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UNILATERAL CONTACT PROBLEMS FOR HOMOGENEOUS HEMITROPIC ELASTIC SOLIDS WITH A FRICTION

Abstract. In the present paper, we study a one-sided contact problem for a micropolar homogeneous elastic hemitropic medium with a friction. Here, on a part of the elastic medium surface with a friction, instead of a normal component of force stress there is prescribed the normal component of the displacement vector. We consider two cases, the so-called coercive case (when the elastic medium is fixed along some part of the boundary) and noncoercive case (without fixed parts). By using the Steklov–Poincaré operator, we reduce this problem to an equivalent boundary variational inequality. Based on our variational inequality approach, we prove the existence and uniqueness theorems for the weak solution. In the coercive case, the problem is unconditionally solvable, and the solution depends continuously on the data of the original problem. In the noncoercive case, we present in a closed-form the necessary condition for the existence of a solution of the contact problem. Under additional assumptions, this condition is also sufficient for the existence of a solution.

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Key words and phrases. Elasticity theory, hemitropic solids, contact problem with a friction, boundary variational inequality.

რეზიუმე. წარმოდგენილ ნაშრომში შესწავლილია ცალმხრივი საკონტაქტო ამოცანა მიკროპოლარული, ერთგვაროვანი, ჰემიტროპული დრეკადი სხეულისთვის ხახუნის გათვალისწინებით. ამ შემთხვევაში დრეკადი სხეულის საზღვრის იმ ნაწილზე, სადაც ხახუნის ეფექტია გათვალისწინებული, ნაცვლად ძაბვის ნორმალური მდგენელისა მოცემულია გადაადგილების ნორმალური მდგენელი. განხილულია ორი შემთხვევა, კოერციტიული (როდესაც სხეული საზღვრის დადებითი ზომის გარკვეული ნაწილით ჩამაგრებულია) და არაკოერციტიული (როდესაც ასეთი ჩამაგრებები არ გვაქვს). სტეკლოვ-პუანკარეს ოპერატორის გამოყენებით განსახილველი ფიზიკური ამოცანა ეკვივალენტურად დაიყვანება სასაზღვრო ვარიაციულ უტოლობაზე. ვარიაციულ უტოლობათა ზოგადი თეორიის საფუძველზე შესწავლილია სუსტი ამონახსნების არსებობისა და ერთადერთობის საკითხი. კერძოდ, დადგენილია, რომ კოერციტიულ შემთხვევაში ამოცანა ამოხსნადია ცალსახად და უპირობოდ, ხოლო არაკოერციტიულ შემთხვევაში ცხადი სახით იწერება ამონახსნის არსებობის აუცილებელი პირობა, რომელიც გარკვეულ დამატებით მოთხოვნებში წარმოადგენს ამონახსნის არსებობის საკმარის პირობასაც.

1 Introduction

In the present paper, we investigate the one-sided contact problem for a homogeneous hemitropic elastic medium with a friction. In the considered model of the theory of elasticity, as distinct from the classical theory, every elementary medium particle undergoes both displacement and rotation. In this case, all mechanical values are expressed in terms of the displacement and rotation vectors.

In their works [2] and [3], E. Cosserat and F. Cosserat created and presented the model of a solid medium in which every material point has six degrees of freedom, three of which are defined by displacement components and the other three by the components of rotation (for the history of the model of elasticity see [5,28,30,31,34,39,40] and the references therein).

A micropolar medium, not possessing symmetry with respect to the inversion, is called a hemitropic or noncentrosymmetric medium.

Improved mathematical models describing hemitropic properties of elastic materials have been obtained and considered in [29] and [1]. The main equations of that model are interrelated and generate a matrix second order differential operator of dimension 6×6. Particular problems for solid media of the hemitropic theory of elasticity have been considered in [35,36,39] and [40]. The basic boundary value problems and also the transmission problems of the hemitropic theory of elasticity with the use of the potential method for smooth and non-smooth Lipschitz domains were studied in [35], the one-sided contact problems of statics of the hemitropic theory of elasticity free from friction were investigated in [16,18,20], and the contact problems of statics and dynamics with a friction were considered in [8–15,17,19,21–24]. Analogous one-sided problems of classical linear theory of elasticity have been considered in many works and monographs (see [4,6,7,26,27,41] and the references therein).

In the present work, we present the basic equations of statics of the elasticity theory for homogeneous hemitropic media in a vector-matrix form, introduce the generalized stress operator and a quadratic form of potential energy. Then we describe mathematical model of boundary conditions which show the contact between a hemitropic medium and a solid body with regard for the friction effect. We will consider the case, where some part of the elastic medium boundary is fixed mechanically. The problem is reduced equivalently to the boundary variational inequality, the question on the existence and uniqueness of a weak solution of the initial problem is treated, and a continuous Lipschitz dependence of the solution on the data of the problem is investigated. Further, we will investigate more complicated cases, where friction is considered on the whole medium boundary. In such cases, the corresponding mathematical problem is, in general, unsolvable. The necessary conditions of solvability are established and the sufficient conditions for the existence of a solution are formulated explicitly.

2 Basic equations and Green's formulas

2.1 Basic equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a C^∞-smooth boundary $S = \partial\Omega$, $\overline{\Omega} = \Omega \cup S$. The domain Ω is assumed to be filled with a homogeneous hemitropic material.

The basic equilibrium equations in the hemitropic theory of elasticity written in components of the displacement and rotation vectors are of the form

$$\begin{aligned}
&(\mu+\alpha)\Delta u(x) + (\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div} u(x) + (\varkappa+\nu)\Delta\omega(x)\\
&\qquad + (\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div}\omega(x) + 2\alpha\operatorname{curl}\omega(x) + \rho F(x) = 0,\\
&(\varkappa+\nu)\Delta u(x) + (\delta+\varkappa-\nu)\operatorname{grad}\operatorname{div} u(x) + 2\alpha\operatorname{curl} u(x) + (\gamma+\varepsilon)\Delta\omega(x)\\
&\qquad + (\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\omega(x) + 4\nu\operatorname{curl}\omega(x) - 4\alpha\omega(x) + \rho\Psi(x) = 0,
\end{aligned} \tag{2.1}$$

where $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator, $\partial_j = \partial/\partial x_j$, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, $\omega = (\omega_1, \omega_2, \omega_3)^\top$ is the vector of rotation, $F = (F_1, F_2, F_3)^\top$ and $\Psi = (\Psi_1, \Psi_2, \Psi_3)^\top$ are the mass force and mass moment calculated per unit of mass, ρ is density of the elastic medium, and α, β, γ, δ, λ, μ, ν, $\varkappa$ and ε are elastic constants (see [1,36]). Here and in what follows, the symbol $(\,\cdot\,)^\top$ denotes transposition.

We introduce a matrix differential operator corresponding to the left-hand side of system (2.1):

$$L(\partial)=\begin{bmatrix} L^{(1)}(\partial) & L^{(2)}(\partial) \\ L^{(3)}(\partial) & L^{(4)}(\partial) \end{bmatrix}_{6\times 6},$$

$$L^{(1)}(\partial):=(\mu+\alpha)\Delta I_3+(\lambda+\mu-\alpha)Q(\partial),$$

$$L^{(2)}(\partial)=L^{(3)}(\partial):=(\varkappa+\nu)\Delta I_3+(\delta+\varkappa-\nu)Q(\partial)+2\alpha R(\partial),$$

$$L^{(4)}(\partial):=\big[(\gamma+\varepsilon)\Delta-4\alpha\big]I_3+(\beta+\gamma-\varepsilon)Q(\partial)+4\nu R(\partial),$$

where I_k is the unit $k\times k$-matrix and

$$Q(\partial)=\big[Q_{kj}(\partial)\big]_{3\times 3},\quad Q_{kj}(\partial)=\partial_k\,\partial_j,\quad R(\partial)=\begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}.$$

The system of equations (2.1) can be rewritten in the matrix form

$$L(\partial)U(x)+\mathcal{G}(x)=0,\ \ x\in\Omega,$$

where $U=(u,\omega)^\top$ and $\mathcal{G}=(\rho F,\rho\Psi)^\top$.

By $T(\partial,n)$ we denote the generalized stress operator of dimension 6×6 (see [36]):

$$T(\partial,n)=\begin{bmatrix} T^{(1)}(\partial,n) & T^{(2)}(\partial,n) \\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n) \end{bmatrix},\quad T^{(j)}=\big[T^{(j)}_{pq}\big]_{3\times 3},\ \ j=\overline{1,4},$$

where

$$T^{(1)}_{pq}(\partial,n):=(\mu+\alpha)\delta_{pq}\partial_n+(\mu-\alpha)n_q\partial_p+\lambda n_p\partial_q,$$

$$T^{(2)}_{pq}(\partial,n):=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,$$

$$T^{(3)}_{pq}(\partial,n):=(\varkappa+\nu)\delta_{pq}\partial_n+(\varkappa-\nu)n_q\partial_p+\delta n_p\partial_q,$$

$$T^{(4)}_{pq}(\partial,n):=(\gamma+\varepsilon)\delta_{pq}\partial_n+(\gamma-\varepsilon)n_q\partial_p+\beta n_p\partial_q-2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.$$

Here, $n(x)=(n_1(x),n_2(x),n_3(x))$ denotes the outward (with respect to Ω) unit normal vector at the point $x\in S$, and $\partial_n=\partial/\partial n$ is the normal derivative in the direction of the vector n. The six-component generalized stress vector has the form

$$T(\partial,n)U=(\mathcal{T}U,\mathcal{M}U)^\top,$$

where $\mathcal{T}U:=T^{(1)}u+T^{(2)}\omega$ is the force stress vector and $\mathcal{M}U:=T^{(3)}u+T^{(4)}\omega$ is the moment stress vector.

