In this paper, we study the asymptotic properties of solutions of a second-order differential equation in the right-hand side of which, apart from the terms with regularly varying nonlinearities, there are also terms with rapidly varying nonlinearities.

Consider the differential equation

$$y'' = \sum_{i=1}^{m} \alpha_i p_i(t) \varphi_i(y), \tag{1.1}$$

where $\alpha_i \in \{-1, 1\}$ $(i = \overline{1,m})$, $p_i : [a, \omega[\to]0, +\infty[$ $(i = \overline{1,m})$ are continuous functions, $-\infty < a < \omega \le +\infty$; $\varphi_i : \Delta_{Y_0} \to]0, +\infty[$ $(i = \overline{1,m})$, where Δ_{Y_0} is a one-sided neighborhood of the point Y_0, Y_0 is equal either to 0 or to $\pm\infty$, are continuous functions for $i = \overline{1,l}$ and twice continuously differentiable for $i = \overline{l+1,m}$, such that for each $i \in \{1, \dots, l\}$ as some $\sigma_i \in \mathbb{R}$

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_i(\lambda y)}{\varphi_i(y)} = \lambda^{\sigma_i} \ \text{ for each } \ \lambda > 0, \tag{1.2}$$

and for each $i \in \{l+1, \dots, m\}$,

$$\varphi_i'(y) \neq 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_i(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_i''(y)\varphi_i(y)}{\varphi_i'^2(y)} = 1. \tag{1.3}$$

The functions φ_i $(i = \overline{1,l})$ that satisfy conditions (1.2) are called regularly varying functions as $y \to Y_0$ of orders σ_i $(i = \overline{1,l})$ (see the monograph by E. Seneta [17, Ch. 1, § 1, pp. 9–10]). For each of them the representations of the form

$$\varphi_i(y) = |y|^{\sigma_i} L_i(y) \ \ (i = \overline{1,l}) \tag{1.4}$$

hold, where L_i are the slowly varying functions as $y \to Y_0$, i.e., such that

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{L_i(\lambda y)}{L_i(y)} = 1 \ \ (i = \overline{1,l}) \ \text{ for each } \ \lambda > 0.$$

We also say that a function L_i $(i \in \{1, \dots, l\})$ satisfies the condition S_0 if

$$L_i(\nu e^{[1+o(1)] \ln |y|}) = L_i(y)[1 + o(1)] \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}),$$

where $\nu = \operatorname{sign} y$.

Examples of functions slowly varying as $y \to Y_0$ are as follows:

$$|\ln|y||^{\gamma_1}, \ \ |\ln|y||^{\gamma_1} \big|\ln|\ln|y||\big|^{\gamma_2} \ \ (\gamma_1, \gamma_2 \neq 0), \ \ e^{\sqrt{|\ln|y||}}.$$

The first two functions satisfy the condition S_0.

From conditions (1.3) it immediately follows that

$$\lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y\varphi_i'(y)}{\varphi_i(y)} = \pm\infty \ \ (i = \overline{l+1,m}),$$

due to which each of the functions φ_i for $i \in \{l+1, \dots, m\}$ and its first derivative are rapidly varying as $y \to Y_0$ (see the monograph by M. Maric [14, Ch. 3, § 3.4, Lemmas 3.2, 3.3, pp. 91–92]).

Definition 1.1. A solution y of the differential equation (1.1) is called a $P_\omega(Y_0,\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on some interval $[t_0,\omega[\subset [a,\omega[$ and satisfies the following conditions:

$$\lim_{t\uparrow\omega} y(t)=Y_0,\qquad \lim_{t\uparrow\omega} y'(t)=\begin{cases}\text{either} & 0,\\ \text{or} & \pm\infty,\end{cases}\qquad \lim_{t\uparrow\omega}\frac{y'^2(t)}{y''(t)y(t)}=\lambda_0. \tag{1.5}$$

In [10], $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1) were studied in the case $\lambda_0\in\mathbb{R}\setminus\{0;1\}$.

