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VARIATION FORMULAS OF SOLUTIONS FOR CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATIONS WITH THE CONTINUOUS INITIAL CONDITION WITH REGARD FOR PERTURBATIONS OF THE INITIAL MOMENT AND SEVERAL DELAYS

Abstract. Variation formulas of solutions for nonlinear controlled functional differential equations are proved which show the effect of perturbations of the initial moment, constant delays and also that of the continuous initial condition.

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1 Introduction and formulation of main results

The term "variation formula of a solution" has been introduced by R. V. Gamkrelidze and proved in [2] for the ordinary differential equation. The effects of perturbation of the initial moment and the discontinuous initial condition in the variation formulas of solutions (shortly, variation formulas) were revealed by T. A. Tadumadze in [4] for the delay differential equation.

In the present paper, for the controlled functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t)\big)$$

with the continuous initial condition

$$x(t) = \varphi(t), \;\; t \le t_0,$$

the variation formulas are proved in the framework of new wide classes of variations of the initial data. The continuity of the initial condition means that the values of the initial function and the trajectory always coincide at the initial moment, i.e., $x(t_0) = \varphi(t_0)$. In [5,9], the variation formulas were proved for the equations

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau)\big), \;\; t \in [t_0, t_1],$$
$$\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t)\big), \;\; t \in [t_0, t_1],$$

respectively, in the case where the initial moment and delay variations had the same signs. In this paper, the essential novelty is that here we consider the equation with several delays, the variation formulas are proved for the controlled functional differential equations with several delays and the variations of the initial moment and delays are, in general, of different signs.

The variation formula plays the basic role in proving of the necessary conditions of optimality [2,3]. The variation formulas for various classes of controlled functional differential equations without perturbation of delays are derived in [1,3,7,8].

Let $I = [a, b]$ be a finite interval and $0 < \theta_{i1} < \theta_{i2}$, $i = 1, \dots, s$, be the given numbers; suppose that $O \subset \mathbb{R}^n$ and $U_0 \subset \mathbb{R}^r$ are the open sets. Let the n-dimensional function $f(t, x, x_1, \dots, x_s, u)$ satisfy the following conditions: for almost all fixed $t \in I$, the function $f(t, \cdot\,) : O^{1+s} \times U_0 \to \mathbb{R}^n$ is continuously differentiable; for each fixed $(x, x_1, \dots, x_s, u) \in O^{1+s} \times U_0$, the functions $f(t, x, x_1, \dots, x_s, u)$, $f_x(t, \cdot\,)$, $f_{x_i}(t, \cdot\,)$, $i = 1, \dots, s$, and $f_u(t, \cdot\,)$ are measurable on I; for arbitrary compact sets $K \subset O$, $U \subset U_0$, there exists a function $m_{K,U}(t) \in L_1(I, \mathbb{R}_+)$, $\mathbb{R}_+ = [0, \infty)$ such that

$$|f(t, x, x_1, \dots, x_s, u)| + |f_x(t, \cdot\,)| + \sum_{i=1}^{s} |f_{x_i}(t, \cdot\,)| + |f_u(t, \cdot\,)| \le m_{K,U}(t)$$

for all $(x, x_1, \dots, x_s, u) \in K^{1+s} \times U$ and for almost all $t \in I$.

Let Φ be a set of continuous functions $\varphi : I_1 = [\widehat{\tau}, b] \to O$, where $\widehat{\tau} = a - \max\{\theta_{12}, \dots, \theta_{s2}\}$ and let Ω be a set of measurable functions $u(t)$, $t \in I$, satisfying the condition $\mathrm{clu}(I) \subset U_0$ and be compact in $\mathbb{R}^r$.

To each element $\mu = (t_0, \tau_1, \dots, \tau_s, \varphi, u) \in \Lambda = [a, b) \times [\theta_{11}, \theta_{12}] \times \dots \times [\theta_{s1}, \theta_{s2}] \times \Phi \times \Omega$ we assign the delay controlled functional differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau_1), \dots, x(t-\tau_s), u(t)\big) \tag{1.1}$$

with the continuous initial condition

$$x(t) = \varphi(t), \;\; t \in [\widehat{\tau}, t_0]. \tag{1.2}$$

Definition 1.1. Let $\mu = (t_0, \tau_1, \dots, \tau_s, \varphi, u) \in \Lambda$. A function $x(t) = x(t; \mu) \in O$, $t \in [\widehat{\tau}, t_1]$, $t_1 \in (t_0, b]$, is called a solution of equation (1.1) with the initial condition (1.2) or a solution corresponding to the element μ and defined on the interval $[\widehat{\tau}, t_1]$ if it satisfies condition (1.2) and is absolutely continuous on the interval $[t_0, t_1]$, and satisfies equation (1.1) almost everywhere (a.e.) on $[t_0, t_1]$.

Let us introduce a set of variations:

$$V=\Big\{\delta\mu=(\delta t_0,\delta\tau_1,\dots,\delta\tau_s,\delta\varphi,\delta u):\ \ |\delta t_0|\le\alpha,|\delta\tau_i|\le\alpha,\ \ i=1,\dots,s,\\ \delta\varphi=\sum_{i=1}^{k}\lambda_i\delta\varphi_i,\ \ |\lambda_i|\le\alpha,\ \ \|\delta u\|\le\alpha,\ \ i=1,\dots,k\Big\},\tag{1.3}$$

where $\delta\varphi_i\in\Phi-\varphi_0$, $i=1,\dots,k$, and $\varphi_0\in\Phi$ are fixed functions; $\alpha>0$ is a fixed number and $\|\delta u\|=\sup\{|\delta u(t)|:\ t\in I\}$.

Let $x_0(t)$ be a solution corresponding to the element $\mu_0=(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,u_0)\in\Lambda$ and defined on the interval $[\widehat{\tau},t_{10}]$, where $t_{00},t_{10}\in(a,b)$, $t_{00}<t_{10}$ and $\tau_{i0}\in(\theta_{i1},\theta_{i2})$, $i=1,\dots,s$.

There exist the numbers $\delta_1>0$ and $\varepsilon_1>0$ such that for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times V$ we have $\mu_0+\varepsilon\delta\mu\in\Lambda$ and a solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$ corresponds to it (see Lemma 2.2).

Due to the uniqueness, the solution $x(t;\mu_0)$ is a continuation of the solution $x_0(t)$ on the interval $[\widehat{\tau},t_{10}+\delta_1]$. Therefore, in the sequel, the solution $x_0(t)$ is assumed to be defined on the interval $[\widehat{\tau},t_{10}+\delta_1]$.