2.2 Green's formulas

For the real-valued vector functions $U=(u,\omega)^\top$ and $U'=(u',\omega')^\top$ of the class $[C^2(\overline{\Omega})]^6$ the following Green's formula [36]

$$\int_{\Omega}\big[L(\partial)U\cdot U'+E(U,U')\big]\,dx=\int_{S}\big\{T(\partial,n)U\big\}^+\cdot\{U'\}^+\,dS \tag{2.2}$$

is valid, where $\{\cdot\}^+$ denotes the trace operator on S from Ω, and $E(\cdot\,,\cdot)$ is a bilinear form defined by the equality

$$\begin{aligned} E(U,U') &= E(U',U) \\ &= \sum_{p,q=1}^{3}\Big\{(\mu+\alpha)u'_{pq}u_{pq} + (\mu-\alpha)u'_{pq}u_{qp} + (\varkappa+\nu)\big(u'_{pq}\omega_{pq} + \omega'_{pq}u_{pq}\big) + (\varkappa-\nu)\big(u'_{pq}\omega_{qp} + \omega'_{pq}u_{qp}\big) \\ &\qquad + (\gamma+\varepsilon)\omega'_{pq}\omega_{pq} + (\gamma-\varepsilon)\omega'_{pq}\omega_{qp} + \delta\big(u'_{pp}\omega_{qq} + \omega'_{qq}u_{pp}\big) + \lambda u'_{pp}u_{qq} + \beta\omega'_{pp}\omega_{qq}\Big\}, \end{aligned}$$

where u_{pq} and ω_{pq} are the so-called tensors of deformation and torsion-bending for the hemitropic media,

$$u_{pq} = u_{pq}(U) = \partial_p u_q - \sum_{k=1}^{3}\varepsilon_{pqk}\omega_k, \quad \omega_{pq} = \omega_{pq}(U) = \partial_p\omega_q, \quad p,q = 1,2,3. \tag{2.3}$$

Here and in the sequel, by $a\cdot b$ we denote the scalar product of two vectors $a, b \in \mathbb{R}^m : a\cdot b = \sum\limits_{j=1}^{m} a_j b_j$.

Under certain assumptions on elastic constants (see [1, 10, 23]), specific energy of deformation $E(U,U)$ is a positive definite quadratic form with respect to $u_{pq}(U)$ and $\omega_{pq}(U)$, i.e., there exists a positive number $C_0 > 0$, depending only on the elastic constants, such that

$$E(U,U) \geq C_0 \sum_{p,q=1}^{3}\big[u_{pq}^2 + \omega_{pq}^2\big].$$

The following assertion describes the null space of the energy quadratic form $E(U,U)$ (see [36]).

Lemma 2.1. *Let $U = (u,\omega)^\top \in [C^1(\overline{\Omega})]^6$ and $E(U,U) = 0$ in Ω. Then*

$$u(x) = [a\times x] + b, \;\; \omega(x) = a, \;\; x\in\Omega,$$

where a and b are arbitrary three-dimensional constant vectors and $[\cdot\times\cdot]$ denotes the cross product of two vectors.

Vectors of the type $([a\times x]+b, a)$ are called generalized rigid vectors. We observe that a generalized rigid displacement vector vanishes, i.e., $a = b = 0$ if it is zero at a single point.

Throughout the paper, $L_p(\Omega)$ $(1\leq p\leq\infty)$, $L_2(\Omega) = H^0(\Omega)$ and $H^s(\Omega) = H_2^s(\Omega)$, $s\in\mathbb{R}$, denote, respectively, the Lebesgue and Bessel potential spaces (see, e.g., [32,42]). The corresponding norms we denote by the symbols $\|\cdot\|_{L_p(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$. By $\mathcal{D}(\Omega)$ we denote the class of $C^\infty(\Omega)$ functions with support in the domain Ω. If M is an open proper part of the manifold $\partial\Omega$, i.e., $M\subset\partial\Omega$, $M\neq\partial\Omega$, then by $H^s(M)$ we denote the restriction of the space $H^s(\partial\Omega)$ on M, $H^s(M) := \{r_{_M}\varphi : \varphi\in H^s(\partial\Omega)\}$, where $r_{_M}$ stands for the restriction operator on the set M. Further, let $\widetilde{H}^s(M) := \{\varphi\in H^s(\partial\Omega) : \operatorname{supp}\varphi\subset\overline{M}\}$.

From the positive definiteness of the energy form $E(\cdot\,,\cdot)$ with respect to the variables (2.3) it follows that

$$B(U,U) := \int_\Omega E(U,U)\,dx \geq 0. \tag{2.4}$$

Moreover, there exist positive constants c_1 and c_2, depending only on the material parameters, such that the following Korn's type inequality (see [7, Part I, § 12])

$$B(U,U) \geq c_1\|U\|^2_{[H^1(\Omega)]^6} - c_2\|U\|^2_{[H^0(\Omega)]^6} \tag{2.5}$$

holds for an arbitrary real-valued vector function $U\in[H^1(\Omega)]^6$.

Remark 2.2. If $U\in[H^1(\Omega)]^6$ and on some part $S^*\subset\partial\Omega$ the trace $\{U\}^+$ vanishes, i.e., $r_{_{S^*}}\{U\}^+ = 0$, we have the strict Korn's inequality $B(U,U)\geq C\|U\|^2_{[H^1(\Omega)]^6}$ with some positive constant $C>0$ which does not depend on the vector U. This follows from (2.5) and the fact that in this case $B(U,U)>0$ for $U\neq 0$ (see, e.g., [33, Ch. 2]; [37, Ch. 3, p. 193]).

Remark 2.3. By the standard limiting arguments, Green's formula (2.2) can be extended to the Lipschitz domains and to the vector function $U \in [H^1(\Omega)]^6$ with $L(\partial)U \in [L_2(\Omega)]^6$ and $U' \in [H^1(\Omega)]^6$ (see [32, 37]),

$$\int\limits_{\Omega} \big[L(\partial)U \cdot U' + E(U,U')\big]\,dx = \big\langle \{T(\partial,n)U\}^+, \{U'\}^+ \big\rangle_{\partial\Omega}\,, \tag{2.6}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega}$ denotes duality between the spaces $[H^{-1/2}(\partial\Omega)]^6$ and $[H^{1/2}(\partial\Omega)]^6$ which generalizes the usual inner product in the space $[L_2(\partial\Omega)]^6$. By virtue of this relation, the generalized trace of the stress operator $\{T(\partial,n)U\}^+ \in [H^{-1/2}(\partial\Omega)]^6$ is determined correctly.

3 Contact problems with a friction

3.1 Pointwise and variational formulation of the contact problem

Let the boundary S of the domain Ω be divided into two open, connected and non overlapping parts S_1 and S_2 of positive measure, $S = \overline{S}_1 \cup \overline{S}_2$, $S_1 \cap S_2 = \varnothing$. Assume that the hemitropic elastic body occupying the domain Ω is in contact with another rigid body along the subsurface S_2.

Definition 3.1. A vector function $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ is said to be a weak solution of the equation

$$L(\partial)U + \mathcal{G} = 0, \ \ \mathcal{G} \in [L_2(\Omega)]^6 \tag{3.1}$$

in the domain Ω if

$$B(U,\Phi) = \int\limits_{\Omega} \mathcal{G}\cdot\Phi\,dx \quad \forall\,\Phi \in [\mathcal{D}(\Omega)]^6,$$

where the bilinear form $B(\,\cdot\,,\,\cdot\,)$ is given by formula (2.4).

For the normal and tangential components of the force stress vector we will use, respectively, the following notation:

$$(\mathcal{T}U)_n := \mathcal{T}U \cdot n, \quad (\mathcal{T}U)_s := \mathcal{T}U - n(\mathcal{T}U)_n.$$

Further, let $\mathcal{G} = (\rho F, \rho\Psi)^\top \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S_2)]^3$, $f \in H^{1/2}(S_2)$, $g \in L_\infty(S_2)$, $g \geq 0$.

Consider the following contact problem of statics with a friction.