In this paper, for $\lambda_0=\pm\infty$, we establish the conditions for the existence of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1.1) and give asymptotic representations, as $t\uparrow\omega$, of such solutions and their first-order derivatives when in each of such solutions the right-hand side of equation is equivalent, as $t\uparrow\omega$, to the s-th item, i.e., when for some $s\in\{1,\dots,l\}$,

$$\lim_{t\uparrow\omega}\frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))}=0\ \text{ for all }\ i\in\{1,\dots,m\}\setminus\{s\}. \tag{1.6}$$

Upon studying the $P_\omega(Y_0,\pm\infty)$-solutions of equation (1.1), some of their a priori asymptotic properties will be used.

We set

$$\pi_\omega(t)=\begin{cases} t & \text{if }\ \omega=+\infty,\\ t-\omega & \text{if }\ \omega<+\infty.\end{cases}$$

Lemma 1.1. *Let $y:[t_0,\omega[\to\mathbb{R}$ be an arbitrary $P_\omega(Y_0,\pm\infty)$-solution of equation* (1.1)*. Then*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y'(t)}{y(t)}=1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}=0. \tag{1.7}$$

The validity of this assertion follows directly from [2] (see Corollary 10.1).

2 Statement of the main results

Here and in the sequel, without loss of generality, we assume that

$$\Delta_{Y_0}=\Delta_{Y_0}(b),$$

where

$$\Delta_{Y_0}(b)=\begin{cases}[b,Y_0[\,, & \text{if }\ \Delta_{Y_0}\ \text{ is a left neighborhood of } Y_0,\\ ]Y_0,b], & \text{if }\ \Delta_{Y_0}\ \text{ is a right neighborhood of } Y_0,\end{cases}$$

and the number b satisfies the inequalities

$$|b|<1\ \text{ as }\ Y_0=0\ \text{ and }\ b>1\ \ (b<-1)\ \text{ as }\ Y_0=+\infty\ \ (Y_0=-\infty).$$

In addition, let us introduce two numbers

$$\nu_0=\operatorname{sign} b,\quad \nu_1=\begin{cases}1, & \text{if }\ \Delta_{Y_0}(b)=[b,Y_0[\,,\\ -1, & \text{if }\ \Delta_{Y_0}(b)=]Y_0,b].\end{cases}$$

According to the definition of the $P_\omega(Y_0,\lambda_0)$-solution of the differential equation (1.1), note that the numbers ν_0 and ν_1 determine the signs of any $P_\omega(Y_0,\lambda_0)$-solution and its first derivative (respectively) in some left neighborhood of ω. The conditions

$$\nu_0\nu_1=-1\ \text{ if }\ Y_0=0,\quad \nu_0\nu_1=1\ \text{ if }\ Y_0=\pm\infty$$

are necessary for the existence of $P_\omega(Y_0,\lambda_0)$-solutions.

Moreover, if for such solutions of (1.1) conditions (1.6) hold, then

$$y''(t)=\alpha_s p_s(t)\varphi_s(y(t))[1+o(1)]\ \text{ as }\ t\uparrow\omega, \tag{2.1}$$

from which it is clear that $\operatorname{sign} y''(t) = \alpha_s$ in some left neighborhood of ω, and in this case

$$\nu_1 \alpha_s = -1 \ \text{ if } \ \lim_{t\uparrow\omega} y'(t) = 0, \quad \nu_1 \alpha_s = 1 \ \text{ if } \ \lim_{t\uparrow\omega} y'(t) = \pm\infty.$$

In the case where $\nu_0 \lim\limits_{t\uparrow\omega} |\pi_\omega(t)| = Y_0$, we choose the number $a_1 \in [a, \omega[$ so that $\nu_0|\pi_\omega(t)| \in \Delta_{Y_0}(b)$ as $t \in [a_1, \omega[$, and for $s \in \{1, \dots, l\}$ set

$$J_s(t) = \int_{A_s}^{t} p_s(\tau)\varphi_s(\nu_0|\pi_\omega(\tau)|)\, d\tau,$$

where

$$A_s = \begin{cases} a_1 & \text{if } \displaystyle\int_{a_1}^{\omega} p_s(\tau)\varphi_s(\nu_0|\pi_\omega(\tau)|)\, d\tau = \pm\infty, \\ \omega & \text{if } \displaystyle\int_{a_1}^{\omega} p_s(\tau)\varphi_s(\nu_0|\pi_\omega(\tau)|)\, d\tau = const. \end{cases}$$