Let us define the increment of the solution $x_0(t)=x(t;\mu_0)$:

$$\Delta x(t)=\Delta x(t;\varepsilon\delta\mu)=x(t;\mu_0+\varepsilon\delta\mu)-x_0(t),\ \ (t,\varepsilon,\delta\mu)\in[\widehat{\tau},t_{10}+\delta_1]\times(0,\varepsilon_1)\times V.\tag{1.4}$$

Theorem 1.1. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous. Let the functions $\dot\varphi_0(t)$ and $f(w,u),(w,u)\in I\times O^{1+s}\times U_0$, be bounded, where $w=(t,x,x_1,\dots,x_s)$. Moreover, there exist the finite limits*

$$\lim_{t\to t_{00}-}\dot\varphi_0(t)=\dot\varphi_0^-,\qquad \lim_{w\to w_0}f(w,u_0(t))=f^-,\ \ w\in(a,t_{00}]\times O^{1+s},$$

where $w_0=(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0}))$. Then there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^-$, we have

$$\Delta x(t;\varepsilon\delta\mu)=\varepsilon\delta x(t;\delta\mu)+o(t;\varepsilon\delta\mu),\tag{1.5}$$ [1]

where $V^-=\{\delta\mu\in V:\delta t_0\le 0\}$ and

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\dot\varphi_0^- - f^-)\delta t_0+\beta(t;\delta\mu),\tag{1.6}$$

$$\beta(t;\delta\mu)=Y(t_{00};t)\delta\varphi(t_{00})+\sum_{i=1}^{s}\int_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi\\ -\int_{t_{00}}^{t}Y(\xi;t)\Big[\sum_{i=1}^{s}f_{x_i}[\xi]\dot x_0(\xi-\tau_{i0})\delta\tau_i\Big]\,d\xi+\int_{t_{00}}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi,\tag{1.7}$$

where $Y(\xi;t)$ is the $n\times n$-matrix function satisfying the equation

$$Y_\xi(\xi;t)=-Y(\xi;t)f_x[\xi]-\sum_{i=1}^{s}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}],\ \ \xi\in[t_{00},t],\tag{1.8}$$

and the condition

$$Y(\xi;t)=\begin{cases}\Upsilon & \text{for } \xi=t,\\ \Theta & \text{for } \xi>t.\end{cases}\tag{1.9}$$

Here,

$$f_{x_i}=\frac{\partial}{\partial x_i}f,\quad f_{x_i}[\xi]=f_{x_i}\big(\xi,x_0(\xi),x_0(\xi-\tau_{10}),\dots,x_0(\xi-\tau_{s0}),u_0(\xi)\big),$$

Υ is the identity matrix and Θ is the zero matrix.

[1] Here and throughout the paper, the symbols $O(t;\varepsilon\delta\mu),o(t;\varepsilon\delta\mu)$ stand for quantities (scalar or vector) having the corresponding order of smallness with respect to ε uniformly with respect to $(t,\delta\mu)$.

Some comments. The function $\delta x(t;\delta\mu)$ is called the first variation of the solution $x_0(t)$, $t\in[t_{00},t_{10}+\delta_2]$, and expression (1.6) is called the variation formula. On the basis of the Cauchy formula for solutions of the linear delay functional differential equation, we conclude that the function

$$\delta x(t)=\begin{cases}\delta\varphi(t), & t\in[\widehat{\tau},t_{00}),\\ \delta x(t;\delta\mu), & t\in[t_{00},t_{10}+\delta_2],\end{cases}$$

is a solution of the equation

$$\dot{\delta x}(t)=f_x[t]\delta x(t)+\sum_{i=1}^{s}f_{x_i}[t]\delta x(t-\tau_{i0})-\sum_{i=1}^{s}f_{x_i}[t]\dot{x}_0(t-\tau_{i0})\delta\tau_i+f_u[t]\delta u(t)$$

with the initial condition

$$\delta x(t)=\delta\varphi(t),\ \ t\in[\widehat{\tau},t_{00}),\ \ \delta x(t_{00})=(\dot{\varphi}_0^- - f^-)\delta t_0+\delta\varphi(t_{00}).$$

The addend $-\int\limits_{t_{00}}^{t}Y(\xi;t)\big[\sum\limits_{i=1}^{s}f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\delta\tau_i\big]\,d\xi$ in formula (1.7) is the effect of perturbations of the delays τ_{i0}, $i=1,\dots,s$.

The expression $Y(t_{00};t)(\dot{\varphi}_0^- - f^-)\delta t_0$ is the effect of the continuous initial condition (1.2) and of the perturbation of the initial moment t_{00}.

The expression $Y(t_{00};t)\delta\varphi(t_{00})+\sum\limits_{i=1}^{s}\int\limits_{t_{00}-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi$ in formula (1.6) is the effect of perturbation of the initial function $\varphi_0(t)$.

The expression $\int\limits_{t_{00}}^{t}Y(\xi;t)\delta u[\xi]\,d\xi$ in formula (1.7) is the effect of perturbation of the control function $u_0(t)$.

Theorem 1.2. *Let the function $\varphi_0(t)$, $t\in I_1$, be absolutely continuous. Let the functions $\dot{\varphi}_0(t)$ and $f(w,u),(w,u)\in I\times O^{1+s}\times U_0$, be bounded. Moreover, there exist the finite limits*

$$\lim_{t\to t_{00}+}\dot{\varphi}_0(t)=\dot{\varphi}_0^+,\quad \lim_{w\to w_0}f(w)=f^+,\ \ w\in[t_{00},b)\times O^{1+s}.$$

Then for each $\widehat{t}_0\in(t_{00},t_{10})$, there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat{t}_0,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^+$, where $V^+=\{\delta\mu\in V:\ \delta t_0\geq 0\}$, formula (1.5) *holds, where*

$$\delta x(t;\delta\mu)=Y(t_{00};t)(\dot{\varphi}_0^+ - f^+)\delta t_0+\beta(t;\delta\mu). \tag{1.10}$$

The following assertion is a corollary to Theorems 1.1 and 1.2.

Theorem 1.3. *Let the assumptions of Theorems* 1.1 *and* 1.2 *be fulfilled. Moreover, $\dot{\varphi}_0^- - f^- = \dot{\varphi}_0^+ - f^+ := \widehat{f}$. Then for each $\widehat{t}_0\in(t_{00},t_{10})$, there exist the numbers $\varepsilon_2\in(0,\varepsilon_1)$ and $\delta_2\in(0,\delta_1)$ such that for arbitrary $(t,\varepsilon,\delta\mu)\in[\widehat{t}_0,t_{10}+\delta_2]\times(0,\varepsilon_2)\times V$ formula* (1.5) *holds, where $\delta x(t;\delta\mu)=Y(t_{00};t)\widehat{f}\delta t_0+\beta(t;\delta\mu)$.*

All assumptions of Theorem 1.3 are satisfied if the function $f(t,x,x_1,\dots,x_s,u)$ is continuous and bounded, the function $\varphi_0(t)$ is continuously differentiable and the function $u_0(t)$ is continuous at the point t_{00}. Clearly, in this case,

$$\widehat{f}=\dot{\varphi}_0(t_{00})-f\big(t_{00},\varphi_0(t_{00}),\varphi_0(t_{00}-\tau_{10}),\dots,\varphi_0(t_{00}-\tau_{s0}),u_0(t_{00})\big).$$

2 Auxiliary assertions

To each element $\mu=(t_0,\tau_1,\dots,\tau_s,\varphi,u)\in\Lambda$ we assign the controlled functional differential equation

$$\dot{y}(t)=f(t_0,\tau_1,\dots,\tau_s,\varphi,y,u)(t) \tag{2.1}$$

with the initial condition

$$y(t_0) = \varphi(t_0), \tag{2.2}$$

where

$$f(t_0,\tau_1,\dots,\tau_s,\varphi,y,u)(t) = f\big(t,y(t),h(t_0,\varphi,y)(t-\tau_1),\dots,h(t_0,\varphi,y)(t-\tau_s),u(t)\big)$$

and $h(t_0,\varphi,y)(t)$ is the operator given by the formula

$$h(t_0,\varphi,y)(t) = \begin{cases} \varphi(t), & t\in[\widehat{\tau},t_0),\\ y(t), & t\in[t_0,b]. \end{cases} \tag{2.3}$$

Definition 2.1. Let $\mu=(t_0,\tau_1,\dots,\tau_s,\varphi,u)\in\Lambda$. An absolutely continuous function $y(t)=y(t;\mu)\in O$, $t\in[r_1,r_2]\subset I$, is called a solution of equation (2.1) with the initial condition (2.2) or a solution corresponding to the element μ and defined on the interval $[r_1,r_2]$ if $t_0\in[r_1,r_2], y(t_0)=\varphi(t_0)$ and the function $y(t)$ satisfies equation (2.1) (a.e.) on $[r_1,r_2]$.