Problem A. Find a vector function $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ which is a weak solution of equation (3.1) and satisfies the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$ and the following conditions:

$$r_{S_1}\{U\}^+ = 0 \ \text{ on } \ S_1, \tag{3.2}$$

$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \ \text{ on } \ S_2, \tag{3.3}$$

$$r_{S_2}\{u_n\}^+ = f \ \text{ on } \ S_2, \tag{3.4}$$

if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| < g$, then $r_{S_2}\{u_s\}^+ = 0$, if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and $\lambda_1 r_{S_2}\{u_s\}^+ = -\lambda_2 r_{S_2}\{(\mathcal{T}U)_s\}^+$, where the symbol $\{\,\cdot\,\}^+$ stands for the trace operator on S_i $(i=1,2)$ from Ω. Conditions (3.2) and (3.4) are understood in the usual trace sense, whereas (3.3) is understood in the generalized functional sense described in Remark 2.3.

To reduce Problem A to a boundary variational inequality, we first reduce the inhomogeneous equation (3.1) to a homogeneous one. In this connection, we consider the following auxiliary linear boundary value problem.

Find a vector function $U_0 = (u_0,\omega_0)^\top \in [H^1(\Omega)]^6$ that is a weak solution of equation (3.1) and satisfies the conditions

$$\begin{gathered} r_{S_1}\{U_0\}^+ = 0 \ \text{ on } \ S_1, \quad r_{S_2}\{\mathcal{M}U_0\}^+ = 0 \ \text{ on } \ S_2, \\ r_{S_2}\{u_{0n}\}^+ = f \ \text{ on } \ S_2, \quad r_{S_2}\{(\mathcal{T}U_0)_s\}^+ = 0 \ \text{ on } \ S_2. \end{gathered} \tag{3.5}$$

It is well known (see [36]) that this problem is uniquely solvable, because S is neither a surface of revolution, nor a ruled surface. Let $V \in [H^1(\Omega)]^6$ be a solution of Problem A, and let $U_0 \in [H^1(\Omega)]^6$ be a solution of the auxiliary problem (3.5); then the difference $U := V - U_0$ is a solution of the following problem.

Problem A$_0$. Find a weak solution $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ of the equation

$$L(\partial)U = 0 \ \text{ in } \ \Omega \tag{3.6}$$

satisfying the inclusion $r_{_{S_2}}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$ and the following conditions:

$$r_{_{S_1}}\{U\}^+ = 0 \ \text{ on } \ S_1, \tag{3.7}$$

$$r_{_{S_2}}\{\mathcal{M}U\}^+ = \varphi \ \text{ on } \ S_2, \tag{3.8}$$

$$r_{_{S_2}}\{u_n\}^+ = 0 \ \text{ on } \ S_2, \tag{3.9}$$

$$\text{if } \ \big|r_{_{S_2}}\{(\mathcal{T}U)_s\}^+\big| < g, \ \text{ then } \ r_{_{S_2}}\{u_s\}^+ = \psi_0, \tag{3.10}$$

if $|r_{_{S_2}}\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, such that

$$\lambda_1\big[r_{_{S_2}}\{u_s\}^+ - \psi_0\big] = -\lambda_2 r_{_{S_2}}\{(\mathcal{T}U)_s\}^+, \tag{3.11}$$

where the symbol $\{\,\cdot\,\}^+$ stands for the trace operator on S_i (i=1,2) from Ω and $\psi_0 = -r_{_{S_2}}\{u_{0s}\}^+ \in [H^{1/2}(S_2)]^3$.

In what follows, we will study Problem A$_0$. Obviously, if a vector function $U \in [H^1(\Omega)]^6$ is a solution of Problem A$_0$, then the sum $U + U_0$ is a solution of Problem A.

3.2 Reduction of **Problem A$_0$** to a boundary variational inequality

To reduce Problem A$_0$ to an equivalent boundary variational inequality, we recall that the vector $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ is a solution of equation (3.6) satisfying the Dirichlet boundary condition $\{U\}^+ = h$ on S with $h \in [H^{1/2}(S)]^6$ and hence can be uniquely represented by the simple layer potential (see [35])

$$U(x) = V(\mathcal{H}^{-1}h)(x) := \int\limits_S \Gamma(x-y)(\mathcal{H}^{-1}h)(y)\,d_yS, \ \ x \in \Omega,$$

where Γ is the fundamental solution matrix of the operator $L(\partial)$ and $\mathcal{H}$ is the boundary integral operator generated by the trace of the simple layer potential on the boundary S (see the closed-form representation of Γ in [35, 36]),

$$\mathcal{H}(h)(x) = \lim_{\Omega\ni z\to x\in S} \int\limits_S \Gamma(z-y)h(y)\,d_yS = \{V(h)\}^+.$$

Note that the simple layer potential V and the integral operator $\mathcal{H}$ have the following properties (see [35, 36]):

$$V : [H^r(S)]^6 \to [H^{r+3/2}(\Omega)]^6, \quad \mathcal{H} : [H^r(S)]^6 \to [H^{r+1}(S)]^6, \quad r \in \mathrm{R}. \tag{3.12}$$

These operators are continuous. Moreover, $\mathcal{H}$ is an invertible operator and

$$\mathcal{H}^{-1} : [H^r(S)]^6 \to [H^{r-1}(S)]^6, \ \ r \in \mathrm{R}. \tag{3.13}$$

The relation

$$\big\{T(\partial,n)V(h)\big\}^+ = (-2^{-1}I_6 + \mathcal{K})h \ \text{ on } \ S \tag{3.14}$$

holds for an arbitrary $h \in [H^{-1/2}(S)]^6$, where $\mathcal{K}$ is the singular integral operator,

$$\mathcal{K}h(x) = \int\limits_S [T(\partial,n)\Gamma(x-y)]h(y)\,d_yS.$$

Note that

$$-\frac{1}{2}\,I_6+\mathcal{K}:[H^{-1/2}(S)]^6\to[H^{-1/2}(S)]^6$$

is a continuous singular operator of normal type with zero index (for details, see [35, 36]).

Next, for the Dirichlet problem we introduce the so-called Green's operator $G:[H^{1/2}(S)]^6\to[H^1(\Omega)]^6$ which is defined by the relation

$$Gh:=V(\mathcal{H}^{-1}h). \tag{3.15}$$

Obviously, $L(\partial)(Gh)=0$ in Ω and $\{Gh\}^+=h$ on S. Taking into account the properties of the trace operator and mappings (3.12), we find that there exist positive numbers C_1 and C_2 such that

$$C_1\|h\|_{[H^{1/2}(S)]^6}\le\|Gh\|_{[H^1(\Omega)]^6}\le C_2\|h\|_{[H^{1/2}(S)]^6} \tag{3.16}$$

for all $h\in[H^{1/2}(S)]^6$.

Now we introduce a generalized operator of the Steklov–Poincaré type by the relation

$$\mathcal{A}h:=\big\{T(\partial,n)(Gh)\big\}^+=\big\{T(\partial,n)V(\mathcal{H}^{-1}h)\big\}^+=(-2^{-1}I_6+\mathcal{K})(\mathcal{H}^{-1}h). \tag{3.17}$$

By $\Lambda(S)$ we denote the set of restrictions of rigid displacement vectors to S, i.e.,

$$\Lambda(S):=\Big\{\chi(x)=(\rho,a)^\top=\big([a\times x]+b,a\big)^\top,\ x\in S\ |\ \ a,b\in\mathrm{R}\Big\}. \tag{3.18}$$

By using the Green's formula (2.6) for $U=U'=V(\mathcal{H}^{-1}h)$, relations (3.14), (3.17) and (3.18), and the uniqueness theorems for the Dirichlet boundary value problem, we obtain $\ker\mathcal{A}=\Lambda(S)$.

Now we state the following lemma describing the properties of the Steklov–Poincaré operator.

Lemma 3.1. *Let $h,\eta\in[H^{1/2}(S)]^6$ and $g\in[\widetilde{H}^{1/2}(S^*)]^6$, where S^* is a regular open subset of the boundary $S=\partial\Omega$. Then the following assertions hold:*

(i) $\langle\mathcal{A}h,\eta\rangle_S=\langle\mathcal{A}\eta,h\rangle_S$;

(ii) $\mathcal{A}:[H^{1/2}(S)]^6\to[H^{-1/2}(S)]^6$ *is a continuous operator;*

(iii) $\langle\mathcal{A}h,h\rangle_S\ge C_1\|h\|^2_{[H^{1/2}(S)]^6}-C_2\|h\|^2_{[L_2(S)]^6}$;

(iv) $\langle\mathcal{A}g,g\rangle_S\ge C\|g\|^2_{[H^{1/2}(S)]^6}$;

(v) $\langle\mathcal{A}h,h\rangle_S\ge C\|h-\mathcal{P}h\|^2_{[H^{1/2}(S)]^6}$.

Here, $\mathcal{P}$ is the operator of orthogonal projection (in the sense of $L_2(S)$) of the space $[H^{1/2}(S)]^6$ onto the space $\Lambda(S)$; the positive constants C, C_1, and C_2 depend on the elasticity constants and on the geometric properties of the surface S and are independent of h and g.