Theorem 2.1. *Let $\sigma_s \neq 1$ for some $s \in \{1, \dots, l\}$ and the function L_s satisfy the condition S_0. Then for the existence of $P_\omega(Y_0, \pm\infty)$-solutions satisfying condition* (1.6) *of the differential equation* (1.1) *it is necessary that*

$$\nu_0 \lim_{t\uparrow\omega} |\pi_\omega(t)| = Y_0, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) J_s'(t)}{J_s(t)} = 0, \tag{2.2}$$

the inequalities

$$\alpha_s \nu_1 (1 - \sigma_s) J_s(t) > 0, \quad \nu_0 \nu_1 \pi_\omega(t) > 0 \ \text{ for } \ t \in]a_1, \omega[\,, \tag{2.3}$$

as well as the conditions

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})} = 0 \tag{2.4}$$

for all $i \in \{1, \dots, l\} \setminus \{s\}$ and

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+\delta_i))}{p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})} = 0 \tag{2.5}$$

for all $i \in \{l+1, \dots, m\}$ hold, where δ_i are arbitrary numbers of some one-sided neighborhood of zero. Moreover, for each of such solutions the following asymptotic representations are valid:

$$y(t) = \nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1 + o(1)] \ \text{ as } \ t \uparrow \omega, \tag{2.6}$$

$$y'(t) = \nu_1|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1 + o(1)] \ \text{ as } \ t \uparrow \omega. \tag{2.7}$$

Proof. Let $y : [t_0, \omega[\to \mathbb{R}$ be an arbitrary $P_\omega(Y_0, \pm\infty)$-solution for some $s \in \{1, \dots, l\}$ satisfying conditions (1.6) of equation (1.1). Then by virtue of (1.1) and (1.6), the asymptotic relation (2.1) holds.

According to Lemma 1.1, the limit relations (1.7) are valid, from which, in particular, it follows that the function y is regularly varying, as $t \uparrow \omega$, function of first order. Therefore, by virtue of the function L_s satisfying the condition S_0, representations (1.4) and the first of the limit relations (1.7), we have

$$\begin{aligned} \varphi_s(y(t)) = |y(t)|^{\sigma_s} L_s(y(t)) &= |y(t)|^{\sigma_s} L_s(\nu_0 e^{[1+o(1)]\ln|\pi_\omega(t)|}) \\ &= |\pi_\omega(t) y'(t)|^{\sigma_s} L_s(\nu_0|\pi_\omega(t)|)[1 + o(1)] \ \text{ as } \ t \uparrow \omega. \end{aligned}$$

Taking into account this asymptotic relation, from (2.1) we obtain

$$\frac{y''(t)}{|y'(t)|^{\sigma_s}} = \alpha_s p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|)[1 + o(1)] \ \text{ for } \ t \uparrow \omega. \tag{2.8}$$

Integrating (2.8) on the interval from t_1 ($t_1 \in [t_0, \omega[$) to t and using the second of conditions (1.5), we get

$$\nu_1 |y'(t)|^{1-\sigma_s} = \alpha_s (1-\sigma_s) J_s(t)[1+o(1)] \text{ as } t \uparrow \omega,$$

which implies representation (2.7) and the equality

$$\nu_1 = \alpha_s \operatorname{sign}[(1-\sigma_s) J_s(t)]. \tag{2.9}$$

From the first relation of (1.7) follows the second of inequalities (2.3), so taking into account (2.9), the first of inequalities (2.3) holds. Taking into account the first of limiting relations (1.7), the second inequality of (2.3) and (2.7), we obtain the asymptotic representation (2.6). The validity of the first limit relation of (2.2) follows from Definition 1.1 and the first equality of (1.7) of Lemma 1.1. The second limit relation of (2.2) follows immediately from (2.8) if we use the above-mentioned representation (2.7) and the second of conditions (1.7).

Since the functions φ_i ($i = \overline{1, l}$) are regularly varying as $y \to Y_0$, we have

$$\varphi_i\big(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)}[1+o(1)]\big) = \varphi_i\big(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)}\big)[1+o(1)] \text{ as } t \uparrow \omega.$$

Then, by virtue of (2.6),

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} = \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)})[1+o(1)]}{p_s(t)\varphi_s(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)})[1+o(1)]}$$
$$= \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)})}{p_s(t)\varphi_s(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)})} \quad (i = \overline{1, l})$$

hence, taking into account (1.6), we find that conditions (2.4) are valid.