Remark 2.1. Let $y(t;\mu)$, $t\in[r_1,r_2]$, be a solution corresponding to the element $\mu=(t_0,\tau_1,\dots,\tau_s,\varphi,u)\in\Lambda$. Then the function

$$x(t;\mu) = h(t_0,\varphi,y(\,\cdot\,;\mu))(t),\ \ t\in[\widehat{\tau},r_2], \tag{2.4}$$

is the solution of equation (1.1) with the initial condition (1.2) (see Definition 1.1 and (2.3)).

Lemma 2.1. *Let $y_0(t)$ be a solution corresponding to the element $\mu_0=(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,u_0)\in\Lambda$ and defined on $[r_1,r_2]\subset(a,b)$; let $t_{00}\in[r_1,r_2)$, $\tau_{i0}\in(\theta_{i1},\theta_{i2})$, $i=1,\dots,s$, and let $K_1\subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1)\cup y_0([r_1,r_2])$. Then there exist the numbers $\varepsilon_1>0$ and $\delta_1>0$ such that, for any $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times V$, we have $\mu_0+\varepsilon\delta\mu\in\Lambda$. In addition, to this element there corresponds a solution $y(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[r_1-\delta_1,r_2+\delta_1]\subset I$. Moreover,*

$$\begin{cases} \varphi(t)=\varphi_0(t)+\varepsilon\delta\varphi(t)\in K_1, & t\in I_1,\\ y(t;\mu_0+\varepsilon\delta\mu)\in K_1, & t\in[r_1-\delta_1,r_2+\delta_1], \end{cases} \tag{2.5}$$

$$\lim_{\varepsilon\to0} y(t;\mu_0+\varepsilon\delta\mu)=y(t;\mu_0)\ \ \textit{uniformly for}\ \ (t,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times V.$$

This lemma is a result of Theorem 3.1 in [6].

Lemma 2.2. *Let $x_0(t)$ be a solution corresponding to the element $\mu_0=(t_{00},\tau_{10},\dots,\tau_{s0},\varphi_0,u_0)\in\Lambda$ and defined on $[\widehat{\tau},t_{10}]$ (see Definition 1.1), let $t_{00},t_{10}\in(a,b)$, $\tau_{i0}\in(\theta_{i1},\theta_{i2})$, $i=1,\dots,s$, and let $K_1\subset O$ be a compact set containing a neighborhood of the set $\varphi_0(I_1)\cup x_0([t_{00},t_{10}])$. Then there exist the numbers $\varepsilon_1>0$ and $\delta_1>0$ such that, for any $(\varepsilon,\delta\mu)\in(0,\varepsilon_1)\times V$, we have $\mu_0+\varepsilon\delta\mu\in\Lambda$. In addition, to this element there corresponds a solution $x(t;\mu_0+\varepsilon\delta\mu)$ defined on the interval $[\widehat{\tau},t_{10}+\delta_1]\subset I_1$. Moreover,*

$$x(t;\mu_0+\varepsilon\delta\mu)\in K_1,\ \ t\in[\widehat{\tau},t_{10}+\delta_1]. \tag{2.6}$$

It is easy to see that if in Lemma 2.1 one put $r_1=t_{00}$, $r_2=t_{10}$, then $x_0(t)=y_0(t)$, $t\in[t_{00},t_{10}]$, and $x(t;\mu_0+\varepsilon\delta\mu)=h(t_0,\varphi,y(\,\cdot\,;\mu_0+\varepsilon\delta\mu))(t)$, $(t,\varepsilon,\delta\mu)\in[\widehat{\tau},t_{10}+\delta_1]\times(0,\varepsilon_1)\times V$ (see (2.4)). Thus, Lemma 2.2 is a simple corollary of Lemma 2.1 (see (2.5)).

Remark 2.2. Due to the uniqueness, the solution $y(t;\mu_0)$ on the interval $[r_1-\delta_1,r_2+\delta_1]$ is a continuation of the solution $y_0(t)$. Therefore, we can assume that the solution $y_0(t)$ is defined on the interval $[r_1-\delta_1,r_2+\delta_1]$.

Lemma 2.1 allows one to define the increment of the solution $y_0(t)=y(t;\mu_0)$:

$$\Delta y(t)=\Delta y(t;\varepsilon\delta\mu)=y(t;\mu_0+\varepsilon\delta\mu)-y_0(t),\ \ (t,\varepsilon,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times(0,\varepsilon_1)\times V. \tag{2.7}$$

Obviously,

$$\lim_{\varepsilon\to0}\Delta y(t;\varepsilon\delta\mu)=0 \tag{2.8}$$

uniformly with respect to $(t,\delta\mu)\in[r_1-\delta_1,r_2+\delta_1]\times V$ (see Lemma 2.1).

Lemma 2.3. *Let the conditions of Theorem* 1.1 *hold. Then there exist the numbers* $\varepsilon_2 \in (0, \varepsilon_1)$ *and* $\delta_2 \in (0, \delta_1)$ *such that*

$$\max_{t \in [t_{00}, r_2+\delta_2]} |\Delta y(t)| \le O(\varepsilon\delta\mu) \tag{2.9}$$

for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2) \times V^-$. *Moreover,*

$$\Delta y(t_{00}) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot{\varphi}_0^- - f^-)\delta t_0\big] + o(\varepsilon\delta\mu). \tag{2.10}$$

Proof. Let $\varepsilon_2' \in (0, \varepsilon_1)$ be so small that for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2') \times V^-$ the inequalities

$$t_0 + \tau_i > t_{00}, \quad i = 1, \dots, s, \tag{2.11}$$

hold, where $t_0 = t_{00}+\varepsilon\delta t_0$, $\tau_i = \tau_{i0}+\varepsilon\delta\tau_i$. On the interval $[t_{00}, r_2+\delta_1]$, the function $\Delta y(t) = y(t)-y_0(t)$ satisfies the equation

$$\Delta\dot{y}(t) = a(t; \varepsilon\delta\mu), \tag{2.12}$$

where

$$\begin{aligned} a(t; \varepsilon\delta\mu) = f\big(t, y_0(t) + \Delta y(t), h(t_0, \varphi, y_0 + \Delta y)(t-\tau_1), \dots, h(t_0, \varphi, y_0 + \Delta y)(t-\tau_s), u(t)\big) \\ - f\big(t, y_0(t), h(t_{00}, \varphi_0, y_0)(t-\tau_{10}), \dots, h(t_{00}, \varphi_0, y_0)(t-\tau_{s0}), u_0(t)\big). \end{aligned} \tag{2.13}$$

We rewrite equation (2.12) in the integral form

$$\Delta y(t) = \Delta y(t_{00}) + \int_{t_{00}}^{t} a(\xi; \varepsilon\delta\mu)\, d\xi.$$

Hence it follows that

$$|\Delta y(t)| \le |\Delta y(t_{00})| + a_1(t; t_{00}, \varepsilon\delta\mu), \tag{2.14}$$

where

$$a_1(t; t_{00}, \varepsilon\delta\mu) = \int_{t_{00}}^{t} |a(\xi; \varepsilon\delta\mu)|\, d\xi, \quad t \in [t_{00}, r_2 + \delta_1].$$