Proof. Let $h,\eta\in[H^{1/2}(S)]^6$. Since the vector Gh is a weak solution of the homogeneous equation $L(\partial)(Gh)=0$, it follows from the Green's formula (2.6) that

$$\begin{aligned}\langle\mathcal{A}h,\eta\rangle_S&=\big\langle\{T(\partial,n)(Gh)\}^+,\{G\eta\}^+\big\rangle_S=B(Gh,G\eta)=B(G\eta,Gh)\\&=\big\langle\{T(\partial,n)(G\eta)\}^+,\{Gh\}^+\big\rangle_S=\langle\mathcal{A}\eta,h\rangle_S.\end{aligned}$$

This implies assertion (i). Assertion (ii) is obvious, because the operator $\mathcal{A}$ is the composition of the continuous operator $\mathcal{H}^{-1}$ and operator $-2^{-1}I_6+\mathcal{K}$ (see relations (3.14) and (3.17)). The proof of (iii) can be carried out as follows. By using condition (2.5), for an arbitrary $h\in[H^{1/2}(S)]^6$, we obtain the inequality

$$\langle\mathcal{A}h,h\rangle_S=B\big(V(\mathcal{H}^{-1}h),V(\mathcal{H}^{-1}h)\big)\ge c_1\|V(\mathcal{H}^{-1}h)\|^2_{[H^1(\Omega)]^6}-c_2\|V(\mathcal{H}^{-1}h)\|^2_{[L_2(\Omega)]^6}.$$

Relations (3.15) and (3.16) imply the inequalities $\|V(\mathcal{H}^{-1}h)\|_{[H^1(\Omega)]^6} \geq C_1\|h\|_{[H^{1/2}(S)]^6}$. On the other hand, since the space $[L_2(S)]^6$ is compactly embedded in $[H^{-1/2}(S)]^6$, it follows from the continuity of operators (3.12) and (3.13) that

$$\|V(\mathcal{H}^{-1}h)\|_{[L_2(\Omega)]^6} \leq C_1^*\|\mathcal{H}^{-1}h\|_{[H^{-3/2}(S)]^6} \leq C_2^*\|h\|_{[H^{-1/2}(S)]^6} \leq C_3^*\|h\|_{[L_2(S)]^6}$$

with some positive constants C_1^*, C_2^* and C_3^* independent of h.

We finally obtain the inequality

$$\langle \mathcal{A}h, h\rangle_S \geq c_1 C_1^2 \|h\|^2_{[H^{1/2}(S)]^6} - c_2 (C_3^*)^2 \|h\|^2_{[L_2(S)]^6},$$

which implies assertion (iii).

Now, assertion (v) follows from assertion (iii) and the nonnegativity of the operator $\mathcal{A}$, and assertion (iv) is a consequence of (iii). The proof of the lemma is complete. ☐

Our aim is to reduce Problem A_0 to an equivalent boundary variational inequality. To this end, on the space $[H^{1/2}(S_2)]^3$ we introduce a convex continuous functional

$$j(v) = \int\limits_{S_2} g|v_s - \psi_0|\, dS, \quad v \in [H^{1/2}(S_2)]^3 \tag{3.19}$$

and the convex closed set

$$\mathcal{K}_0 = \big\{h = (h^{(1)}, h^{(2)})^\top \in [H^{1/2}(S)]^6 :\ r_{S_1} h = 0,\ r_{S_2} h_n^{(1)} = 0\big\}. \tag{3.20}$$

On the set $\mathcal{K}_0$, we consider the following boundary variational inequality.

Find a function $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ such that the boundary variational inequality

$$\langle \mathcal{A}h_0, h - h_0\rangle_S + j(h^{(1)}) - j(h_0^{(1)}) \geq \big\langle \varphi, r_{S_2}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2} \tag{3.21}$$

holds for all $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0$.

4 Equivalence theorem

Let us prove the equivalence of the boundary variational inequality (3.21) and the contact Problem A_0.

Theorem 4.1. *The boundary variational inequality* (3.21) *and the contact* Problem A_0 *are equivalent in the following sense: if $U \in [H^1(\Omega)]^6$ is a solution of* Problem A_0*, then $h_0 = \{U\}^+ \in [H^{1/2}(S)]^6$ is a solution of the variational inequality* (3.21) *and vice versa, if $h_0 \in \mathcal{K}_0$ is a solution of the variational inequality* (3.21)*, then $U := Gh_0 \in [H^1(\Omega)]^6$ is a solution of* Problem A_0.

Proof. Let $U = (u, \omega)^\top \in [H^1(\Omega)]^6$ be a solution of Problem A_0, and let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top := \{U\}^+$. Since $U \in [H^1(\Omega)]^6$ is a solution of Problem A_0, it readily follows from conditions (3.7) and (3.9) that $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$, and by virtue of the definition of the operator G (see relation (3.15)), the solution U in the domain Ω can be uniquely represented in the form $U = Gh_0$. By taking into account the definition of the Steklov–Poincaré operator, we obtain

$$\begin{aligned}\langle \mathcal{A}h_0, h - h_0\rangle_S &+ j(h^{(1)}) - j(h_0^{(1)}) - \big\langle \varphi, r_{S_2}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2}\\ &= \big\langle \{T(\partial,n)(Gh_0)\}^+, h - h_0\big\rangle_S + j(h^{(1)}) - j(h_0^{(1)}) - \big\langle \varphi, r_{S_2}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2}\end{aligned}$$

for each $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0$. Since h and h_0 are elements of the set $\mathcal{K}_0$ and conditions (3.7) and (3.8) are satisfied, we have

$$\begin{aligned}\langle \mathcal{A}h_0, h - h_0\rangle_S &+ j(h^{(1)}) - j(h_0^{(1)}) - \big\langle \varphi, r_{S_2}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2}\\ &= \big\langle \{T(\partial,n)(Gh_0)\}^+, r_{S_2}(h - h_0)\big\rangle_{S_2} + \big\langle g, r_{S_2}(|h^{(1)} - \psi_0| - |h_0^{(1)} - \psi_0|)\big\rangle_{S_2}\\ &= \big\langle \{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h_s^{(1)} - h_{0s}^{(1)})\big\rangle_{S_2} + \big\langle g, |r_{S_2} h_s^{(1)} - \psi_0| - |r_{S_2} h_{0s}^{(1)} - \psi_0|\big\rangle_{S_2} := I.\end{aligned} \tag{4.1}$$

Let

$$\left|r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+\right| < g,$$

then $r_{S_2}\{h_{0s}^{(1)}\}^+ = \psi_0$ and it is obvious that $I \geq 0$. If

$$\left|\{(\mathcal{T}(Gh_0))_s\}^+\right| = g,$$

then

$$\lambda_1\big[r_{S_2}\{h_{0s}^{(1)}\}^+ - \psi_0\big] = -\lambda_2 r_{S_2}\big\{(\mathcal{T}(Gh_0))_s\big\}^+$$

and when $\lambda_1 \neq 0$, we obtain

$$\begin{aligned} I = \int\limits_{S_2} (\mathcal{T}(Gh_0))_s \cdot \big(h_s^{(1)} - \psi_0 - (h_{0s}^{(1)} - \psi_0)\big)\, ds \\ + \int\limits_{S_2} g\big(|h_s^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\, ds = \int\limits_{S_2} (\mathcal{T}(Gh_0))_s \cdot (h_s^{(1)} - \psi_0)\, ds \\ + \int\limits_{S_2} g|h_s^{(1)} - \psi_0|\, ds - \bigg\{ \int\limits_{S_2} \Big[-\frac{\lambda_2}{\lambda_1}\, |(\mathcal{T}(Gh_0))_s|^2 + \frac{\lambda_2}{\lambda_1} g^2 \Big] \bigg\}\, ds \geq 0. \end{aligned}$$

The case $\lambda_2 \neq 0$ is proved similarly.

Therefore, the right-hand side of equation (4.1) is non-negative and, consequently, we find that inequality (3.21) is satisfied. The proof of the first part of Theorem 4.1 is thereby complete.

Now assume that $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ is a solution of the variational inequality (3.21). Let us show that the vector function $U = (u, \omega)^\top := Gh_0 \in [H^1(\Omega)]^6$ is a solution of Problem A_0. By the definition of Green's operator G, the vector Gh_0 is a weak solution of the equation $L(\partial)U = 0$ in Ω; since $h_0 \in \mathcal{K}_0$, we have $r_{S_1}\{U\}^+ = r_{S_1}\{Gh_0\}^+ = r_{S_1} h_0 = 0$; i.e., condition (3.7) is satisfied. Condition (3.9) is automatically satisfied, since $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ and $r_{S_2}\{u_n\}^+ = r_{S_2} h_{0n}^{(1)} = 0$.