For $i \in \{l+1, \dots, m\}$, from (2.6) we have

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} = \lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)}[1+o(1)])}{p_s(t)\varphi_s(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)})}. \tag{2.10}$$

By the monotony of function φ_i ($i \in \{l+1, \dots, m\}$) on the interval $\Delta_{Y_0}(b)$ for each of δ_i from some one-sided neighborhood of zero there exists $t_2 \in [t_1, \omega[$ such that for $t \in [t_2, \omega[$, we have

$$\frac{p_i(t)\varphi_i(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)}[1+o(1)])}{p_s(t)\varphi_s(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)})} \geq \frac{p_i(t)\varphi_i(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)}[1+\delta_i])}{p_s(t)\varphi_s(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)})} > 0.$$

Thus, by virtue of (1.6) and (2.10), we find that conditions (2.5) are valid. The proof of the theorem is complete. □

Now we clarify the question of the actual existence of $P_\omega(Y_0, \pm\infty)$-solutions with the asymptotic representations (2.6) and (2.7) for equation (1.1).

Theorem 2.2. *Let for some $s \in \{1, \dots, l\}$ the function L_s satisfy the condition S_0, the inequality $\sigma_s \neq 1$ and conditions (2.2)–(2.4) hold, and for any $i \in \{l+1, \dots, m\}$,*

$$\lim_{t\uparrow\omega} \frac{p_i(t)\varphi_i(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)}(1+u))}{p_s(t)\varphi_s(\nu_0 |\pi_\omega(t)|\, |(1-\sigma_s) J_s(t)|^{1/(1-\sigma_s)})} = 0 \tag{2.11}$$

uniformly with respect to $u \in [-\delta, \delta]$ for some $0 < \delta < 1$. Then the differential equation (1.1) has at least one $P_\omega(Y_0, \pm\infty)$-solution that admits asymptotic representations (2.6) and (2.7). Moreover, if $\omega = +\infty$ and $A_s = +\infty$, there exists a one-parameter family with such representations, and if $A_s = a_1$, there is a two-parameter family.

Proof. By virtue of conditions (2.2) and (2.3), the function

$$Y(t)=\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}$$

is a first-order function that varies regularly as $t\uparrow\omega$,

$$\lim_{t\uparrow\omega}Y(t)=Y_0$$

and there exists a number $t_0\in[a_1,\omega[$ such that

$$Y(t)[1+u]\in\Delta_{Y_0}(b)\ \text{ for }\ t\in[t_0,\omega[\ \text{ and }\ |u|\le\delta.$$

By virtue of the properties of slowly varying functions, taking into account the fact that the function L_s satisfies the condition S_0, we have

$$\varphi_s(Y(t)(1+u))=|Y(t)(1+u)|^{\sigma_s}L_s(\nu_0|\pi_\omega(t)|)[1+R(t,u)],$$

where the function R is such that

$$\lim_{t\uparrow\omega}R(t,u)=0\ \text{ uniformly with respect to }\ u\in[-\delta,\delta].$$

Now applying to equation (1.1) the transformation

$$\begin{aligned}y(t)&=\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1+u_1(t)],\\ y'(t)&=\nu_1|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}[1+u_2(t)],\end{aligned}\tag{2.12}$$

taking into account inequalities (2.3), we obtain a system of differential equations

$$\begin{cases}u_1'=h_1(t)[f_1(t,u_1)-u_1+u_2],\\ u_2'=h_2(t)\big[f_2(t,u_1)+\sigma_su_1-u_2+V(u_1)\big],\end{cases}\tag{2.13}$$

where

$$h_1(t)=\frac{1}{\pi_\omega(t)}\,,\quad h_2(t)=\frac{J_s'(t)}{(1-\sigma_s)J_s(t)}\,,$$

$$f_1(t,u_1)=-\frac{\pi_\omega(t)J_s'(t)}{(1-\sigma_s)J_s(t)}\,(1+u_1),$$

$$f_2(t,u_1)=(1+u_1)^{\sigma_s}R(t,u_1)+(1+u_1)^{\sigma_s}(1+R(t,u_1))R_1(t,u_1),$$

$$R_1(t,u_1)=\sum_{\substack{i=1\\ i\neq s}}^{m}\frac{\alpha_ip_i(t)\varphi_i(Y(t)(1+u_1))}{\alpha_sp_s(t)\varphi_s(Y(t)(1+u_1))}\,,\quad V(u_1)=(1+u_1)^{\sigma_s}-1-\sigma_su_1.$$