Let us prove formula (2.10). We have

$$\begin{aligned} \Delta y(t_{00}) &= y(t_{00}; \mu_0 + \varepsilon\delta\mu) - y_0(t_{00}) \\ &= \varphi_0(t_0) + \varepsilon\delta\varphi(t_0) + \int_{t_0}^{t_{00}} f\big(t, y_0(t) + \Delta y(t), \varphi(t-\tau_1), \dots, \varphi(t-\tau_s), u(t)\big)\, dt - \varphi_0(t_{00}) \end{aligned} \tag{2.15}$$

(see (2.11) and (2.3)). Since

$$\int_{t_{00}}^{t_0} \dot{\varphi}_0(t)\, dt = \varepsilon\dot{\varphi}_0^-\delta t_0 + o(\varepsilon\delta\mu),$$

$$\lim_{\varepsilon\to 0} \delta\varphi(t_0) = \delta\varphi(t_{00}) \ \text{ uniformly with respect to } \ \delta\mu \in V^-$$

(see (1.3)), we get

$$\begin{aligned} \varphi_0(t_0) + \varepsilon\delta\varphi(t_0) - \varphi_0(t_{00}) &= \int_{t_{00}}^{t_0} \dot{\varphi}_0(t)\, dt + \varepsilon\delta\varphi(t_{00}) + \varepsilon\big[\delta\varphi(t_0) - \delta\varphi(t_{00})\big] \\ &= \varepsilon\big[\dot{\varphi}^-\delta t_0 + \delta\varphi(t_{00})\big] + o(\varepsilon\delta\mu). \end{aligned} \tag{2.16}$$

It is clear that if $t \in [t_0, t_{00}]$, then

$$\lim_{\varepsilon\to 0}\big(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1),\dots,\varphi(t-\tau_s)\big) = \lim_{t\to t_{00}-}\big(t, y_0(t), \varphi_0(t-\tau_{10}),\dots,\varphi_0(t-\tau_{s0})\big) = w_0$$

(see (2.8)). Consequently,

$$\lim_{\varepsilon\to 0}\sup_{t\in[t_0,t_{00}]}\big|f(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1),\dots,\varphi(t-\tau_s), u(t)) - f^-\big| = 0.$$

This relation implies that

$$\begin{aligned}\int_{t_0}^{t_{00}} & f\big(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1),\dots,\varphi(t-\tau_s), u(t)\big)\,dt \\ &= -\varepsilon f^-\delta t_0 + \int_{t_0}^{t_{00}}\big[f(t, y_0(t)+\Delta y(t), \varphi(t-\tau_1),\dots,\varphi(t-\tau_s), u(t)) - f^-\big]\,dt \\ &= -\varepsilon f^-\delta t_0 + o(\varepsilon\delta\mu). \qquad (2.17)\end{aligned}$$

From (2.15), by virtue of (2.16) and (2.17), we obtain (2.10).

Now, let us prove inequality (2.9). First, we note that for any compact set $K_1 \subset O$ and $U_1 \subset U_0$, there exists a function $L_{K_1,U_1}(t) \in L_1(I, R_+)$ such that

$$\big|f(t,x,x_1,\dots,x_s,u_1) - f(t,y,y_1,\dots,y_s,u_2)\big| \le L_{K_1,U_1}(t)\Big(|x-y| + \sum_{i=1}^{s}|x_i-y_i| + |u_1-u_2|\Big)$$

for almost all $t \in I$ and for any $(x,y) \in K^2$, $(x_i,y_i) \in K^2$, $i = 1,\dots,s$, $u_1, u_2 \in U_1$.

Now, we estimate $a_1(t; t_{00}, \varepsilon\delta\mu)$, $t \in [t_{00}, r_2+\delta_1]$. Obviously,

$$a_1(t;t_{00},\varepsilon\delta\mu) \le \int_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\Delta y(\xi)|\,d\xi + \sum_{i=1}^{s} a_{2i}(t;t_{00},\varepsilon\delta\mu) + \varepsilon\int_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\delta u(\xi)|\,d\xi, \qquad (2.18)$$

where

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{K_1,U_1}(\xi)\big|h(t_0,\varphi,y_0+\Delta y)(\xi-\tau_i) - h(t_{00},\varphi_0,y_0)(\xi-\tau_{i0})\big|\,d\xi$$

(see (2.13)).

Evidently,

$$\varepsilon\int_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\delta u(\xi)|\,d\xi \le \varepsilon\alpha\int_{I} L_{K_1,U_1}(t)\,dt = O(\varepsilon).$$

Let $t_{00}+\tau_{i0} \le r_2$ and let ε_2' be so small that $t_{00}+\tau_i < r_2+\delta_1$. Furthermore, let $\rho_{i1} = \min\{t_0+\tau_i, t_{00}+\tau_{i0}\}$, $\rho_{i2} = \max\{t_{00}+\tau_i, t_{00}+\tau_{i0}\}$. It is easy to see that $\rho_{i2} \ge \rho_{i1} > t_{00}$ and $\rho_{i2}-\rho_{i1} = O(\varepsilon\delta\mu)$. Let $t \in [t_{00}, \rho_{i1})$. Then for $\xi \in [t_{00}, t]$, we have $\xi - \tau_i < t_0$ and $\xi - \tau_{i0} < t_{00}$. Therefore,

$$a_{2i}(t;t_{00},\varepsilon\delta\mu) = \int_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})|\,d\xi.$$

From the boundedness of the function $\dot\varphi_0(t)$, $t \in I_1$, it follows that

$$\begin{aligned}|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})| &= \big|\varphi_0(\xi-\tau_i) + \varepsilon\delta\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})\big| \\ &= O(\varepsilon\delta\mu) + \bigg|\int_{\xi-\tau_{i0}}^{\xi-\tau_i}\dot\varphi_0(t)\,dt\bigg| \le O(\varepsilon\delta\mu). \qquad (2.19)\end{aligned}$$

Thus, for $t \in [t_{00}, \rho_{i1}]$, we have

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu), \quad i = 1, \dots, s. \tag{2.20}$$

Let $t \in [\rho_{i1}, \rho_{i2}]$, then

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \le a_{2i}(\rho_{i1}; t_{00}, \varepsilon\delta\mu) + a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu).$$

Let $\rho_{i1} = t_0 + \tau_i$ and $\rho_{i2} = t_{00} + \tau_i$, i.e. $t_0 + \tau_i < t_{00} + \tau_{i0} < t_{00} + \tau_i$. We have

$$\begin{aligned}
a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) &\le \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi_0(\xi-\tau_{i0})\big|\,d\xi \\
&\quad + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - y_0(\xi-\tau_{i0})\big|\,d\xi \\
&\le \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi(\xi-\tau_i)\big|\,d\xi \\
&\quad + \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{K_1,U_1}(\xi)|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})|\,d\xi + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi(\xi-\tau_i)\big|\,d\xi \\
&\quad + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})|\,d\xi + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)|\varphi_0(\xi-\tau_{i0}) - y_0(\xi-\tau_{i0})|\,d\xi \\
&\le o(\varepsilon\delta\mu) + \int_{t_0+\tau_i}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi(\xi-\tau_i)\big|\,d\xi \\
&\quad + \int_{t_{00}+\tau_{i0}}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)|\varphi_0(\xi-\tau_{i0}) - y_0(\xi-\tau_{i0})|\,d\xi \\
&= o(\varepsilon\delta\mu) + \int_{t_0}^{t_{00}} L_{K_1,U_1}(\xi+\tau_i)|y(\xi; \mu_0+\varepsilon\delta\mu) - \varphi(\xi)|\,d\xi + \int_{t_{00}}^{t_{00}+\tau_i-\tau_{i0}} L_{K_1,U_1}(\xi+\tau_{i0})|\varphi_0(\xi) - y_0(\xi)|\,d\xi
\end{aligned}$$