Let $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0$, $h^{(1)} = h_0^{(1)}$, and $h^{(2)} = h_0^{(2)} \pm \chi$, where $\chi \in [\widetilde{H}^{1/2}(S_2)]^3$ is an arbitrary vector function. Since $r_{S_1}(h - h_0) = 0$, it follows from inequality (3.21) that

$$\langle \{\mathcal{M}(Gh_0)\}^+ - \varphi, r_{S_2}\chi\rangle_{S_2} = 0 \quad \forall\, \chi \in [\widetilde{H}^{1/2}(S_2)]^3,$$

so $\{\mathcal{M}(Gh_0)\}^+ = \varphi$; i.e., condition (3.8) is satisfied. Therefore, inequality (3.21) can be represented in the form

$$\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h_s^{(1)} - h_{0s}^{(1)})\big\rangle_{S_2} + j(h^{(1)}) - j(h_0^{(1)}) \geq 0 \;\; \forall\, h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0,$$

i.e.,

$$\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h_s^{(1)} - \psi_0 - (h_{0s}^{(1)} - \psi_0))\big\rangle_{S_2} + \big\langle g, r_{S_2}\big(|h_s^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\big\rangle_{S_2} \geq 0.$$

Let $\chi \in [\widetilde{H}^{1/2}(S_2)]^3$. Since

$$\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi_s\big\rangle_{S_2} = \big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi\big\rangle_{S_2}$$

and $|r_{S_2}\chi_s| \leq |r_{S_2}\chi|$, taking $r_{S_2}(h_s^{(1)} - \psi_0)$ instead of $r_{S_2}\chi_s$, we obtain

$$\begin{aligned} \big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi\big\rangle_{S_2} + \big\langle g, r_{S_2}|\chi|\big\rangle_{S_2} \\ - \Big\{\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h_{0s}^{(1)} - \psi_0)\big\rangle_{S_2} + \big\langle g, r_{S_2}|h_{0s}^{(1)} - \psi_0|\big\rangle\Big\} \geq 0 \;\; \forall\, \chi \in [\widetilde{H}^{1/2}(S_2)]^3. \end{aligned} \tag{4.2}$$

Further, let $t \geq 0$ be an arbitrary number and take $\pm t\chi$ for χ in (4.2)

$$\begin{aligned} t\Big\{ \pm \big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}\chi\big\rangle_{S_2} + \big\langle g, r_{S_2}|\chi|\big\rangle_{S_2} \Big\} \\ - \Big\{\big\langle r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+, r_{S_2}(h_{0s}^{(1)} - \psi_0)\big\rangle_{S_2} + \big\langle g, r_{S_2}|h_{0s}^{(1)} - \psi_0|\big\rangle_{S_2}\Big\} \geq 0 \;\; \forall\, \chi \in [\widetilde{H}^{1/2}(S_2)]^3, \end{aligned}$$

whence, by making t tending first to $+\infty$ and then to 0, we easily derive

$$\big\langle r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+, r_{_{S_2}}(h_{0s}^{(1)}-\psi_0)\big\rangle_{S_2} + \big\langle g, r_{_{S_2}}|h_{0s}^{(1)}-\psi_0|\big\rangle_{S_2} \le 0, \tag{4.3}$$

$$\big|\big\langle r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+, r_{_{S_2}}\chi\big\rangle_{S_2}\big| \le \big\langle g, r_{_{S_2}}|\chi|\big\rangle_{S_2} \quad \forall\,\chi\in[\widetilde{H}^{1/2}(S_2)]^3. \tag{4.4}$$

Now we prove that $r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+\in[L_\infty(S_2)]^3$. To this end, on the space $[\widetilde{H}^{1/2}(S_2)]^3$ we consider the linear functional

$$\Phi(\chi) = \big\langle r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+, r_{_{S_2}}\chi\big\rangle_{S_2} \quad \forall\,\chi\in[\widetilde{H}^{1/2}(S_2)]^3.$$

Inequality (4.4) shows that the functional Φ is continuous on the space $r_{_{S_2}}[\widetilde{H}^{1/2}(S_2)]^3$ with respect to the topology induced by the space $[L_1(S_2)]^3$. Since the space $r_{_{S_2}}[\widetilde{H}^{1/2}(S_2)]^3$ is dense in $[L_1(S_2)]^3$, the functional Φ can be continuously extended to the space $[L_1(S_2)]^3$ preserving the norm. Therefore, by the Riesz theorem, there is a functional $\Phi^*\in[L_\infty(S_2)]^3$ such that

$$\Phi(\chi) = \int\limits_{S_2}\Phi^*\cdot\chi\,dS \quad \forall\,\chi\in[L_1(S_2)]^3.$$

Thus,

$$\big\langle r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+, r_{_{S_2}}\chi\big\rangle_{S_2} = \int\limits_{S_2}\Phi^*\cdot\chi\,dS \quad \forall\,\chi\in[L_1(S_2)]^3,$$

i.e.,

$$\big\langle r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+ - \Phi^*, r_{_{S_2}}\chi\big\rangle_{S_2} = 0 \quad \forall\,\chi\in[\widetilde{H}^{1/2}(S_2)]^3,$$

which implies

$$r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+ = \Phi^*\in[L_\infty(S_2)]^3.$$

It is well known that for an arbitrary essentially bounded function $\widetilde{\psi}\in L_\infty(S_2)$ there is a sequence $\widetilde{\varphi}_l\in C^\infty(S_2)$ with $\operatorname{supp}\widetilde{\varphi}_l\subset S_2$ such that (see, e.g., [38, Lemma 1.4.2])

$$\lim_{l\to\infty}\widetilde{\varphi}_l(x) = \widetilde{\psi}(x) \ \text{ for almost all } \ x\in S_2,$$
$$|\widetilde{\varphi}_l(x)| \le \operatorname*{ess\,sup}_{y\in S_2}|\widetilde{\psi}(y)| \ \text{ for almost all } \ x\in S_2.$$

Therefore, by the Lebesgue dominated convergence theorem, it follows from inequality (4.4) that

$$\int\limits_{S_2}\big[\pm\{(\mathcal{T}(Gh_0))_s\}^+\cdot\chi - g|\chi|\big]\,dS \le 0 \quad \forall\,\chi\in[L_\infty(S_2)]^3.$$

Instead of χ we can put $\gamma(S^*)\chi$, where $\chi\in[L_\infty(S_2)]^3$ and $\gamma(S^*)$ is the characteristic function of an arbitrary measurable subset $S^*\subset S_2$. As a result, we arrive at the inequality $\pm\{(\mathcal{T}(Gh_0))_s\}^+\cdot\chi\le g|\chi|$ on S_2 for all $\chi\in[L_\infty(S_2)]^3$ and, by choosing $\chi=\{(\mathcal{T}(Gh_0))_s\}^+$, we finally get

$$\big|r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+\big| \le g \ \text{ on } \ S_2. \tag{4.5}$$

In view of (4.3) and (4.5), we obtain

$$r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+\cdot r_{_{S_2}}(h_{0s}^{(1)}-\psi_0) + g|r_{_{S_2}}(h_{0s}^{(1)}-\psi_0)| = 0 \ \text{ on } \ S_2. \tag{4.6}$$

Now, it is evident that if $|r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+|<g$, then (4.6) implies $r_{_{S_2}}h_{0s}^{(1)}=\psi_0$. Also, if $|r_{_{S_2}}\{(\mathcal{T}(Gh_0))_s\}^+|=g$, then (4.6) can be rewritten as

$$g|r_{_{S_2}}(h_{0s}^{(1)}-\psi_0)|(\cos\alpha+1) = 0 \ \text{ on } \ S_2,$$

where α is the angle between the vectors $r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+$ and $r_{S_2}(h_{0s}^{(1)} - \psi_0)$ at a point $x \in S_2$. Therefore, there exist the functions $\lambda_1(x) \geq 0$ and $\lambda_2(x) \geq 0$ such that$\lambda_1(x) + \lambda_2(x) > 0$ and

$$\lambda_1(x) r_{S_2}(h_{0s}^{(1)} - \psi_0) = -\lambda_2(x) r_{S_2}\{(\mathcal{T}(Gh_0))_s\}^+ \ \text{ on } \ S_2.$$

Moreover, we can assume that λ_1 belongs to the same class as $\{(\mathcal{T}(Gh_0))_s\}^+$ and λ_2 belongs to the same class as $r_{S_2}(h_{0s}^{(1)} - \psi_0)$.

Thus, conditions (3.10) and (3.11) of Problem A_0 hold as well, and the proof of Theorem 4.1 is complete. □

5 The existence and uniqueness of a solution

5.1 Uniqueness

Let us prove the following uniqueness theorem.