We consider system (2.13) on the set

$$\Omega=[t_0,\omega[\times D,\ \text{ where }\ D=\big\{(u_1,u_2):\ |u_i|\le\delta,\ i=1,2\big\}.$$

We show that the function R_1 is such that

$$\lim_{t\uparrow\omega}R_1(t,u_1)=0\ \text{ uniformly with respect to }\ u_1\in[-\delta,\delta].\tag{2.14}$$

Since the functions φ_i with $i\in\{1,\dots,l\}$ are regularly varying of orders σ_i as $y\to Y_0$, by virtue of (1.4), taking into account the properties of slowly varying functions, we have

$$\begin{aligned}\varphi_i(Y(t)(1+u_1))&=\varphi_i\big(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+u_1)\big)\\ &=|\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+u_1)|^{\sigma_i}L_i\big(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+u_1)\big)\\ &=|\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}(1+u_1)|^{\sigma_i}L_i\big(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)}\big)(1+r_i(t,u_1))\big)\\ &=\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})(1+u_1)^{\sigma_i}(1+r_i(t,u_1))\ \ (i=\overline{1,l})\end{aligned}$$

where the functions r_i are such that

$$\lim_{t\uparrow\omega} r_i(t,u_1)=0 \ \text{ uniformly with respect to } \ u_1\in[-\delta,\delta].$$

By virtue of the above conditions,

$$\lim_{t\uparrow\omega}\sum_{\substack{i=1\\ i\neq s}}^{l}\frac{\alpha_i p_i(t)\varphi_i(Y(t)(1+u_1))}{\alpha_s p_s(t)\varphi_s(Y(t)(1+u_1))}=0 \tag{2.15}$$

uniformly with respect to $u_1\in[-\delta,\delta]$, since due to (2.4),

$$\lim_{t\uparrow\omega}\sum_{\substack{i=1\\ i\neq s}}^{l}\frac{\alpha_i p_i(t)\varphi_i(Y(t)(1+u_1))}{\alpha_s p_s(t)\varphi_s(Y(t)(1+u_1))}$$

$$=\lim_{t\uparrow\omega}\sum_{\substack{i=1\\ i\neq s}}^{l}\frac{\alpha_i p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})(1+r_i(t,u_1))}{\alpha_s p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})(1+r_s(t,u_1))}$$

$$=\lim_{t\uparrow\omega}\sum_{\substack{i=1\\ i\neq s}}^{l}\frac{\alpha_i p_i(t)\varphi_i(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})}{\alpha_s p_s(t)\varphi_s(\nu_0|\pi_\omega(t)|\,|(1-\sigma_s)J_s(t)|^{1/(1-\sigma_s)})}=0 \ \text{ uniformly with respect to } \ u_1\in[-\delta,\delta].$$

From (2.11) and (2.15), by virtue of the form of function R_1, we find that (2.14) is valid. In the system of equations (2.13) the functions h_1, $h_2:[t_0,\omega[\to\mathbb{R}$ are continuous and are such that

$$h_1(t)h_2(t)\neq 0 \ \text{ for } \ t\in[t_0,\omega[\,,$$

$$\int_{t_0}^{\omega} h_2(\tau)\,d\tau=\frac{1}{1-\sigma_s}\int_{t_0}^{\omega}\frac{J_s'(\tau)}{J_s(\tau)}\,d\tau=\frac{1}{1-\sigma_s}\ \ln|J_s(\tau)|\Bigg|_{t_0}^{\omega}=\pm\infty.$$

In addition, by virtue of the second of conditions (2.2), we have

$$\lim_{t\uparrow\omega}\frac{h_2(t)}{h_1(t)}=\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_s'(t)}{(1-\sigma_s)J_s(t)}=0.$$

Further, by the form of the functions V, f_k $(k=1,2)$, we have

$$\frac{h_1(t)}{h_2(t)}\,f_1(t,u_1) \ \text{ is bounded on the set } \Omega,$$

$$\lim_{u_1\to 0}\frac{dV(u_1)}{du_1}=0,$$

$$\lim_{t\uparrow\omega} f_2(t,u_1)=0 \ \text{ uniformly with respect to } \ u_1\in[-\delta,\delta].$$