(see (2.19)) with $t_{00} + \tau_i - \tau_{i0} > t_{00} + \tau_{i0} - \tau_{i0} = t_{00}$. The functions $f(w, u), (w, u) \in I \times O^{1+s} \times U_0$, and $\dot{\varphi}_0(t)$, $t \in I_1$, are bounded; therefore, we have

$$\begin{aligned}
&|y(\xi; \mu_0+\varepsilon\delta\mu) - \varphi(\xi)| \\
&\qquad = \left|\varphi(t_0) + \int_{t_0}^{\xi} f(t_0, \tau_1, \dots, \tau_s, \varphi, y_0+\Delta y, u)(t)\,dt - \varphi(\xi)\right| \le O(\varepsilon\delta\mu), \quad \xi \in [t_0, t_{00}],
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
|\varphi_0(\xi) - y_0(\xi)| = \left|\varphi_0(\xi) - \varphi_0(t_{00}) - \int_{t_{00}}^{\xi} f(t_{00}, \tau_{10}, \dots, \tau_{s0}, \varphi_0, y_0, u_0)(t)\,dt\right| \le O(\varepsilon\delta\mu), \\
\xi \in [t_{00}, t_{00}+\tau_i-\tau_{i0}].
\end{aligned}$$

Thus, $a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) = o(\varepsilon\delta\mu)$. Let $\rho_{i1} = t_0 + \tau_i$ and $\rho_{i2} = t_{00} + \tau_{i0}$, then

$$a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) = \int_{t_0+\tau_i}^{t_{00}+\tau_{i0}} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i; \mu_0+\varepsilon\delta\mu) - \varphi_0(\xi-\tau_{i0})\big|\,d\xi = o(\varepsilon\delta\mu).$$

Let $\rho_{i1} = t_{00} + \tau_{i0}$ and $\rho_{i2} = t_{00} + \tau_i$, i.e., $t_{00} + \tau_{i0} < t_0 + \tau_i < t_{00} + \tau_i$. We have

$$a_{2i}(\rho_{i2}; \rho_{i1}, \varepsilon\delta\mu) \le \int\limits_{t_{00}+\tau_{i0}}^{t_0+\tau_i} L_{K_1,U_1}(\xi)|\varphi(\xi-\tau_i) - y_0(\xi-\tau_{i0})|\, d\xi \\ + \int\limits_{t_0+\tau_i}^{t_{00}+\tau_i} L_{K_1,U_1}(\xi)\big|y(\xi-\tau_i;\mu_0+\varepsilon\delta\mu) - y_0(\xi-\tau_{i0})\big|\, d\xi = o(\varepsilon\delta\mu).$$

Consequently, for $t \in [t_{00}, \rho_{i2}]$, inequality (2.20) holds.

Let $t \in [\rho_{i2}, r_2 + \delta_1]$, then $t - \tau_i \ge t_0$ and $t - \tau_{i0} \ge t_{00}$. Therefore,

$$\begin{aligned} a_{2i}(t; t_{00}, \varepsilon\delta\mu) &= a_{2i}(\rho_{i2}; t_{00}, \varepsilon\delta\mu) + \int\limits_{\rho_{i2}}^{t} L_{K_1,U_1}(\xi)\big|y_0(\xi-\tau_i) + \Delta y(\xi-\tau_i) - y_0(\xi-\tau_{i0})\big|\, d\xi \\ &\le O(\varepsilon\delta\mu) + \int\limits_{\rho_{i2}-\tau_i}^{t-\tau_i} L_{K_1,U_1}(\xi+\tau_i)|\Delta y(\xi)|\, d\xi + \int\limits_{\rho_{i2}}^{t} L_{K_1,U_1}(\xi)|y_0(\xi-\tau_i) - y_0(\xi-\tau_{i0})|\, d\xi \\ &\le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \chi(\xi+\tau_i)L_{K_1,U_1}(\xi+\tau_i)|\Delta y(\xi)|\, d\xi + \int\limits_{\rho_{i2}}^{r_2+\delta_1} L_{K_1,U_1}(\xi)|y_0(\xi-\tau_i) - y_0(\xi-\tau_{i0})|\, d\xi, \end{aligned}$$

where $\chi(\xi)$ is the characteristic function of the interval I.

Further, for $\xi \in [\rho_{i2}, r_2 + \delta_1]$,

$$|y_0(\xi-\tau_i) - y_0(\xi-\tau_{i0})| \le \int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i} \big|f(t_{00}, \tau_{10}, \dots, \tau_{s0}, y_0, u_0)(t)\big|\, dt \le O(\varepsilon\delta\mu).$$

Thus, for $t \in [t_{00}, r_2 + \delta_1]$, we get

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \chi(\xi+\tau_i)L_{K_1,U_1}(\xi+\tau_i)|\Delta y(\xi)|\, d\xi. \tag{2.22}$$

We now consider the case where $t_{00} + \tau_{i0} > r_2$. Let $\delta_2 \in (0, \delta_1)$ and $\varepsilon_2'' \in (0, \varepsilon_1)$ be so small numbers that $t_{00} + \tau_{i0} > r_2 + \delta_2$ and $t_0 + \tau_i > r_2 + \delta_2$ for arbitrary $(\varepsilon, \delta\mu) \in (0, \varepsilon_2'') \times V^-$.

It is easy to see that

$$a_{2i}(t; t_{00}, \varepsilon\delta\mu) \le \int\limits_{t_{00}}^{t} L_{K_1,U_1}(\xi)|\varphi(\xi-\tau_i) - \varphi_0(\xi-\tau_{i0})|\, dt \le O(\varepsilon\delta\mu).$$

Thus, for arbitrary $(t, \varepsilon, \delta\mu) \in [t_{00}, r_2 + \delta_2] \times (0, \varepsilon_2) \times V^-$ and $i = 1, \dots, s$, where $\varepsilon_2 = \min(\varepsilon_2', \varepsilon_2'')$, inequality (2.22) holds.

Consequently, we have

$$a_1(t; t_{00}, \varepsilon\delta\mu) \le O(\varepsilon\delta\mu) \\ + \int\limits_{t_{00}}^{t} \Big[L_{K_1,U_1}(\xi) + \sum_{i=1}^{s} \chi(\xi+\tau_i)L_{K_1,U_1}(\xi+\tau_i)\Big]\, |\Delta y(\xi)|\, d\xi, \ \ t \in [t_{00}, r_2 + \delta_1] \tag{2.23}$$

(see (2.18)).

According to (2.10) and (2.23), inequality (2.14) directly implies

$$|\Delta y(t)| \le O(\varepsilon\delta\mu) + \int\limits_{t_{00}}^{t} \Big[L_{K_1,U_1}(\xi) + \sum_{i=1}^{s} \chi(\xi+\tau_i) L_{K_1,U_1}(\xi+\tau_i)\Big] |\Delta y(\xi)|\, d\xi, \ \ t \in [t_{00}, r_2+\delta_2]$$

from which, by the Gronwall lemma, we get (2.9). □

The following lemma, with a minor modification can be proved analogously to Lemma 2.3.