Theorem 5.1. Problem A_0 *has at most one solution.*

Proof. Let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ and $\widetilde{h}_0 = (\widetilde{h}_0^{(1)}, \widetilde{h}_0^{(2)})^\top \in \mathcal{K}_0$ be two arbitrary solutions of the variational inequality (3.21). Then

$$\begin{aligned}
&\big\langle \mathcal{A}h_0, \widetilde{h}_0 - h_0 \big\rangle_S + j(\widetilde{h}_0^{(1)}) - j(h_0^{(1)}) \geq \big\langle \varphi, r_{S_2}(\widetilde{h}_0^{(2)} - h_0^{(2)}) \big\rangle_{S_2}, \\
&\big\langle \mathcal{A}\widetilde{h}_0, h_0 - \widetilde{h}_0 \big\rangle_S + j(h_0^{(1)}) - j(\widetilde{h}_0^{(1)}) \geq \big\langle \varphi, r_{S_2}(h_0^{(2)} - \widetilde{h}_0^{(2)}) \big\rangle_{S_2}.
\end{aligned}$$

By summing these inequalities, we obtain $\langle \mathcal{A}(h_0 - \widetilde{h}_0), h_0 - \widetilde{h}_0 \rangle_S \leq 0$. Since $\mathcal{A}$ is a positive definite operator, it follows that $\langle \mathcal{A}(h_0 - \widetilde{h}_0), h_0 - \widetilde{h}_0 \rangle_S = 0$. By virtue of relation (3.17) and Lemma 2.1, we have

$$\begin{aligned}
0 &= \big\langle \mathcal{A}(h_0 - \widetilde{h}_0), h_0 - \widetilde{h}_0 \big\rangle_S \\
&= \big\langle \{T(\partial, n) V(\mathcal{H}^{-1}(h_0 - \widetilde{h}_0))\}^+, h_0 - \widetilde{h}_0 \big\rangle_S = \big\langle \{T(\partial, n) G(h_0 - \widetilde{h}_0)\}^+, h_0 - \widetilde{h}_0 \big\rangle_S \\
&= \big\langle \{T(\partial, n) G(h_0 - \widetilde{h}_0)\}^+, \{G(h_0 - \widetilde{h}_0)\}^+ \big\rangle_S = B\big(G(h_0 - \widetilde{h}_0), G(h_0 - \widetilde{h}_0)\big).
\end{aligned}$$

Hence we derive the relation $G(h_0 - \widetilde{h}_0) = V(\mathcal{H}^{-1}(h_0 - \widetilde{h}_0)) = \big([a \times x] + b, a\big)^\top$ in Ω. Since $h_0, \widetilde{h}_0 \in \mathcal{K}_0$, we have $r_{S_1}\{G(h_0 - \widetilde{h}_0)\}^+ = r_{S_1}(h_0 - \widetilde{h}_0) = 0$; i.e., $([a \times x] + b, a)^\top = 0$ on S_1. Consequently, $a = b = 0$ and $V(\mathcal{H}^{-1}(h_0 - \widetilde{h}_0)) = 0$ in Ω. Therefore, $h_0 = \widetilde{h}_0$ on S. □

5.2 Existence of a solution

To prove the existence of a solution, on the set $\mathcal{K}_0$ we introduce the functional

$$\mathcal{I}(h) = \frac{1}{2} \langle \mathcal{A}h, h \rangle_S + j(h^{(1)}) - \langle \varphi, r_{S_2} h^{(2)} \rangle_{S_2} \ \ \forall\, h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0. \tag{5.1}$$

Since $\mathcal{A}$ is a symmetric operator (see Lemma 3.1(i)), it follows that the existence of a solution of the variational inequality (3.21) is equivalent to the existence of an element of the set $\mathcal{K}_0$ minimising the functional (5.1); i.e., the variational inequality (3.21) is equivalent to the following minimization problem:

$$\mathcal{I}(h_0) = \inf_{h \in \mathcal{K}} \mathcal{I}(h). \tag{5.2}$$

By the general theory of variational inequalities (see [4,25]), the solvability of the minimization problem (5.2) readily follows from the coerciveness of the functional $\mathcal{I}$, i.e., from the property

$$\mathcal{I}(h) \to \infty \ \text{ as } \ \|h\|_{[H^{1/2}(S)]^6} \to \infty, \ \ h \in \mathcal{K}_0.$$

Since $\mathcal{A}$ is a coercive operator on the set $\mathcal{K}_0$ (see Lemma 3.1(iv)) and $j(h^{(1)}) \geq 0$, we find that the coerciveness of the consequence of the obvious estimate

$$\mathcal{I}(h) \geq C_1\|h\|^2_{[H^{1/2}(S)]^6} - C_2\|h\|_{[H^{1/2}(S)]^6}, \quad h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0,$$

where C_1 and C_2 are the positive constants independent of h. Consequently, functional (5.1) is coercive on the closed set $\mathcal{K}_0$. In addition, $\mathcal{I}$ is a convex continuous functional. By the general theory of variational inequalities (see [4, 25]), we find that the variational inequality (3.21) has a unique solution. Therefore, from Theorem 4.1 we obtain the following assertion of the existence of the solution of Problem A_0.

Theorem 5.2. *Let* $\operatorname{mes} S_1 > 0$, $\varphi \in [H^{-1/2}(S_2)]^3$, $g \in L_\infty(S_2)$ *and* $g \geq 0$. *Then the variational inequality* (3.21) *has a unique solution* $h_0 \in [H^{1/2}(S)]^6$, *and* $U = Gh_0$ *is a solution of* Problem A_0.

Remark 5.3. Let $\operatorname{mes} S_1 > 0$, $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S_2)]^3$, $f \in H^{1/2}(S_2)$, $g \in L_\infty(S_2)$ and $g \geq 0$. Then Problem A has a unique solution which can be represented in the form $U + U_0$, where U is a solution of Problem A_0 and U_0 is a solution of the auxiliary problem (3.5).

5.3 Lipschitz continuous dependence of the solution on the problem data

Let $U \in [H^1(\Omega)]^6$ and $\widetilde{U} \in [H^1(\Omega)]^6$ be two solutions of Problem A_0 corresponding to the data φ, g and $\widetilde{\varphi}$, $\widetilde{g}$, respectively. Further, let $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ and $\widetilde{h}_0 = (\widetilde{h}_0^{(1)}, \widetilde{h}_0^{(2)})^\top \in \mathcal{K}_0$ be the traces of the vector functions U and $\widetilde{U}$, respectively, on the boundary S. By Theorem 4.1, the vectors h_0 and $\widetilde{h}_0$ are the solutions of the corresponding variational inequalities (3.21) for the above-introduced data. Therefore, we have two variational inequalities of form (3.21), one for h_0 and another for $\widetilde{h}_0$. By substituting $\widetilde{h}_0$ for h into the first inequality and h_0 into the second one, we obtain the inequalities:

$$\langle \mathcal{A}h_0, \widetilde{h}_0 - h_0\rangle_S + \int\limits_{S_2} g\big(|\widetilde{h}_{0s}^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\,dS \geq \big\langle \varphi, r_{S_2}(\widetilde{h}_0^{(2)} - h_0^{(2)})\big\rangle_{S_2},$$

$$\langle \mathcal{A}\widetilde{h}_0, h_0 - \widetilde{h}_0\rangle_S + \int\limits_{S_2} \widetilde{g}\big(|h_{0s}^{(1)} - \psi_0| - |\widetilde{h}_{0s}^{(1)} - \psi_0|\big)\,dS \geq \big\langle \widetilde{\varphi}, r_{S_2}(h_0^{(2)} - \widetilde{h}_0^{(2)})\big\rangle_{S_2}.$$

By summing these inequalities, we obtain

$$\big\langle \mathcal{A}(h_0 - \widetilde{h}_0), \widetilde{h}_0 - h_0\big\rangle_S + \int\limits_{S_2} \big(g - \widetilde{g})(|\widetilde{h}_{0s}^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\,dS \geq \big\langle \varphi - \widetilde{\varphi}, r_{S_2}(\widetilde{h}_0^{(2)} - h_0^{(2)})\big\rangle_{S_2},$$

i.e.,

$$\big\langle \mathcal{A}(h_0 - \widetilde{h}_0), h_0 - \widetilde{h}_0\big\rangle_S \leq \int\limits_{S_2} (g - \widetilde{g})\big(|\widetilde{h}_{0s}^{(1)} - \psi_0| - |h_{0s}^{(1)} - \psi_0|\big)\,dS + \big\langle \widetilde{\varphi} - \varphi, r_{S_2}(\widetilde{h}_0^{(2)} - h_0^{(2)})\big\rangle_{S_2}.$$

This inequality, together with (3.16), property (iv) of the operator $\mathcal{A}$ (see Lemma 3.1(iv)), and the continuous inclusion $H^{1/2}(S) \subset L_2(S)$ implies the Lipschitz estimate

$$\|U - \widetilde{U}\|_{[H^1(\Omega)]^6} \leq C_1\|h_0 - \widetilde{h}_0\|_{[H^{1/2}(S)]^6} \leq C_2\big(\|\varphi - \widetilde{\varphi}\|_{[H^{-1/2}(S)]^6} + \|g - \widetilde{g}\|_{L_2(S)}\big),$$

where C_1 and C_2 are the positive constants independent of U and $\widetilde{U}$ and the data of the problem under consideration.

6 The semicoercive case

Let $S_1 = \varnothing$, $S_2 = S$, $\mathcal{G} \in [L_2(\Omega)]^6$, $\varphi \in [H^{-1/2}(S)]^3$, $g \in L_\infty(S)$ and $g \geq 0$. Consider the boundary contact problem.