Coefficient at u_1 in square brackets of the first equation of system (2.13) is nonzero. In addition, the sum of the coefficients of u_1 and u_2 in the square brackets of the first equation of system (2.13) is zero, and in the second equation is equal to the number σ_s-1, which is nonzero. This implies that system (2.13) satisfies all the assumptions of Theorem 2.7 of [11]. According to this theorem, the system of differential equations (2.13) has at least one solution $u=(u_1,u_2):[t_*,\omega[\to\mathbb{R}^2$ $(t_*\geq t_0)$, tending to zero as $t\uparrow\omega$. Each solution of this kind of system (2.13), by virtue of transformations (2.12), corresponds to the solution of the differential equation (1.1) that admits, as $t\uparrow\omega$, asymptotic representations (2.6), (2.7), and this solution is the $P_\omega(Y_0,\pm\infty)$-solution of equation (1.1). Moreover, if $\omega=+\infty$, then there exists a one-parameter family of such solutions if $\frac{J_s'(t)}{J_s(t)}<0$ on $]a_1,+\infty[$ (this inequality holds when J_s is chosen for the integration limit of A_s to be equal to $+\infty$), and a two-parameter family if the inequality $\frac{J_s'(t)}{J_s(t)}>0$ holds (i.e., when $A_s=a_1$). The proof of the theorem is complete. □

Remark. In the case when there are no terms in equation (1.1) with rapidly varying nonlinearity, i.e., when $m = l$, the assertion of Theorems 2.1 and 2.2 remains true without conditions (2.5) and (2.11).

3 Example

As an example illustrating the results obtained in this paper, we consider a differential equation of the form

$$y'' = \alpha_1 p_1(t)|y|^\sigma + \alpha_2 p_2(t) e^{\mu y}, \tag{3.1}$$

in which $\alpha_i \in \{-1, 1\}$ $(i = 1, 2)$, $p_i : [a, \omega[\to]0, +\infty[$ $(i = 1, 2)$ are continuous functions, $-\infty < a < \omega \le +\infty$, $\mu \ne 0$.

For equation (3.1) let us clarify the existence of $P_\omega(Y_0, \pm\infty)$-solutions for which

$$\lim_{t\uparrow\omega} y(t) = \pm\infty \ \ (Y_0 = \pm\infty), \quad \lim_{t\uparrow\omega} \frac{p_2(t) e^{\mu y(t)}}{p_1(t)|y(t)|^\sigma} = 0. \tag{3.2}$$

From Theorems 2.1 and 2.2 we have

Corollary 3.1. *Suppose that inequality $\sigma \ne 1$ holds. Then for the existence of $P_\omega(Y_0, \pm\infty)$-solutions of the differential equation* (3.1) *satisfying conditions* (3.2) *it is necessary, and if*

$$p_2(t) = o\Big(\frac{p_1(t) t^\sigma |(1-\sigma) J_1(t)|^{\frac{\sigma}{1-\sigma}}}{e^{\mu \nu_0 t |(1-\sigma) J_1(t)(1+u)|^{\frac{1}{1-\sigma}}}} \Big) \ \ as \ \ t \to +\infty$$

uniformly with respect to $u \in [-\delta, \delta]$ for some $0 < \delta < 1$, it is sufficient that the conditions

$$\omega = +\infty, \quad \lim_{t\to+\infty} \frac{t J_1'(t)}{J_1(t)} = 0,$$
$$\nu_0 \nu_1 > 0, \quad \alpha_1 \nu_1 (1-\sigma) J_1(t) > 0 \ \ for \ \ t \in]a_1, +\infty[$$

hold. Moreover, each solution of that kind admits the asymptotic representations

$$y(t) = \nu_0 t |(1-\sigma) J_1(t)|^{\frac{1}{1-\sigma}} [1 + o(1)] \ \ as \ \ t \to +\infty,$$
$$y'(t) = \nu_1 |(1-\sigma) J_1(t)|^{\frac{1}{1-\sigma}} [1 + o(1)] \ \ as \ \ t \to +\infty.$$

Moreover, if $A_s = +\infty$, there exists a one-parameter family with such representations, and in case $A_s = a_1$, there is a two-parameter family.

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