Lemma 2.4. *Let the conditions of Theorem 1.2 hold. Then there exist the numbers $\varepsilon_2 \in (0,\varepsilon_1)$ and $\delta_2 \in (0,\delta_1)$ such that $\max\limits_{t\in[t_0,r_2+\delta_2]} |\Delta y(t)| \le O(\varepsilon\delta\mu)$ for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_2)\times V^+$. Moreover,*

$$\Delta y(t_0) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot\varphi_0^+ - f^+)\delta t_0\big] + o(\varepsilon\delta\mu).$$

3 Proof of Theorem 1.1

Let $r_1 = t_{00}$ and $r_2 = t_{10}$ in Lemma 2.1, then

$$x_0(t) = \begin{cases} \varphi_0(t), & t \in [\widehat\tau, t_{00}),\\ y_0(t), & t \in [t_{00}, t_{10}], \end{cases}$$

and for arbitrary $(\varepsilon,\delta\mu) \in (0,\varepsilon_1)\times V^-$,

$$x(t;\mu_0+\varepsilon\delta\mu) = \begin{cases} \varphi(t) := \varphi_0(t) + \varepsilon\delta\varphi(t), & t \in [\widehat\tau, t_0),\\ y(t;\mu_0+\varepsilon\delta\mu), & t \in [t_0, t_{10}+\delta_1] \end{cases}$$

(see (2.4)).

We note that $\delta\mu \in V^-$, i.e., $t_0 < t_{00}$, therefore, we have

$$\Delta x(t) = \begin{cases} \varepsilon\delta\varphi(t) & \text{for } \ t \in [\widehat\tau, t_0),\\ y(t;\mu_0+\varepsilon\delta\mu) - \varphi_0(t) & \text{for } \ t \in [t_0, t_{00}),\\ \Delta y(t) & \text{for } \ t \in [t_{00}, t_{10}+\delta_1] \end{cases}$$

(see (1.4) and (2.7)). By Lemma 2.3 and the relation $|y(t;\mu_0+\varepsilon\delta\mu) - \varphi_0(t)| \le O(\varepsilon\delta\mu)$, $t \in [t_0, t_{00}]$, we have

$$|\Delta x(t)| \le O(\varepsilon\delta\mu) \ \ \forall\, (t,\varepsilon,\delta\mu) \in [\widehat\tau, t_{10}+\delta_2]\times(0,\varepsilon_2)\times V^-, \tag{3.1}$$

$$\Delta x(t_{00}) = \varepsilon\big[\delta\varphi(t_{00}) + (\dot\varphi_0^- - f^-)\delta t_0\big] + o(\varepsilon\delta\mu). \tag{3.2}$$

The function $\Delta x(t)$ satisfies the equation

$$\begin{aligned}\dot\Delta x(t) &= f\Big(t, x_0(t)+\Delta x(t), x_0(t-\tau_1)+\Delta x(t-\tau_1), \dots, x_0(t-\tau_s)+\Delta x(t-\tau_s), u(t)\Big) - f[t]\\ &= f_x[t]\Delta x(t) + \sum_{i=1}^{s} f_{x_i}[t]\Delta x(t-\tau_{i0}) + \varepsilon f_u[t]\delta u(t) + r(t;\varepsilon\delta\mu) \end{aligned}\tag{3.3}$$

on the interval $[t_{00}, t_{10}+\delta_2]$, where

$$\begin{aligned} r(t;\varepsilon\delta\mu) = f\Big(t, x_0(t)+\Delta x(t), x_0(t-\tau_1)+\Delta x(t-\tau_1), \dots, x_0(t-\tau_s)+\Delta x(t-\tau_s), u(t)\Big)\\ -f[t] - f_x[t]\Delta x(t) - \sum_{i=1}^{s} f_{x_i}[t]\Delta x(t-\tau_{i0}) - \varepsilon f_u[t]\delta u(t), \end{aligned}\tag{3.4}$$

$$f[t] = f\big(t, x_0(t), x_0(t-\tau_{10}), \dots, x_0(t-\tau_{s0}), u_0(t)\big),$$

By using the Cauchy formula, one can represent the solution of equation (3.3) in the form

$$\Delta x(t) = Y(t_{00};t)\Delta x(t_{00}) + \varepsilon \int\limits_{t_{00}}^{t} Y(\xi;t) f_u[t]\delta u(t)\,dt + \sum_{p=0}^{1} R_p(t;t_{00},\varepsilon\delta\mu), \quad t \in [t_{00}, t_{10}+\delta_2], \tag{3.5}$$

where

$$\begin{cases} R_0(t;t_{00},\varepsilon\delta\mu) = \displaystyle\sum_{i=1}^{s} R_{i0}(t;t_{00},\varepsilon\delta\mu), \\ R_{i0}(t;t_{00},\varepsilon\delta\mu) = \displaystyle\int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi, \\ R_1(t;t_{00},\varepsilon\delta\mu) = \displaystyle\int\limits_{t_{00}}^{t} Y(\xi;t) r(\xi;\varepsilon\delta\mu)\,d\xi \end{cases} \tag{3.6}$$

and $Y(\xi;t)$ is the matrix function satisfying equation (1.8) and condition (1.9). The function $Y(\xi;t)$ is continuous on the set $\Pi = \{(\xi,t) :\ t_{00}-\delta_2 \le \xi \le t,\ t \in [t_{00}, t_{10}+\delta_2]\}$ by Lemma 2.1.7 in [3, p. 22]. Therefore,

$$Y(t_{00};t)\Delta x(t_{00}) = \varepsilon Y(t_{00};t)\big[\delta\varphi(t_{00}) + (\dot\varphi_0^- - f^-)\delta t_0\big] + o(t;\varepsilon\delta\mu) \tag{3.7}$$

(see (3.2)), where $o(t;\varepsilon\delta\mu) = Y(t_{00};t)o(\varepsilon\delta\mu)$. One can readily see that

$$\begin{aligned} R_{i0}(t;t_{00},\varepsilon\delta\mu) &= \varepsilon \int\limits_{t_{00}-\tau_{i0}}^{t_0} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi + \int\limits_{t_0}^{t_{00}} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi \\ &= \varepsilon \int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi + o(t;\varepsilon\delta\mu) \end{aligned} \tag{3.8}$$

(see (3.1)). Thus,

$$R_0(t;t_{00},\varepsilon\delta\mu) = \varepsilon \sum_{i=1}^{s} \int\limits_{t_{00}-\tau_{i0}}^{t_{00}} Y(\xi+\tau_{i0};t) f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi + o(t;\varepsilon\delta\mu).$$

We introduce the notations:

$$\begin{aligned} f[t;\theta,\varepsilon\delta\mu] = f\Big(&t, x_0(t)+\theta\Delta x(t), x_0(t-\tau_{10}) + \theta\big(x_0(t-\tau_1) - x_0(t-\tau_{10}) + \Delta x(t-\tau_1)\big), \dots, \\ & x_0(t-\tau_{s0}) + \theta\big(x_0(t-\tau_s) - x_0(t-\tau_{s0}) + \Delta x(t-\tau_s)\big), u_0(t) + \theta\varepsilon\delta u(t)\Big), \end{aligned}$$

$\sigma(t;\theta,\varepsilon\delta\mu) = f_x[t;\theta,\varepsilon\delta\mu] - f_x[t]$, $\varrho_i(t;\theta,\varepsilon\delta\mu) = f_{x_i}[t;\theta,\varepsilon\delta\mu] - f_{x_i}[t]$,
$\vartheta(t;\theta,\varepsilon\delta\mu) = f_u[t;\theta,\varepsilon\delta\mu] - f_u[t]$.