Problem B. Find a vector function $U=(u,\omega)^\top\in[H^1(\Omega)]^6$ which is a weak solution of equation (3.1) in the domain Ω, satisfying the inclusion $\{(\mathcal{T}U)_s\}^+\in[L_\infty(S)]^3$ and the following boundary conditions on the surface S:

$$\{\mathcal{M}U\}^+=\varphi,\ \ \{u_n\}^+=0,$$

if $|\{(\mathcal{T}U)_s\}^+|<g$, then $\{u_s\}^+=0$, if $|\{(\mathcal{T}U)_s\}^+|=g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and $\lambda_1\{u_s\}^+=-\lambda_2\{(\mathcal{T}U)_s\}^+$.

To reduce Problem B to an equivalent boundary variational inequality, we first reduce the inhomogeneous equation (3.1) to a homogeneous one. In this connection, we consider the following auxiliary linear boundary value problem.

Find a weak solution $U_0=(u_0,\omega_0)^\top\in[H^1(\Omega)]^6$ of equation (3.1) in the domain Ω under the conditions

$$\{u_0\}^+=0,\quad \{\mathcal{M}U_0\}^+=0 \tag{6.1}$$

on S. It is well known (see [23]) that the problem is uniquely solvable. Let $W\in[H^1(\Omega)]^6$ be a solution of Problem B, and let $U_0\in[H^1(\Omega)]^6$ be a solution of the auxiliary problem (6.1), then the difference $U:=W-U_0$ is a solution of the following problem.

Problem $\mathsf{B_0}$. Find a vector function $U=(u,\omega)^\top\in[H^1(\Omega)]^6$ that is a weak solution of the homogeneous equation

$$L(\partial)U=0\ \text{ in }\ \Omega$$

satisfying the inclusion $\{(\mathcal{T}U)_s\}^+\in[L_\infty(S)]^3$ and the following conditions on S:

$$\{\mathcal{M}U\}^+=\varphi,\ \ \{u_n\}^+=0;$$

if $|\{(\mathcal{T}U)_s\}^++\varphi_0|<g$, then $\{u_s\}^+=0$, if $|\{(\mathcal{T}U)_s\}^++\varphi_0|=g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and

$$\lambda_1\{u_s\}^+=-\lambda_2\big(\{(\mathcal{T}U)_s\}^++\varphi_0\big),$$

where $\varphi_0=\{(\mathcal{T}U_0)_s\}^+$.

By analogy with the preceding coercive case (see Theorem 4.1), one can show that Problem B_0 is equivalent to the following boundary variational inequality.

Find a vector $h_0=(h_0^{(1)},h_0^{(2)})^\top\in\mathcal{K}$ such that the inequality

$$\langle\mathcal{A}h_0,h-h_0\rangle_S+j_1(h^{(1)})-j_1(h_0^{(1)})\ge\langle\varphi,h^{(2)}-h_0^{(2)}\rangle_S \tag{6.2}$$

holds for all $h=(h^{(1)},h^{(2)})^\top\in\mathcal{K}$, where

$$j_1(v)=\int\limits_S g|v_s|\,dS+\langle\varphi_0,v_s\rangle_S,\ \ v\in[H^{1/2}(S)]^3,$$

$$\mathcal{K}=\big\{h=(h^{(1)},h^{(2)})^\top\in[H^{1/2}(S)]^6:\ h_n^{(1)}=0\big\}. \tag{6.3}$$

Note that the variational inequality (6.2) is equivalent to Problem B_0 in the following sense: if $U\in[H^1(\Omega)]^6$ is a solution of Problem B_0, then $h_0=\{U\}^+\in\mathcal{K}$ is a solution of the variational inequality (6.2); conversely, if $h_0\in\mathcal{K}$ is a solution of the variational inequality (6.2), then $Gh_0\in[H^1(\Omega)]^6$ is a weak solution of Problem B_0 (here the operator G is defined by relation (3.15)).

Let $h_0=(h_0^{(1)},h_0^{(2)})^\top\in\mathcal{K}$ be a solution of the variational inequality (6.2). By substituting first $h=0$ and then $h=2h_0$ into inequality (6.2), we obtain the relation $\langle\mathcal{A}h_0,h_0\rangle_S+j_1(h_0^{(1)})=\langle\varphi,h_0^{(2)}\rangle_S$ which, together with (6.2), implies that

$$\langle\mathcal{A}h_0,h\rangle_S+j_1(h^{(1)})\ge\langle\varphi,h^{(2)}\rangle_S. \tag{6.4}$$

Let $\xi=(\rho,a)^\top\in\Lambda(S)$ and $\rho_n=0$ on S. By substituting $\pm\xi\in\Lambda(S)$ for h into inequality (6.4) ($\Lambda(S)$ is defined by relation (3.18)) and taking into account the relation $\ker\mathcal{A}=\Lambda(S)$, we obtain the inequality

$$\int\limits_S g|\rho_s|\,dS-\big|\langle\varphi,a\rangle_S-\langle\varphi_0,\rho_s\rangle_S\big|\ge0. \tag{6.5}$$

Inequality (6.5) is a necessary condition for the solvability of the variational inequality (6.2).

Consider the case in which inequality (6.5) is strict. Taking into account the fact that the space $\Lambda(S)$ has finite dimension ($\dim\Lambda(S)=6$), one can readily see that inequality (6.5) is equivalent to the relation

$$\int\limits_S g|\rho_s|\,dS - \big|\langle\varphi,a\rangle_S - \langle\varphi_0,\rho_s\rangle_S\big| \geq \|\xi\|_{[L_2(\Omega)]^6} \tag{6.6}$$

with some positive constant C and with an arbitrary $\xi=(\rho,a)^\top\in\Lambda(S)$. Let $\mathcal{P}$ be the operator of orthogonal projection of the space $[H^{1/2}(S)]^6$ onto $\Lambda(S)$ in the sense of $[L_2(S)]^6$; i.e., any function $h\in[H^{1/2}(S)]^6$ can be represented in the form $h=\xi+\chi$, where $\xi=(\rho,a)^\top=\mathcal{P}h\in\Lambda(S)$ and $\chi=(\eta,\zeta)^\top\in\Lambda^\perp(S):=\{h\in[H^{1/2}(S)]^6:(h,\xi)_{[L_2(S)]^6}=0\ \forall\,\xi\in\Lambda(S)\}$.

One can readily see that the norm $\|h\|_{[H^{1/2}(S)]^6}$ is equivalent to the norm $\|\chi\|_{[H^{1/2}(S)]^6}+\|\xi\|_{[L_2(S)]^6}$. On the convex closed set $\mathcal{K}$ we introduce the continuous convex functional

$$\mathcal{I}_1(h)=\frac12\langle\mathcal{A}h,h\rangle_S+j_1(h^{(1)})-\langle\varphi,h^{(2)}\rangle_S,\ \ h=(h^{(1)},h^{(2)})^\top\in\mathcal{K}$$

$\forall\,h=\chi+\xi\in[H^{1/2}(S)]^6$ with $\chi=(\eta,\zeta)^\top$ and $\xi=(\rho,a)^\top$, we obtain

$$\begin{aligned}\mathcal{I}_1(h)=\mathcal{I}_1(\chi+\xi)&=\frac12\big\langle\mathcal{A}(\chi+\xi),\chi+\xi\big\rangle_S+j_1(\eta+\rho)-\langle\varphi,\zeta+a\rangle_S\\&=\frac12\langle\mathcal{A}\chi,\chi\rangle_S-\langle\varphi,\zeta\rangle_S+j_1(\rho)-\langle\varphi,a\rangle_S+j_1(\eta+\rho)-j_1(\rho)\\&\qquad\geq C_1\|\chi\|^2_{[H^{1/2}(S)]^6}-C_2\|\chi\|_{[H^{1/2}(S)]^6}+C\|\xi\|_{[L_2(S)]^6}+j_1(\eta+\rho)-j_1(\rho),\end{aligned}$$

with some positive constants C, C_1 and C_2. Now let us estimate the difference $j_1(\eta+\rho)-j_1(\rho)$. We have

$$\begin{aligned}j_1(\eta+\rho)-j_1(\rho)&=\int\limits_S g|\eta_s+\rho_s|\,dS+\langle\varphi_0,\eta_s+\rho_s\rangle_S-\int\limits_S g|\rho_s|\,dS-\langle\varphi_0,\rho_s\rangle_S\\&=\int\limits_S g\big(|\eta_s+\rho_s|-|\rho_s|\big)\,dS+\langle\varphi_0,\eta_s\rangle_S\geq-\int\limits_S g|\eta_s|\,dS-C_3\|\chi\|_{[H^{1/2}(S)]^6}\geq-C_4\|\chi\|_{[H^{1/2}(S)]^6},\end{aligned}$$

where C_4 is a positive constant independent of η and ρ. By taking into account this inequality, we finally obtain the estimate

$$\mathcal{I}_1(h)\geq C_1\|\chi\|^2_{[H^{1/2}(S)]^6}+C\|\xi\|_{[L_2(S)]^6}-C_5\|\chi\|_{[H^{1/2}(S)]^6},$$

which implies that

$$\mathcal{I}_1(h)\to+\infty\ \text{ as }\ \|h\|_{[H^{1/2}(S)]^6}\to\infty,\ \ h\in\mathcal{K}.$$

We have thereby shown that the functional $\mathcal{I}_1$ is coercive and the minimization problem is solvable for this functional. Consequently, the corresponding variational inequality (6.2) is solvable (see [4,25]). By virtue of the symmetry of the operator $\mathcal{A}$, the problem of minimization of the functional $\mathcal{I}_1$ on the space $[H^{1/2}(S)]^6$ is equivalent to the solvability of the variational inequality (6.2). Next, note that $\langle\mathcal{A}(h_0-\widetilde{h}_0),h_0-\widetilde{h}_0\rangle_S=0$ for two possible solutions h_0 and $\widetilde{h}_0$ of the variational inequality (6.2) in the set $\mathcal{K}$. Hence it follows that $h_0-\widetilde{h}_0=([a\times x]+b,a)^\top$, $a,b\in\mathrm{R}^3$. We have thereby proved the following theorem on the existence and uniqueness of the solution.