It is easy to see that

$$\begin{aligned} f\Big(t, x_0(t)+\Delta x(t), x_0(t-\tau_1)+\Delta x(t-\tau_1), \dots, x_0(t-\tau_s)+\Delta x(t-\tau_s), u_0(t)+\varepsilon\delta u(t)\Big) - f[t] \\ = \int\limits_0^1 \frac{d}{d\theta} f[t;\theta,\varepsilon\delta\mu]\,d\theta \end{aligned}$$

$$= \int\limits_0^1 \Big\{ f_x[t;\theta,\varepsilon\delta\mu]\Delta x(t) + \sum_{i=1}^{s} f_{x_i}[t;\theta,\varepsilon\delta\mu]\big(x_0(t-\tau_i) - x_0(t-\tau_{i0}) + \Delta x(t-\tau_i)\big) + \varepsilon f_u[t;\theta,\varepsilon\delta\mu]\delta u(t) \Big\}\,d\theta$$

$$= \Big[\int_0^1 \sigma(t;\theta,\varepsilon\delta\mu)\,d\theta\Big]\Delta x(t) + \sum_{i=1}^{s}\Big[\int_0^1 \varrho_i(t;\theta,\varepsilon\delta\mu)\,d\theta\Big]\big(x_0(t-\tau_i)-x_0(t-\tau_{i0})+\Delta x(t-\tau_i)\big)$$

$$+\varepsilon\Big[\int_0^1 \vartheta(t;\theta,\varepsilon\delta\mu)\,d\theta\Big]\delta u(t)+f_x[t]\Delta x(t)+\sum_{i=1}^{s} f_{x_i}[t]\big(x_0(t-\tau_i)-x_0(t-\tau_{i0})+\Delta x(t-\tau_i)\big)+\varepsilon f_u[t]\delta u(t).$$

Taking into account the last relation for $t\in[t_{00},t_{10}+\delta_2]$, we have

$$R_1(t;t_{00},\varepsilon\delta\mu)=\sum_{p=2}^{6} R_p(t;t_{00},\varepsilon\delta\mu),$$

where

$$R_2(t;t_{00},\varepsilon\delta\mu)=\int_{t_{00}}^{t} Y(\xi;t)\sigma_1(\xi;\varepsilon\delta\mu)\Delta x(\xi)\,d\xi,\quad \sigma_1(\xi;\varepsilon\delta\mu)=\int_0^1 \sigma(\xi;\theta,\varepsilon\delta\mu)\,d\theta,$$

$$R_3(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^{s}\int_{t_{00}}^{t} Y(\xi;t)\varrho_{i1}(\xi;\varepsilon\delta\mu)\big[x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})+\Delta x(\xi-\tau_i)\big]\,d\xi,$$

$$\varrho_{i1}(\xi;\varepsilon\delta\mu)=\int_0^1 \varrho_i(\xi;\theta,\varepsilon\delta\mu)\,d\theta,$$

$$R_4(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^{s}\int_{t_{00}}^{t} Y(\xi;t)f_{x_i}[\xi]\big[x_0(\xi-\tau_i)-x_0(\xi-\tau_{i0})\big]\,d\xi,$$

$$R_5(t;t_{00},\varepsilon\delta\mu)=\sum_{i=1}^{s}\int_{t_{00}}^{t} Y(\xi;t)f_{x_i}[\xi]\big[\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big]\,d\xi,$$

$$R_6(t;t_{00},\varepsilon\delta\mu)=\varepsilon\int_{t_{00}}^{t} Y(\xi;t)\vartheta_1(\xi;\varepsilon\delta\mu)\delta u(\xi)\,d\xi,\quad \vartheta_1(\xi;\varepsilon\delta\mu)=\int_0^1 \vartheta(\xi;\theta,\varepsilon\delta\mu)\,d\theta$$

(see (3.4)). The function $x_0(t)$, $t\in[\widehat{\tau},t_{10}+\delta_2]$, is absolutely continuous, then for each fixed Lebesgue point $\xi_i\in(t_{00},t_{10}+\delta_2)$ of function $\dot{x}_0(\xi-\tau_{i0})$, we get

$$x_0(\xi_i-\tau_i)-x_0(\xi_i-\tau_{i0})=\int_{\xi_i}^{\xi_i-\varepsilon\delta\tau_i} \dot{x}_0(\varsigma-\tau_{i0})\,d\varsigma=-\varepsilon\dot{x}_0(\xi_i-\tau_{i0})\delta\tau_i+\gamma_i(\xi_i;\varepsilon\delta\mu) \tag{3.9}$$

where

$$\lim_{\varepsilon\to0}\frac{\gamma_i(\xi_i;\varepsilon\delta\mu)}{\varepsilon}=0 \ \text{ uniformly for } \ \delta\mu\in V^-. \tag{3.10}$$

Thus, (3.9) is valid for almost all points of the interval $(t_{00},t_{10}+\delta_2)$. From (3.9), taking into account the boudedness of the function

$$\dot{x}_0(t)=\begin{cases}\dot{\varphi}_0(t), & t\in[\widehat{\tau},t_{00}],\\ f\big(t,x_0(t),x_0(t-\tau_{10}),\dots,x_0(t-\tau_{s0}),u_0(t)\big), & t\in(t_{00},t_{10}+\delta_2],\end{cases}$$

it follows that

$$|x_0(\xi_i-\tau_i)-x_0(\xi_i-\tau_{i0})|\le O(\varepsilon\delta\mu) \ \text{ and } \ \Big|\frac{\gamma_i(\xi_i;\varepsilon\delta\mu)}{\varepsilon}\Big|\le const. \tag{3.11}$$

Clearly,

$$|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})|=\begin{cases} o(\xi;\varepsilon\delta\mu) & \text{for } \ \xi\in[t_{00},\rho_{i1}],\\ O(\xi;\varepsilon\delta\mu) & \text{for } \ \xi\in[\rho_{i1},\rho_{i2}]\end{cases} \tag{3.12}$$

(see (3.1)).

Let $\xi\in[\rho_{i2},t_{10}+\delta_1]$, then $\xi-\tau_i\geq t_{00}$, $\xi-\tau_{i0}\geq t_{00}$. Therefore,

$$\begin{aligned}\big|\Delta x(\xi-\tau_i)-\Delta x(\xi-\tau_{i0})\big| &\leq \int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i}|\dot{\Delta} x(s)|\,ds\leq \int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i}L_{K_1,U_1}(\varsigma)\Big[|\Delta x(\varsigma)|\\ &+\sum_{i=1}^{s}|x_0(\varsigma-\tau_i)-x_0(\varsigma-\tau_{i0})|+|\Delta x(\varsigma-\tau_i)|\Big]\,d\varsigma+\varepsilon\alpha\int\limits_{\xi-\tau_{i0}}^{\xi-\tau_i}L_{K_1,U_1}(\varsigma)\,d\varsigma=o(\xi;\varepsilon\delta\mu)\end{aligned} \tag{3.13}$$