Theorem 6.1. *Let $S_1=\varnothing$, $\varphi\in[H^{-1/2}(S)]^3$, $g\in L_\infty(S)$, $g\geq0$ and let inequality* (6.6) *be satisfied. Then the variational inequality* (6.2)*is solvable and if $h_0\in\mathcal{K}$ is a solution of inequality* (6.2)*, then $U=Gh_0$ is a solution of* Problem B_0. *Moreover, two solutions can differ from each other only by a rigid displacement vector.*

Remark 6.2. Let $S_1=\varnothing$, $\mathcal{G}\in[L_2(\Omega)]^6$, $\varphi\in[H^{-1/2}(S)]^3$, $g\in L_\infty(S)$, $g\geq0$ and let inequality (6.6) be satisfied. Then Problem B has a solution which can be represented in the form $U+U_0$, where U is a solution of Problem B_0 and U_0 is a solution of the auxiliary problem (6.1).

Remark 6.3. Let the boundary $S = \partial\Omega$ fall into three mutually disjoint parts S_1, S_T and S_2 such that $\overline{S}_1 \cup \overline{S}_T \cup \overline{S}_2 = S$, $\overline{S}_1 \cap \overline{S}_2 = \varnothing$. By analogy with the coercive case, we can study the problem, when on S_T the traction boundary condition $r_{S_T}\{T(\partial,n)U\}^+ = Q$ is assigned, where $Q \in [H^{-1/2}(S_T)]^6$. The conditions on the parts S_1 and S_2 in this case remain the same as in Problem A.

To reduce this problem to a boundary variational inequality, we first consider the following auxiliary problem.

Find a vector function $U_0 = (u_0,\omega_0)^\top \in [H^1(\Omega)]^6$ that is a weak solution of equation (3.1) in the domain Ω and satisfies the boundary conditions

$$r_{S_2}\{U_0\}^+ = 0,\ \ r_{S_T}\{T(\partial,n)U_0\}^+ = 0,$$
$$r_{S_2}\{\mathcal{M}U_0\}^+ = 0,\ \ r_{S_2}\{u_{0n}\}^+ = f,\ \ r_{S_2}\{(\mathcal{T}U_0)_s\}^+ = 0.$$

It is well known that this problem has a unique weak solution (see [4,25]), because S is neither a surface of revolution, nor a ruled surface. Obviously, if V is a solution of the above-considered problem and U_0 is a solution of the auxiliary problem, then the difference $U := V - U_0$ is a solution of the following problem.

Find a weak solution $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ of the equation

$$L(\partial)U = 0 \ \text{ on } \ \Omega,$$

which satisfies the inclusion $r_{S_2}\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S_2)]^3$ and the following conditions:

$$r_{S_1}\{U\}^+ = 0 \ \text{ on } \ S_1, \quad r_{S_T}\big\{T(\partial,n)U\big\}^+ = Q \ \text{ on } \ S_T,$$
$$r_{S_2}\{\mathcal{M}U\}^+ = \varphi \ \text{ on } \ S_2, \quad r_{S_2}\{u_n\}^+ = 0 \ \text{ on } \ S_2,$$

if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| < g$, then $r_{S_2}\{u_s\}^+ = \psi_0$, whereas if $|r_{S_2}\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and $\lambda_1(r_{S_2}\{u_s\}^+ - \psi_0) = -\lambda_2 r_{S_2}\{(\mathcal{T}U)_s\}^+$, where $\psi_0 = -r_{S_2}\{u_{os}\}^+$. Just as above, this problem can be reduced to an equivalent boundary variational inequality.

Find a vector $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}_0$ such that the inequality

$$\langle \mathcal{A}h_0, h - h_0\rangle_S + j(h^{(1)}) - j(h_0^{(1)}) \geq \langle Q, r_{S_T}(h-h_0)\rangle_{S_T} + \langle \varphi, h^{(2)} - h_0^{(2)}\rangle_S$$

holds for all $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}_0$, where the functional j and the convex set $\mathcal{K}_0$ are defined by relations (3.19) and (3.20), respectively. The proof of the existence, uniqueness and Lipschitz continuous dependence of the solution on the problem data in this case can be carried out just as in Problem A_0 in the coercive case.

Remark 6.4. By analogy with the non-coercive case, we can study the problem when on the part S_1 of the boundary instead of the Dirichlet condition (3.7) there is assigned the tractional boundary condition $r_{S_1}\{T(\partial,n)U\}^+ = Q$, where $Q \in [\widetilde{H}^{-1/2}(S_1)]^6$. Moreover, we assume that the vector φ appearing in condition (3.8) belongs to the space $[\widetilde{H}^{-1/2}(S_2)]^3$ and the conditions imposed on the part S_2 are the same as in Problem A_0.

To reduce the above problem to the equivalent boundary variational inequality, we preliminarily reduce the inhomogeneous equation (3.1) to a homogeneous one. In this connection, we consider the following auxiliary problem.

In the domain Ω, find a weak solution $U_0 = (u_0,\omega_0)^\top \in [H^1(\Omega)]^6$ of equation (3.1) with the following condition on S:

$$r_{S_1}\big\{T(\partial,n)U_0\big\}^+ = 0,\ \ r_{S_2}\{u_0\}^+ = 0,\ \ r_{S_2}\{\mathcal{T}U_0\}^+ = 0.$$

By [23], this problem is uniquely solvable. In this regard, we also consider the following problem.

Problem C_0. Find a vector function $U = (u,\omega)^\top \in [H^1(\Omega)]^6$ which is a weak solution of the homogeneous equation

$$L(\partial)U = 0 \ \text{ in } \ \Omega$$

satisfying the inclusion $\{(\mathcal{T}U)_s\}^+ \in [L_\infty(S)]^3$ and the following conditions on S:

$$r_{S_1}\{T(\partial,n)U\}^+ = Q, \;\; r_{S_2}\{\mathcal{M}U\}^+ = \varphi - \varphi_0,$$
$$\varphi_0 = r_{S_2}\{\mathcal{M}U_0\}^+ \in [H^{-1/2}(S_2)]^3, \;\; r_{S_2}\{u_n\}^+ = 0;$$

if $|\{(\mathcal{T}U)_s\}^+| < g$, then $r_{S_2}\{u_s\}^+ = 0$, if $|\{(\mathcal{T}U)_s\}^+| = g$, then there exist nonnegative functions λ_1 and λ_2 which do not vanish simultaneously, and $\lambda_1 r_{S_2}\{u_s\}^+ = -\lambda_2 r_{S_2}\{(\mathcal{T}U)_s\}^+$. In this case, we obtain the following boundary variational inequality.

Find a function $h_0 = (h_0^{(1)}, h_0^{(2)})^\top \in \mathcal{K}$ such that the inequality

$$\langle \mathcal{A}h_0, h - h_0\rangle_S + j_1(h^{(1)}) - j_1(h_0^{(1)}) \geq \big\langle r_{S_1}Q, r_{S_1}(h - h_0)\big\rangle_{S_1} + \big\langle r_{S_2}(\varphi - \varphi_0), r_{S_2}(h^{(2)} - h_0^{(2)})\big\rangle_{S_2} \quad (6.7)$$

holds for all $h = (h^{(1)}, h^{(2)})^\top \in \mathcal{K}$, where $j_1(h^{(1)}) = \int\limits_{S_2} g|h_s^{(1)}|\,dS$ and $\mathcal{K}$ defined by formula (6.3).

Now the necessary condition for the solvability of the variational inequality acquires the form

$$\int\limits_{S_2} g|\rho_s|\,dS - \Big|\big\langle r_{S_2}(\varphi - \varphi_0), a\big\rangle_{S_2} + \big\langle r_{S_1}Q, r_{S_1}\xi\big\rangle_{S_1}\Big| \geq 0 \quad (6.8)$$

for all $\xi = (\rho, a)^\top \in \Lambda(S)$, $r_{S_2}\rho_n = 0$. When inequality (6.8) is strict, then, just as in the non-coercive case, one can show that condition (6.8) is sufficient for the solvability of inequality (6.7).

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