(see (2.6), (3.1), (3.3) and (3.11)). According to (3.1), (3.9) and (3.11)–(3.13) for the expressions $R_p(t;t_{00},\varepsilon\delta\mu)$, $p=2,\dots,6$, we have

$$|R_2(t;t_{00},\varepsilon\delta\mu)|\leq\|Y\|O(\varepsilon\delta\mu)\sigma_2(\varepsilon\delta\mu),\quad \sigma_2(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_1}|\sigma_1(\xi;\varepsilon\delta\mu)|\,d\xi,$$

$$|R_3(t;t_{00},\varepsilon\delta\mu)|\leq\|Y\|O(\varepsilon\delta\mu)\sum_{i=1}^{s}\rho_{i2}(\varepsilon\delta\mu),\quad \rho_{i2}(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_1}|\rho_{i1}(\xi;\varepsilon\delta\mu)|\,d\xi,$$

$$R_4(t;t_{00},\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i+\sum_{i=1}^{s}\gamma_{i1}(t;\varepsilon\delta\mu),$$

$$|R_5(t;t_{00},\varepsilon\delta\mu)|=o(t;\varepsilon\delta\mu),$$

$$|R_6(t;t_{00},\varepsilon\delta\mu)|\leq\varepsilon\|Y\|\vartheta_2(\varepsilon\delta\mu),\quad \vartheta_2(\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t_{10}+\delta_1}|\vartheta_1(\xi;\varepsilon\delta\mu)|\,d\xi,$$

where

$$\|Y\|=\sup\big\{|Y(\xi;t)|:\ (\xi,t)\in\Pi\big\},\quad \gamma_{i1}(t;\varepsilon\delta\mu)=\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\gamma_i(\xi;\varepsilon\delta\mu)\,d\xi.$$

Obviously,

$$\Big|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|\leq\|Y\|\int\limits_{t_{00}}^{t_{10}+\delta_1}\big|f_{x_i}[\xi]\big|\,\Big|\frac{\gamma_i(\xi;\varepsilon\delta\mu)}{\varepsilon}\Big|\,d\xi.$$

By the Lebesgue theorem on the passage to the limit under the integral sign, we have

$$\lim_{\varepsilon\to0}\sigma_2(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\rho_{i2}(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\vartheta_2(\varepsilon\delta\mu)=0,\quad \lim_{\varepsilon\to0}\Big|\frac{\gamma_{i1}(t;\varepsilon\delta\mu)}{\varepsilon}\Big|=0$$

uniformly for $(t,\delta\mu)\in[t_{00},t_{10}+\delta_2]\times V^-$ (see (3.10)). Thus,

$$R_p(t;t_{00},\varepsilon\delta\mu)=o(t;\varepsilon\delta\mu),\ \ p=2,3,5,6, \tag{3.14}$$

$$R_4(t;t_{00},\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i \tag{3.15}$$

On the basis of (3.14), (3.15), we obtain

$$R_1(t;t_{00},\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\bigg[\int\limits_{t_{00}}^{t}Y(\xi;t)f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\,d\xi\bigg]\delta\tau_i+o(t;\varepsilon\delta\mu). \tag{3.16}$$

From (3.5), by virtue of (3.7), (3.8) and (3.16), we obtain (1.5), where $\delta x(t;\delta\mu)$ has the form (1.6).

4 Proof of Theorem 1.2

First of all, we note that $\delta\mu\in V^+$, i.e., $t_{00}<t_0$, therefore, we have

$$\Delta x(t)=\begin{cases}\varepsilon\delta\varphi(t) & \text{for } t\in[\widehat{\tau},t_{00}),\\ \varphi(t)-y_0(t) & \text{for } t\in[t_{00},t_0),\\ \Delta y(t) & \text{for } t\in[t_0,t_{10}+\delta_1].\end{cases}$$

In a similar way (see (2.21)), one can prove $|\varphi(t)-y_0(t)|=O(t;\varepsilon\delta\mu),\ \ t\in[t_{00},t_0]$. According to the last relation and Lemma 2.4, we have

$$|\Delta x(t)|\le O(\varepsilon\delta\mu)\ \ \forall\,(t,\varepsilon,\delta\mu)\in[\widehat{\tau},t_{10}+\delta_2]\times[0,\varepsilon_2]\times V^+,$$
$$\Delta x(t_0)=\varepsilon\big[\delta\varphi(t_{00})+(\dot{\varphi}_0^+-f^+)\delta t_0\big]+o(\varepsilon\delta\mu).$$

Let $\widehat{t}\in(t_{00},t_{10})$ be a fixed point, and let $\varepsilon_2\in(0,\varepsilon_1)$ be so small that $t_0<\widehat{t}$ for arbitrary $(\varepsilon,\delta\mu)\in(0,\varepsilon_2)\times V^+$. The function $\Delta x(t)$ satisfies equation (3.3) on the interval $[\widehat{t},t_{10}+\delta_2]$; therefore, by using the Cauchy formula, we can represent it in the form

$$\Delta x(t)=Y(t_0;t)\Delta x(t_0)+\varepsilon\int\limits_{t_0}^{t}Y(\xi;t)f_u[\xi]\delta u(\xi)\,d\xi+\sum_{i=0}^{1}R_i(t;t_0,\varepsilon\delta\mu) \tag{4.1}$$

(see (3.6)). The matrix function $Y(\xi;t)$ is continuous on $[t_{00},\widehat{t}]\times[\widehat{t},t_{10}+\delta_2]$; therefore,

$$Y(t_0;t)\Delta x(t_0)=\varepsilon Y(t_{00};t)[\delta\varphi(t_{00})+(\dot{\varphi}_0^+-f^+)\delta t_0]+o(t;\varepsilon\delta\mu), \tag{4.2}$$

where $o(t;\varepsilon\delta\mu)=Y(t_0,t)o(\varepsilon\delta\mu)$. Let us now transform

$$\begin{aligned}R_{i0}(t;t_0,\varepsilon\delta\mu)&=\varepsilon\int\limits_{t_0-\tau_{i0}}^{t_{00}}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\delta\varphi(\xi)\,d\xi+\int\limits_{t_{00}}^{t_0}Y(\xi+\tau_{i0};t)f_{x_i}[\xi+\tau_{i0}]\Delta x(\xi)\,d\xi\\&=\varepsilon\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(\xi+\tau_0;t)f_{x_i}[\xi+\tau_{i0};t]\delta\varphi(\xi)\,d\xi+o(t;\varepsilon\delta\mu).\end{aligned}$$

Thus,

$$R_0(t;t_0,\varepsilon\delta\mu)=\varepsilon\int\limits_{t_{00}-\tau_0}^{t_{00}}Y(\xi+\tau_0;t)f_{x_i}[\xi+\tau_{i0};t]\delta\varphi(\xi)\,d\xi+o(t;\varepsilon\delta\mu). \tag{4.3}$$

In a similar way, with nonessential changes, for $t\in[\widehat{t},t_{10}+\delta_2]$ one can prove

$$R_1(t;t_0,\varepsilon\delta\mu)=-\varepsilon\sum_{i=1}^{s}\int\limits_{t_{00}}^{t}Y(\xi;t)\big[f_{x_i}[\xi]\dot{x}_0(\xi-\tau_{i0})\delta\tau_i\big]\,d\xi+o(t;\varepsilon\delta\mu). \tag{4.4}$$

Finally, note that

$$\varepsilon\int_{t_0}^{t} Y(\xi;t)\delta f_u[\xi]\delta u(\xi)\,d\xi=\varepsilon\int_{t_{00}}^{t} Y(\xi;t)\delta f_u[\xi]\delta u(\xi)\,d\xi+o(t;\varepsilon\delta\mu) \tag{4.5}$$

for $t\in[\widehat{t},t_{10}+\delta_2]$. Taking into account (4.2)–(4.5), from (4.1), we obtain (1.5), where $\delta x(t;\varepsilon\delta\mu)$ has the form (1.10).

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